Some Theoretical Aspects of Car Trips within Urban Area

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OD trips in urban area are described by an ergodic Markov chain for each kind of car and have a certain steady pattern for each trip. The pattern of OD trips is effected by the total number of cars independently of initial distribution of registered cars and strongly by the transition probabilities from zone to zone and their limiting vector. The limiting vector make a display of potentiality of generating trips for each kind of car. In estimating future traffic volume, it plays an important role in determination of the pattern of car trips, because the limiting vector is given by land use planning. Transition probabilities are introduced from maximizing entropy per unit time under the fixed limiting vector. Thus OD trips in the future are easily estimated, using the transition probabilities. It is proved that car trips estimated by maximizing entropy are symmetrical.

1. Introduction

Traffic in urban area will be characterized by what most of his trip-ends are included in urban area. It means that urban area is approximately treated as a closed system. Characteristics of traffic in such a closed system are investigated in this paper.

We have studied car trips through Random Walk within urban area.[1][2][3] It is noted that most of cars take their destinations according to certain probabilities as seen in operation of taxi-cabs. Such a probability is called transition probability. We assume that a car staying at zone i will select his next destination j with transition probability p_{ij} $(j=1, 2, \cdots, r)$ independent of his previous choice of zone.

In the case of except periodical cars, transition matrix $\boldsymbol{P}_0$ whose entries are p_{ij}'s $(i, j=1, 2, \cdots, r)$ is regular. Trips of cars having regular transition matrix are taken into considerations.

2. Mathematical Description of Car Trips as a Markov Chain

Our considerations will be confined to trips of cars in which all their trip-

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ends are included in the urban area. We call it a closed region.

Let us consider a group of cars having the same transition probabilities $\boldsymbol{P}_0$ and number of trips a day. Let $\boldsymbol{T}$ be a row vector whose entry T_i shows the number of registered cars concerned in zone i, that is

$$\boldsymbol{T} = (T_1, T_2, \cdots, T_r) \tag{1}$$

where, r is number of zone in urban area.

We denote by $\boldsymbol{P}_0$ the transition matrix whose entry shows the transition probability from zone i to j.

We assume that all cars start from their registered zones in urban area for the first trips and select their destinations according to p_{ij}. Then traffic flow for the first trips from zone i to j, n_{ij} is shown by the ij-entry of OD matrix

$$\{n_{ij}\} \equiv \begin{pmatrix} T_1 p_{11}, & T_1 p_{12}, & \cdots, & T_1 p_{1r} \\ T_2 p_{21}, & T_2 p_{22}, & \cdots, & T_2 p_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ T_r p_{r1}, & T_r p_{r2}, & \cdots, & T_r p_{rr} \end{pmatrix}. \tag{2}$$

Denoting the number of cars which converge to zone i after the first trips by $T_i^{(1)}$, we can write OD matrix for the second trips in the form

$$\{n_{ij}^{(1)}\} \equiv \begin{pmatrix} T_1^{(1)} p_{11}, & T_1^{(1)} p_{12}, & \cdots, & T_1^{(1)} p_{1r} \\ T_2^{(1)} p_{21}, & T_2^{(1)} p_{22}, & \cdots, & T_2^{(1)} p_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ T_r^{(1)} p_{r1}, & T_r^{(1)} p_{r2}, & \cdots, & T_r^{(1)} p_{rr} \end{pmatrix}, \tag{3}$$

where

$$T_i^{(1)} = T_1 p_{1i} + T_2 p_{2i} + \cdots + T_r p_{ri}. \tag{4}$$

Considering the new row vector

$$\boldsymbol{T}^{(1)} = (T_1^{(1)}, T_2^{(1)}, \cdots, T_r^{(1)}),$$

we may examine the following relation

$$\boldsymbol{T}^{(1)} = \boldsymbol{T}\boldsymbol{P}_0. \tag{5}$$

Similarly, number of cars concentrated to each zone after the $(N-1)$th trips is described by the entries of the row vetor

$$\boldsymbol{T}^{(N-1)} = \boldsymbol{T}\boldsymbol{P}_0^{N-1}. \tag{6}$$

Then OD matrix for the $(N-1)$th trips is given by

$$\{n_{ij}^{(N-2)}\} \equiv \begin{pmatrix} T_1^{(N-2)} p_{11}, & T_1^{(N-2)} p_{12}, & \cdots, & T_1^{(N-2)} p_{1r} \\ T_2^{(N-2)} p_{21}, & T_2^{(N-2)} p_{22}, & \cdots, & T_2^{(N-2)} p_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ T_r^{(N-2)} p_{r1}, & T_r^{(N-2)} p_{r2}, & \cdots, & T_r^{(N-2)} p_{rr} \end{pmatrix}. \tag{7}$$

Now, we assume that each vehicle takes N trips a day that include the stochastic trips of $(N-1)$ times and the final trips going back to the starting zone in the first trips.

