Stresses in Rectangular Blocks Compressed Between Rough Plates

By

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In this paper non-uniform elastic compression of rectangular blocks between parallel end-blocks is considered. The friction between the end-blocks and the end surfaces of the rectangular blocks is assumed to be sufficient to prevent slippage. Approximate solution through variational approach is obtained and numerical results are shown for height-width ratios of the test block 0.5, 1.0, 2.0 and 4.0.

1. Introduction

The distribution of stress in a rectangular block subjected to compression is a problem of considerable importance in testing the strength of materials. There always exists friction between the end-blocks and the end surfaces of the test block. The friction prevents the uniform distribution of stresses in the specimen. The exact determination of the stresses requires the solution of a mixed boundary value problem. However, it is very difficult to obtain the exact solution.

The authors solved the problem approximately through the same variational approach as developed in references,[1),3),4)] using the self-equilibrating ortho-normal polynomials derived by G. Horvay.[1,2)]

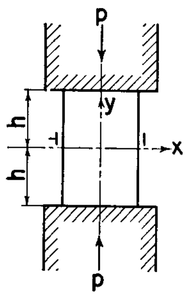

Fig. 1. General boundary conditions.

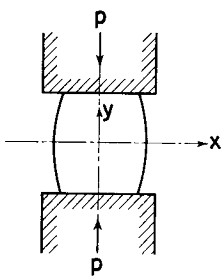

Fig. 2. Rectangular block after compression.

2. Statement of Problem

Consider a rectangular block with thickness d compressed between parallel end-blocks as shown in Fig. 1. The rectangular test block assumed to be elastic and in the state

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of plane stress. The further assumption is made to simplify the problem that the endblocks are rigid and that there is sufficient friction between the end-blocks and the end faces of the specimen to prevent slippage. Thus, the points on the end faces of the specimen are permitted only vertical displacements, but no horizontal displacements. After deformation the specimen will appear as shown in Fig. 2.

In the Cartesian co-ordinates, the boundary conditions of the original problem are

$$\left.\begin{array}{ll} u(x, \pm h) = 0\,, & v(x, \pm h) = \text{const.}\,, \\ \sigma_x(\pm 1, y) = 0\,, & \tau_{xy}(\pm 1, y) = 0\,, \end{array}\right\} \tag{2·1}$$

where u and v are the displacements in the x- and y-directions respectively, and σ_x and τ_{xy} are the normal and tangential components of stresses transmitted through the surfaces.

If the specimen is restrained in such a way that the points on the side faces are prevented displacements, then stresses in both directions x and y must be uniform (see Fig. 3a).

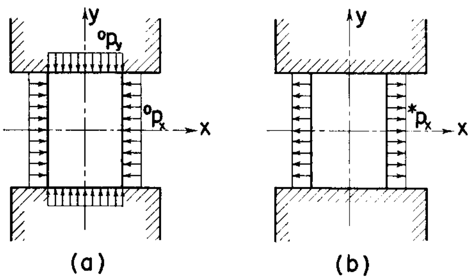

Fig. 3. Imaginary states.

No surface traction, however, acts on the side faces actually. The actual conditions will be fulfilled by superposing the state of Fig. 3a on the state such that the surface tractions along the side faces have the same magnitude as those in Fig. 3a but opposite sense (Fig. 3b).

If we denote the stress components corresponding to the two states as $^\circ\sigma_x$, $^\circ\sigma_x$, $^\circ\tau_{xy}$ and $^*\sigma_x$, $^*\sigma_y$, $^*\tau_{xy}$, respectively, the true stresses are

$$\sigma_x = {}^\circ\sigma_x + {}^*\sigma_y\,, \quad \sigma_y = {}^\circ\sigma_y + {}^*\sigma_y\,, \quad \tau_{xy} = {}^\circ\tau_{xy} + {}^*\tau_{xy}\,. \tag{2·2}$$

Thus, the central part of the present problem is how to determine stresses $^*\sigma_x$, $^*\sigma_y$ and $^*\tau_{xy}$.

Suppose that the stress function is expressed in the form

$$\Phi(x, y) = \phi(x, y) + \sum_n f_n(x)\cdot g_n(y) \qquad (n=2, 3, \cdots)\,, \tag{2·3}$$

then stresses are

$$\left.\begin{array}{l} {}^*\sigma_x = \Phi_{yy} = \phi_{yy} + \sum\limits_n f_n(x)\cdot g_n''(y)\,, \\ {}^*\sigma_y = \Phi_{xx} = \phi_{xx} + \sum\limits_n f_n''(x)\cdot g_n(y)\,, \\ {}^*\tau_{xy} = -\Phi_{xy} = -\phi_{xy} - \sum\limits_n f_n'(x)\cdot g_n'(y) \\ \qquad\qquad\qquad\qquad (n = 2, 3, \cdots)\,. \end{array}\right\} \tag{2·4}$$

If we choose ϕ as to satisfy the following boundary conditions

$$\left.\begin{aligned}\phi_{yy}(\pm 1, y) &= {}^*p_x\,, \\ \phi_{xy}(\pm 1, y) &= 0\,,\end{aligned}\right\} \tag{2·5}$$

then the remaining part of the stress function must satisfy conditions

$$\left.\begin{aligned}&\sum_n f_n(\pm 1)\cdot g_n''(y) = 0\,, \\ &\sum_n f_n'(\pm 1)\cdot g_n'(y) = 0 \\ &\qquad (n = 2, 3, \cdots)\,.\end{aligned}\right\} \tag{2·6}$$

The conditions (2·6) are always satified term by term, if we choose functions such that

$$f_n(\pm 1) = f_n'(\pm 1) = 0\,. \tag{2·7}$$

The functions which meet the above conditions are derived by G. Horvay.[1),2)] Thus, the unknown functions are $g_n(y)$ only, which are determined by conditions on the end faces. The assumed stress function Φ does not always satisfy the biharmonic field equation, the solution must be sought in the mean through variational approach.