Traffic flow from zone i to j on the stochastic trips of $(N-1)$ times after the first trips is expressed as

$$n_{ij}+n_{ij}^{(1)}+\cdots+n_{ij}^{(N-2)} = (T_i+T_i^{(1)}+\cdots+T_i^{(N-2)})p_{ij}. \quad (8)$$

For the final trips, we may write

$$n_{ij}^{(N-1)} = T_j p_{ji}^{(N-1)},$$

where, $p_{ji}^{(N-1)}$ is the ji-entry of the matrix $\boldsymbol{P}_0^{N-1}$.

Therefore, we have traffic volume from zone i to j,

$$(T_i+T_i^{(1)}+\cdots+T_i^{(N-2)})p_{ij}+T_j p_{ji}^{(N-1)}.$$

By use of Eq. (8), we may calculate traffic volume in the urban area.

We consider the trips of passenger cars in Kyoto City as an example, though they may not select their destinations according to the common transition matrix. The initial distribution $\boldsymbol{T}$ of passenger cars is given in Table 1.

Table 1. Number of registered passenger cars in Kyoto City (1962 year)

Zone i	1	2	3	4	5	6	7	8	9
T_i	1337	1945	1519	2754	1974	2459	1808	2947	1600

The ratio of trips discharging to zone j to the total trips generated from zone i, p_{ij}^*, are conveniently taken in place of real transition probability p_{ij}. Because the real transition probability is not investigated.

The OD trips of passenger cars of Kyoto City in 1962 are shown in Table 2.

Table 2. OD table of passenger cars in Kyoto City (1962 year)

O \ D	1	2	3	4	5	6	7	8	9
1	2119	2113	1429	2813	309	1772	439	810	80
2	1887	4708	2770	4441	1220	3931	375	827	207
3	1334	2435	6975	3967	3051	3423	110	212	22
4	2807	4936	3935	11822	4446	10864	1473	1374	768
5	450	1257	2709	4601	8230	4987	809	395	568
6	1724	3653	2937	10259	5441	16480	3224	1754	945
7	307	294	310	1379	681	3499	2930	227	870
8	905	736	330	1955	169	1391	411	2173	22
9	118	297	222	542	616	922	780	73	2234

Table 3. Transition probabilities of passenger cars

O \ D	1	2	3	4	5	6	7	8	9
1	.178	.178	.120	.237	.026	.149	.037	.068	.007
2	.093	.231	.136	.218	.060	.193	.018	.041	.010
3	.062	.113	.324	.184	.142	.159	.005	.010	.001
4	.066	.116	.093	.279	.105	.256	.035	.032	.018
5	.019	.052	.113	.192	.343	.208	.034	.016	.024
6	.037	.079	.063	.221	.117	.355	.069	.038	.021
7	.029	.028	.030	.131	.065	.333	.279	.022	.083
8	.112	.091	.041	.242	.021	.172	.051	.267	.003
9	.020	.051	.038	.093	.106	.159	.134	.013	.386

Table 4. Evaluated OD trips in transient state (passenger cars)

O \ D	1	2	3	4	5	6	7	8	9
1	2073	2073	1401	2957	302	1730	513*	799	82
2	1830	4580	2693	4316	1184	3824	348	813	197
3	1239	2275	6529	3763	2860	3202	96*	198	120**
4	2587	4558	3698	10959	4120	10054	1370	1260	633
5	441	1203	2617	4443	7975	4820	783	369	536
6	1628	3442	2771	9738	5145	15641	3085	1674	927
7	337	325*	354	1532	758	3889	3273	256	958
8	1173	951	427	2539	454*	1852	534	2799	231**
9	152*	386	286	696*	793	1198	1004	201**	2892

Table 3 shows the values of p_{ij}^* obtained from Table 2. We have OD trips as shown in Table 4 using Eq. (8).

Number of trips a car N is assumed 10.4 as a mean value and calculation of Eq. (8) is performed by interpolation of between two values of $N=10$ and $N=11$. It means that the days of $N=10$ and $N=11$ will occur at the percentages of 60 and 40 respectively.

As is known from Table 2 and Table 4, OD trips will be nearly expressed by Eq. (8) except several cells (note a mark*). The values in such marked cells become greater than actual ones.

It will be clear in the following discussion that difference in number of trips result from biasedness of initial distribution of passenger cars.

Now, we take an assumption that the initial distribution of cars is independent of number of registered cars in each zone, and that is

$$\boldsymbol{T}^{(0)} = T\boldsymbol{w},$$

or

$$T_i^{(0)} = Tw_i, \quad (\sum_i T_i = T). \tag{9}$$

where, T and $T_i^{(0)}$ are the total number of cars in urban area and an entry of row vector of initial distribution of cars $\boldsymbol{T}^{(0)}$, w_i is an entry of the limiting vector $\boldsymbol{w}$ of transition matrix $\boldsymbol{P}_0$.

The limiting vector is evaluated by

$$\boldsymbol{w} = \boldsymbol{w}\boldsymbol{P}_0\,, \quad (\sum_i w_i = 1)\,, \tag{10}$$

or by the limiting matrix

$$\boldsymbol{W} = \lim_{N\to\infty} \boldsymbol{P}_0{}^N\,, \tag{11}$$

where

$$\boldsymbol{W} = \begin{pmatrix} w_1, w_2, \cdots, w_r \\ w_1, w_2, \cdots, w_r \\ \cdots\cdots\cdots\cdots \\ w_1, w_2, \cdots, w_r \end{pmatrix}.$$

Then OD traffic volume is invariable for each trip and the pattern of OD trips is fixed in the form

$$T\begin{pmatrix} w_1p_{11}, w_1p_{12}, \cdots, w_1p_{1r} \\ w_2p_{21}, w_2p_{22}, \cdots, w_2p_{2r} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ w_rp_{r1}, w_rp_{r2}, \cdots, w_rp_{rr} \end{pmatrix},$$

where, sum of entries in the ith row is equal to w_i, sum of entries in the ith column. Such unvariable flow for OD trips may be called steady.

If $N>5$, number of the final trips from zone i to j will be expressed as

$$Tw_iw_j\,,$$

because of

$$p_{ji}{}^{(N-1)} \doteqdot w_i\,,$$

or

$$\boldsymbol{P}_0{}^{N-1} \doteqdot \boldsymbol{W}.$$

Table 5 shows the values of $\boldsymbol{P}_0{}^6$ is nearly equal to the limiting matrix $\boldsymbol{W}$. The values of w_i $(i=1, 2, \cdots, r)$ are shown in Table 6.

Table 5. Value of $P_0{}^6$ (passenger cars)

O \ D	1	2	3	4	5	6	7	8	9
1	.061	.107	.114	.219	.127	.248	.055	.040	.029
2	.061	.107	.114	.219	.127	.248	.055	.040	.029
3	.061	.107	.114	.219	.127	.248	.055	.040	.029
4	.061	.107	.113	.219	.127	.248	.055	.041	.029
5	.060	.106	.114	.219	.128	.249	.055	.040	.030
6	.060	.106	.113	.218	.127	.249	.056	.041	.030
7	.060	.106	.112	.218	.127	.249	.057	.040	.030
8	.061	.107	.113	.219	.126	.248	.055	.041	.030
9	.059	.105	.111	.217	.128	.249	.058	.040	.033

Table 6. Limiting vector of passenger cars in Kyoto City

Zone i	1	2	3	4	5	6	7	8	9
w_i	.0604	.1065	.1132	.2185	.1273	.2482	.0554	.0405	.0300

Thus we have traffic volume from zone i to j a day

$$TNw_ip_{ij}+Tw_i(w_j-p_{ij}). \tag{12}$$

The second term is less than the first term of Eq. (12) as N becomes larger. Therefore, we have a matrix of OD trips

$$TN\begin{pmatrix} w_1p_{11}, w_1p_{12}, \cdots, w_1p_{1r} \\ w_2p_{21}, w_2p_{22}, \cdots, w_2p_{2r} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_rp_{r1}, w_rp_{r2}, \cdots, w_rp_{rr} \end{pmatrix}. \tag{13}$$

The result of computation of OD trips of the passenger cars in Kyoto City by Eq. (13) is shown in Table 7. Table 7 approaches more remarkably to Table 2 than Table 4.[4)]

Table 7. Evaluated OD trips in steady state (passenger cars)

O \ D	1	2	3	4	5	6	7	8	9
1	2063	2063	1375	2731	306	1719	420	783	78
2	1881	4699	2770	4431	1222	3935	363	840	210
3	1337	2445	7001	3973	3075	3438	115	210	21
4	2751	4833	3878	11652	4374	10677	1452	1337	745
5	458	1261	2751	4661	8347	5062	821	382	592
6	1757	3763	2999	10506	5558	16847	3285	1814	993
7	306	306	325	1394	688	3515	2961	229	879
8	890	707	325	1872	172	1337	401	2063	21
9	105	287	210	535	611	917	780	78	2216

It should be greatly noted that the initial distribution of registered cars do not effect the pattern of OD trips but the total number of cars in urban area and the limiting vector govern the pattern of OD trips. Such properties of steady flow also appears in the pattern of trips of trucks in Kyoto City. As another illustration of steady state, we take light passenger cars in Amagasaki City. OD trips in 1962 year and transition probability p_{ij}* are shown in Table 8 and and Table 9. Table 8 shows a symmetric matrix owing to be introduced from a triangular OD table. Table 10 and Table 11 are calculated from Eq. (8) and Eq. (13) respectively. Table 11 comes close to actual trips.

Thus the asumption of Eq. (9) will be held. Namely, the initial distribution

Table 8. OD trips of light passenger cars in Amagasaki City and number of registered cars

O \ D	1	2	3	4	5	6	number of cars
1	195	33	300	718	100	12	106
2	33	156	105	407	43	17	429
3	300	105	1557	1594	591	105	1434
4	718	407	1594	4903	703	372	2262
5	100	43	591	703	1021	86	1615
6	12	17	105	372	86	164	753

Table 9. Transition probabilities in Amagasaki City

D \ O	1	2	3	4	5	6
1	.143	.024	.221	.529	.074	.009
2	.043	.205	.138	.535	.057	.022
3	.071	.025	.366	.374	.139	.025
4	.083	.047	.183	.563	.081	.043
5	.039	.017	.232	.276	.402	.034
6	.016	.022	.139	.492	.114	.217

Table 10. Evaluated OD trips in transient state (Amagasaki City)

D \ O	1	2	3	4	5	6
1	125	22	194	462	65	8
2	40	189	127	496	53	20
3	296	105	1520	1543	555	109
4	733	357	1382	4284	616	326
5	139	61	825	984	1431	122
6	22	30	192	664	154	128

Table 11. Evaluated OD trips in steady state (Amagasaki City)

O \ D	1	2	3	4	5	6
1	194	33	299	718	100	12
2	33	155	104	409	43	23
3	301	107	1574	1598	596	107
4	718	409	1596	4953	705	378
5	100	45	596	706	1034	84
6	12	23	107	380	84	166

of cars should be estimated by Eq. (9) but not number of registered cars.

Now we introduce an idea of cycle of car trips. We consider number of going back to his starting place without stochastic choice of his destination during a day. We shall call it number of cycle. Figure 1 illustrates number of cycle $c=1$, 2 and 4.

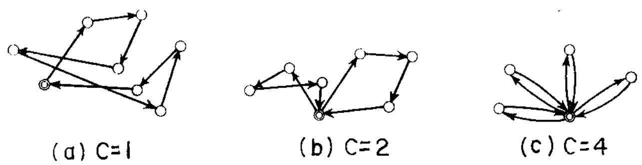

Fig. 1. Illustrations of number of cycle for $N=8$

Let c be number of cycle, a car. Since c is equal to number of returning trip, we have

$$TNw_i p_{ij} + Tcw_i(w_j - p_{ij}) \tag{14}$$

in place of Eq. (12) as long as number of trips a cycle is not less than five. The values of N and c in Amagasaki City are shown in Table 12.

Table 12. Observed values of N and c

Kind of cars	(1)*	N	c
Passenger cars (private)	0.392	5.4	2.0
Passenger cars (business)	0.909	37.0	10.2
Trucks (private)	0.279	7.0	1.2
Trucks (business)	0.306	7.6	2.0
Light cars	0.661	6.7	3.1
Special cars	0.259	7.6	3.3
Average	0.387	7.6	2.1

*(1): Difference ratio between registered place and starting place

Next, we consider the false transition probability p_{ij}^*. They are calculated as ratio of number of trips diverging to zone j to the total trips generated from zone i. Hence the following relation is derived by use of Eq. (14).

$$p_{ij}^* = p_{ij} + \frac{c(w_j - p_{ij})}{N}. \tag{15}$$

Finally we may use p_{ij}^* in place of p_{ij} proved that (c/N) is small.

Thus the distribution of OD trips will be expressed by Eq. (13) or Eq. (14).

3. Evaluation of Transition Probabilities

In estimating car trips in the future, we need to evaluate T, N and $\boldsymbol{P}_0$ at the horizontal year.

Since total number of cars within urban area and number of trips a car is taking a simple trend in past years, the future values of T and N are easily extrapolated. In this paragraph, we discuss a method of evaluation of transition probabilities.

From the fact that a number of cars converged into or diverged from zone i for each trip becomes Tw_i in steady state, we believe that w_i means the fraction of generated and concentrated traffic demand in zone i. Hence the limiting vector $\boldsymbol{w}$ have an intimate relationship to land use patterns.

In the future planning of the urban area, however, the value of $\boldsymbol{w}$ should be given previously through land use plan. Hence transition matrix $\boldsymbol{P}_0$ have to be determined to have the same limiting vector $\boldsymbol{w}$.

Our concern comes to the determination of such transition probabilities as to satisfy Eq. (10), that is

$$\sum_i w_i p_{ij} = w_j, \quad \sum_i w_i = 1, \qquad (i = 1, 2, \cdots, r) \tag{16}$$

There are innumerable matrices to satisfy Eq. (16) under the constraint vector $\boldsymbol{w}$.

As a simple case, we can take

$$\boldsymbol{P}_0 = \boldsymbol{W},$$

or

$$p_{ij} = w_j. \tag{17}$$

Eq. (17) is a special solution of Eq. (16).

We may use Eq. (17) for estimating of future OD trips in the small urban area. Such convenient method to evaluate the values of p_{ij} was applied to estimation of future trips in Fukuchiyama City in Kyoto Prefecture. There is a drawback in the convenient technique that the values of diagonal entries or numbers of intrazonal trips are small to compare with actual trips.

Now, we consider the case when total number of certain "kind of cars" T is divided into $r \times r$ cells and each cell is filled up with cars of the same OD. We shall call the cell filled up with trips from zone i to j the ij-cell.

Let p_{ij}' be the probability that any trip falls into the ij-cell. Then we have

$$\sum_i \sum_j p_{ij}' = 1$$

By polynomial theorem, the joint probability that n_{ij} is assigned to the ij-cell respectively for each trip is written as follows,

$$p = \frac{T!}{n_{11}!\, n_{12}! \cdots n_{rr}!} (p_{11}')^{n_{11}} (p_{12}')^{n_{12}} \cdots (p_{rr}')^{n_{rr}}, \tag{18}$$

where,

$$T=\sum_i\sum_j n_{ij}\,.$$

If the traveling time to all the other zones j from any zone i is similar, cars will be attracted to zone j proportional to value of w_j.

Since the probability taking a trip at zone i is stated as w_i, then we have

$$p_{ij}'=w_iw_j\,. \tag{19}$$

Hence Eq. (18) becomes

$$p=\frac{T!}{\prod_{i,j}(n_{ij}!)}\prod_{i,j}(w_iw_j)^{n_{ij}}\,. \tag{20}$$

We would determine p_{ij} so as to maximize Eq. (20) under the constraints

$$\sum_j p_{ij}=1\,,\quad \sum_i w_ip_{ij}=w_j\,. \tag{21}$$

However, since

$$\prod_{i,j}(w_iw_j)^{n_{ij}}$$

becomes a constant value under the fixed w, namely,

$$\begin{aligned}\prod_{i,j}(w_iw_j)^{n_{ij}}&=\prod_i(w_i)^{\sum_j n_{ij}}\prod_j(w_j)^{\sum_i n_{ij}}\\&=\prod_i(w_i)^{Tw_i}\prod_j(w_j)^{Tw_j}\,,\end{aligned}$$

because of the property in the steady state

$$\sum_j n_{ij}=Tw_i\,,\quad \sum_i n_{ij}=T\sum_i w_ip_{ij}=Tw_i\,,$$

to maximize Eq. (20) is equivalent to maximize the value of

$$\frac{T!}{\prod_{i,j}(Tw_ip_{ij})!}\,. \tag{22}$$

Logarithm of Eq. (22) is easily written as

$$-T\sum_i w_i\log w_i-T\sum_i\sum_j w_ip_{ij}\log p_{ij}\,, \tag{23}$$

by Stirling Formula

$$\log(n!)=n\log n-n\,.$$

We may maximize Eq. (23) or

$$H\equiv-\sum_i\sum_j w_ip_{ij}\log p_{ij}\,, \tag{24}$$

in place of Eq. (20) under constraints of Eq. (21). Eq. (24) is described in the form of entropy. We shall call it entropy of OD trips.

Hence, most probable pattern of OD trips is represented by such a set of p_{ij} as to maximize entropy H as long as selection of a destination is independent of distance to the destination.

Here, we will find a solution of maximizing entropy. Lagrange's function is

$$F = H+\sum_j \lambda_j(\sum_i w_i p_{ij}-w_j)+\sum_i \mu_i(\sum_j p_{ij}-1),$$

where λ_j and μ_i are indeterminate coefficients. Then we have

$$p_{ij} = \alpha_i\beta_j e^{-1}, \tag{25}$$

where,

$$\alpha_i = \exp(\mu_i/w_i),$$
$$\beta_j = \exp(\lambda_j),$$

from

$$\partial F/\partial p_{ij} = -w_i(1+\log p_{ij})+w_i\lambda_j+\mu_i = 0.$$

From the relation of

$$\sum_j p_{ij} = 1,$$

we have

$$\alpha_i = e/\sum_j \beta_j,$$

so that α_i is independent of i. Thus Eq. (25) becomes

$$p_{ij} = \beta_j/\sum_j \beta_j.$$

Hence, we can find

$$\beta_j/\sum_j \beta_j = w_j,$$

from the relation

$$\sum_i w_i p_{ij} = w_j.$$

Finally we have

$$p_{ij} = w_j.$$

It is a very interesting property that the convenient estimation method of Eq. (17) requires maximum entropy.

Next, we consider entropy per unit time

$$R \equiv H/\sum_i \sum_j w_i p_{ij} t_{ij}, \tag{26}$$

according to information theory, although we have not theoretical interpretation. t_{ij} in Eq. (26) shows the traveling time required from zone i to j.

It is difficult to determine a set of p_{ij} explicitly so as to maximize R when w_i and t_{ij} are fixed. To find the solution, let us consider Lagrange's function

$$F = H/\bar{t}+\sum_j \lambda_j(\sum_i w_i p_{ij}-w_j)+\sum_i \mu_i(\sum_j p_{ij}-1), \tag{27}$$

where

$$\bar{t} = \sum_i \sum_j w_i p_{ij} t_{ij}. \tag{28}$$

Differentiating Eq. (27) with respect to p_{ij}, λ_j and μ_i, we have the simultaneous equations

$$\frac{\partial F}{\partial p_{ij}} = \frac{-w_i(1+\log p_{ij})}{\bar{t}} - \frac{w_i t_{ij} R}{\bar{t}} + \lambda_j w_i + \mu_i = 0, \tag{29}$$

$$\frac{\partial F}{\partial \lambda_j} = \sum_i w_i p_{ij} - w_j = 0, \tag{30}$$

$$\frac{\partial F}{\partial \lambda_i} = \sum_j p_{ij} - 1 = 0. \tag{31}$$

From Eq. (29) we have

$$p_{ij} = \alpha_i \beta_j \exp(-1 - t_{ij} R), \tag{32}$$

where,

$$\alpha_i = \exp(\mu_i \bar{t}/w_i),$$

$$\beta_j = \exp(\lambda_j \bar{t}).$$

From Eqs. (30), (41) and (32), we obtain

$$\alpha_i = e/\sum_j \beta_j \exp(-t_{ij} R), \tag{33}$$

$$\beta_j = e w_j / \sum_i w_i \alpha_i \exp(-t_{ij} R). \tag{34}$$

To obtain the solution maximizing Eq. (26), we should repeat the following operations:

(i) to calculate α_i from Eq. (33) assuming values of R and β_j, $(j=1, 2, \cdots, r)$,

(ii) to calculate β_j from Eq. (34) using α_i and R,

(iii) to obtain p_{ij} from Eq. (32) using α_i and β_j,

(iv) to calculate R from Eq. (26) using p_{ij},

(v) to return to (i) and repeat these operations to converge.

We shall take some examples to illustrate determination of transition probabilities.

For passenger cars of Kyoto City, the entries of the limiting vector and the traveling time from zone to zone are shown in Table 6 and Table 13 respectively. A set of transition probabilities to give the maximum R is shown in Table 14. Comparing Table 14 with Table 3, we notice that the actual transition probabilities are fairly approximate to ones maximizing entropy per unit time.

Table 15 and Table 16 show a variation of traveling time and the resulted transition probabilities. Transition probabilities are influenced through variation of traveling time t_{ij}.

Similarly, Table 17 and Table 18 show the limiting vector and the transition probabilities calculated by maximizing R for the trucks of Kyoto City. OD trips of trucks in 1962 year are shown in Table 19 and traveling time is assumed the same as that of passenger cars.

It should be noted that transition probabilities change sensitively through variation of traveling time particularly t_{ii}.

Table 13. Traveling time in Kyoto City (min)

O \ D	1	2	3	4	5	6	7	8	9
1	10	11	15	15	25	20	28	20	35
2	11	8	13	10	22	15	20	18	32
3	15	13	8	15	18	15	25	30	30
4	15	10	15	10	15	9	15	15	25
5	25	22	18	15	12	15	18	30	20
6	20	15	15	9	15	7	9	18	20
7	28	20	25	15	18	9	5	25	15
8	20	18	30	15	30	18	25	10	40
9	35	32	30	25	20	20	15	40	12

Table 14. Evaluated transition probabilities (passenger cars)

O \ D	1	2	3	4	5	6	7	8	9
1	.215	.213	.152	.204	.050	.104	.010	.046	.006
2	.121	.214	.132	.274	.051	.140	.022	.040	.006
3	.081	.125	.328	.160	.108	.167	.013	.009	.009
4	.056	.134	.083	.229	.117	.279	.037	.052	.013
5	.024	.042	.096	.200	.324	.211	.044	.011	.048
6	.025	.060	.077	.246	.108	.346	.083	.031	.024
7	.011	.042	.026	.148	.100	.371	.213	.016	.073
8	.069	.106	.024	.280	.033	.191	.022	.271	.004
9	.011	.020	.033	.093	.202	.202	.133	.005	.300

Table 15. Traveling time in variation (min)

O \ D	1	2	3	4	5	6	7	8	9
1	10	10	15	15	28	25	28	20	35
2	10	8	10	10	22	15	20	18	32
3	15	10	8	15	18	15	25	30	30
4	15	10	15	10	15	9	15	15	25
5	28	22	18	15	12	13	18	30	20
6	25	15	15	9	13	7	9	18	20
7	28	20	25	15	18	9	5	25	15
8	20	18	30	15	30	18	25	10	40
9	35	32	30	25	20	20	15	40	12

Table 16. Evaluated transition probabilities from Table 15.

O \ D	1	2	3	4	5	6	7	8	9
1	.240	.242	.153	.217	.032	.051	.011	.048	.006
2	.137	.186	.183	.259	.044	.127	.020	.037	.005
3	.082	.173	.306	.154	.099	.158	.012	.008	.008
4	.060	.126	.080	.236	.113	.281	.038	.053	.013
5	.015	.037	.088	.194	.302	.267	.042	.010	.046
6	.012	.055	.072	.247	.137	.340	.083	.031	.024
7	.012	.038	.024	.150	.096	.371	.219	.016	.074
8	.072	.098	.022	.285	.031	.188	.022	.278	.004
9	.011	.018	.031	.093	.194	.199	.137	.005	.312

Table 17. Limiting vector of trucks in Kyoto City

Zone i	1	2	3	4	5	6	7	8	9
w_i	.0547	.1235	.0852	.2315	.0726	.2177	.0712	.0789	.0647

Table 18. Evaluated transition probabilities of trucks

O \ D	1	2	3	4	5	6	7	8	9
1	.193	.243	.124	.217	.031	.092	.013	.077	.010
2	.107	.241	.107	.289	.031	.123	.026	.066	.010
3	.080	.155	.293	.186	.074	.162	.017	.015	.018
4	.051	.154	.068	.247	.074	.250	.046	.087	.023
5	.024	.053	.087	.235	.223	.205	.059	.019	.095
6	.023	.069	.064	.265	.068	.311	.103	.052	.045
7	.010	.046	.020	.150	.060	.314	.248	.026	.126
8	.053	.103	.017	.255	.018	.144	.023	.381	.006
9	.009	.019	.023	.084	.107	.151	.139	.007	.461

Table 19. Transition probabilities of trucks (1962 year)

O \ D	1	2	3	4	5	6	7	8	9
1	.227	.225	.082	.157	.025	.113	.039	.074	.012
2	.107	.367	.082	.196	.040	.115	.021	.053	.019
3	.039	.092	.391	.184	.087	.121	.030	.042	.014
4	.042	.099	.061	.364	.051	.229	.040	.088	.026
5	.025	.082	.098	.153	.296	.207	.055	.034	.049
6	.027	.073	.045	.240	.065	.363	.073	.068	.046
7	.042	.042	.033	.146	.052	.199	.350	.061	.076
8	.056	.079	.041	.247	.043	.200	.056	.258	.020
9	.014	.037	.010	.084	.059	.155	.084	.037	.520

4. Symmetry of Car Trips

OD trips are briefly represented by Eq. (13), so that symmetry in distribution of OD trips requires

$$w_i p_{ij} = w_j p_{ji}. \tag{35}$$

Property of symmetry is held in most cases of observations. When the relation of Eq. (35) exists, we call the OD trips a symmetrical flow.

The transition probabilities of Table 14 form a symmetrical flow as shown in Table 20. In general, we prove that such the transition probabilities as to maximize entropy per unit time make a symmetrical distribution of OD trips under the relation of $t_{ij}=t_{ji}$.

Table 20. Illustration of symmetry of $w_i p_{ij}$ (passenger cars)

O \ D	1	2	3	4	5	6	7	8	9
1	$.130\times10^{-1}$	$.129\times10^{-1}$	$.092\times10^{-1}$	$.123\times10^{-1}$	$.030\times10^{-1}$	$.063\times10^{-1}$	$.006\times10^{-1}$	$.028\times10^{-1}$	$.003\times10^{-1}$
2	.129	.228	.141	.292	.054	149	.023	.043	.006
3	.092	.141	.372	.181	.123	.190	.014	.010	.009
4	.123	.293	.180	.501	.255	.610	.082	.113	.028
5	.030	.054	.123	.255	.414	.268	.056	.013	.060
6	.063	.149	.190	.610	.268	.858	.206	.077	.061
7	.006	.023	.014	.082	.056	.206	.118	.009	.040
8	.028	.043	.010	.114	.013	.077	.009	.110	.001
9	.003	.006	.010	.028	.060	.061	.040	.001	.091

From Eq. (32), we have

$$p_{ij} = K\alpha_i\beta_j x_{ij}, \tag{36}$$

where,

$$K = 1/e \quad \text{and} \quad x_{ij} = \exp(-t_{ij}R).$$

Then the symmetry of OD trips require the relation of

$$w_i\alpha_i\beta_j x_{ij} = w_j\alpha_j\beta_i x_{ji}. \tag{37}$$

Since, $t_{ij}=t_{ji}$ or $x_{ij}=x_{ji}$, it follows that

$$w_i\alpha_i/\beta_i = w_j\alpha_j/\beta_j. \tag{38}$$

If we put

$$M_i = w_i\alpha_i/\beta_i,$$

it is sufficient to prove the symmetry that M_i is a constant value independent of i.

From Eq. (33) and Eq. (34), we have

$$\sum_j \beta_j x_{ij} = 1/K\alpha_i \qquad (i=1, 2, \cdots, r), \tag{39}$$

and

$$\sum_j w_j \alpha_j x_{ji} = w_i/K\beta_i \qquad (i=1, 2, \cdots, r). \tag{40}$$

Hence, we can write

$$M_i = \sum_j w_j \alpha_j x_{ji} / \sum_j \beta_j x_{ij} \tag{41}$$

Using Eq. (41) and relation $x_{ij}=x_{ji}$, we have

$$\sum_j (w_j \alpha_j - \beta_j M_i) x_{ij} = 0. \tag{42}$$

By definition of $M_j = w_j \alpha_j / \beta_j$, Eq. (42) is reduced to

$$\sum_j \beta_j (M_j - M_i) x_{ij} = 0. \tag{43}$$

Let us now assume that M_i's are not always equal to each others, i.e., M_i is dependent of i. Then, there exists a maximum value among the set of M_i such as M_{i_0}. We know already that

$$\beta_j > 0 \quad \text{and} \quad x_{ij} > 0.$$

Therefore, we have

$$\beta_j (M_j - M_{i_0}) x_{i_0 j} \leqq 0 \qquad (j=1, 2, \cdots, r). \tag{44}$$

By the above assumption, we may find M_{j_0} such as $M_{j_0} < M_{i_0}$. Then we have

$$\beta_{j_0} (M_{j_0} - M_{i_0}) x_{i_0 j_0} < 0. \tag{45}$$

Hence

$$\sum_j \beta_j (M_j - M_{i_0}) x_{i_0 j} < 0. \tag{46}$$

This is inconsistent with Eq. (43).

Therefore, M_i is a constant value for all i.

In most cases, OD trips are given by a triangular OD table. We have to make OD table from it. Then OD trips are clearly symmetrical. For example, such OD trips of light passenger cars in Amagasaki City are shown in Table 8. The limiting vector and the traveling time are shown respectively in Table 21 and Table 22. Thus we can find transition probabilities of maximum entropy as shown in Table 23.

The actual distribution of OD trips comes near the situation of maximum entropy per unit time.

Table 21. Limiting vector (Amagasaki City)

Zone i	1	2	3	4	5	6
w_i	.0739	.0414	.2315	.4735	.1385	.0412

Table 22. Traveling time (min)

O \ D	1	2	3	4	5	6
1	2.5	8	7	10	15	18
2	8	3	14	6	22	15
3	7	14	5	9	12	17
4	10	6	9	5	16	10
5	15	22	12	16	7	11
6	18	15	17	10	11	6

Table 23. Evaluated transition probabilities (Amagasaki City)

O \ D	1	2	3	4	5	6
1	.220	.050	.302	.332	.082	.014
2	.090	.116	.096	.649	.026	.023
3	.096	.017	.387	.361	.124	.015
4	.052	.057	.177	.615	.057	.042
5	.044	.008	.207	.193	.480	.069
6	.025	.023	.085	.487	.232	.148

5. Conclusion

Urban traffic flow from origin to destination forms an ergodic Markov chain of steady state, and number of trips from zone i to j are described by TNw_ip_{ij} as known from Eq. (13). Hence, OD trips will be fixed stationarily for each trip and be influenced by total number of cars independently of distribution of registered cars.

It should be noted that the transition probabilities control the pattern of car trips in accompany with their limiting vector w.

The limiting vector shows a distribution of relative potentiality of generating or concentrating traffic for each kind of cars and should be given through land use planning in estimation of future trips.

Although there are innumerable transition matrices that converge to the same limiting matrix, we could select a transition matrix through maximizing entropy per unit time. Difference between such transition matrix and actual one will be caused by speciality of interzonal connection excluding traveling time required on a trip.

Car trips of maximum entropy will discharge from every zone in a symmetrical flow when the traveling time is symmetrical.

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