3. Solutions

In the equilibrium condition, the complementary energy of a elastic body must be minimum. The complementary energy Ω stored in the rectangular specimen (Fig. 3b) is

$$\Omega = \frac{d}{2E}\int_{-h}^{h}\int_{-1}^{+1}\{{}^*\sigma_x^2 + {}^*\sigma_y^2 - 2\nu\,{}^*\sigma_x^*\sigma_y + 2(1+\nu)\,{}^*\tau_{xy}^2\}dxdy\,, \tag{3·1}$$

where E is Young's modulus. The first variation expressed in terms of stress function (2·3) is

$$\begin{aligned}\frac{E}{d}\delta\Omega = &\int_{-h}^{h}\int_{-1}^{+1}\nabla^4\Phi\delta\Phi dxdy + 2(1+\nu)[\{\Phi_{xy}\delta\Phi\}_{-1}^{+1}]_{-h}^{h} \\ &+\int_{-h}^{h}[(\Phi_{xx}-\nu\Phi_{yy})\delta\Phi_x - (\Phi_{xxx}+(2+\nu)\Phi_{xyy})\delta\Phi]_{-1}^{+1}dy \\ &+\int_{-1}^{+1}[(\Phi_{yy}-\nu\Phi_{xx})\delta\Phi_y - (\Phi_{yyy}+(2+\nu)\Phi_{xxy})\delta\Phi]_{-h}^{h}dx\end{aligned} \tag{3·2}$$

and must be zero, i.e. $\delta\Omega = 0$. Variation $\delta\Phi$ and $\delta\Phi_x$ must be equal to zero on the surface $x = \pm 1$, since the surface tractions are assigned there. Thus, the Eq. (3·2) becomes simpler

$$\int_{-h}^{h}\int_{-1}^{1}\nabla^4\Phi\delta\Phi dxdy + \int_{-1}^{+1}(\Phi_{yy}-\nu\Phi_{xx})\delta\Phi_y - \{\Phi_{yyy}+(2+\nu)\Phi_{xxy}\}\delta\Phi]_{-h}^{h}dx = 0\,. \tag{33}$$

Substituting derivatives of Φ and

$$\delta\Phi = \sum_i f_i(x)\cdot\delta g_i(y)\,,$$
$$\delta\Phi_y = \sum_i f_i(x)\cdot\delta g_i'(y)$$

into Eq. (3·3) and considering that $\delta g_i(y)$ and $\delta g_i'(y)$ can be chosen arbitrarily, we obtain

$$\int_{-1}^{+1}\{\nabla^4\phi+\sum_n f_n^{(IV)}(x)\cdot g_n(y)+2\sum_n f_n''(x)\cdot g_n''(y)+\sum_n f_n(x)\cdot g_n^{(IV)}(y)\}$$
$$\times\sum_i f_i(x)dx = 0 \qquad (i=2,3,\cdots) \tag{3·4}$$

from the first term and

$$\left.\begin{aligned}&\int_{-1}^{+1}[\{\phi_{yy}(x,y)+\sum_n f_n(x)\cdot g_n''(y)-\nu(\phi_{xx}(x,y)+\sum_n f_n''\cdot g_n(y))\}\sum_i f_i(x)]_{y=\pm h}dx = 0\,,\\&\int_{-1}^{1}[\{(\phi_{yyy}(x,y)+\sum_n f_n(x)\cdot g_n'''(y)+(2+\nu)(\phi_{xxy}(x,y)+\sum_n f_n''(x)\cdot g_n'(y)\}\\&\quad\times\sum_i f_i(x)]_{y=\pm h}dx = 0\,,\\&\qquad\qquad\qquad\qquad(i=2,3,\cdots)\end{aligned}\right\} \tag{3·5}$$

from the second term of Eq. (3·3).

Since $f_n(x)$ are ortho-normal functions,

$$(f_n\cdot f_i) = \delta_{ni} \qquad \text{(Kronecker's delta)}$$

and

$$(f_n''\cdot f_i) = -(f_n'\cdot f_i')\,,$$
$$(f_n^{(IV)}\cdot f_i) = (f_n''\cdot f_i'')\,,$$

where

$$(f_n^{(p)}\cdot f_i^{(q)}) = \int_{-1}^{+1} f_n^{(p)}(x)\cdot f_i^{(q)}(x)dx\,.$$

Although the derivatives of $f_n(x)$ are not orthogonal each other, the values $(f_n'\cdot f_i')$ and $(f_n''\cdot f_i'')$ are negligibly small for $n\neq i$ as compared with those for $n=i$. Here we assume

$$(f_n'\cdot f_i') = (f_n''\cdot f_i'') = 0 \qquad \text{for} \quad n\neq i\,.$$

With this assumption and ortho-normal conditions the eqeations (3·4) and (3·5) becomes simpler

$$g_n^{(IV)}(y)-2(f_n'\cdot f_n')g_n''(y)+(f_n''\cdot f_n'')g_n(y) = -\int_{-1}^{+1}\nabla^4\phi\cdot f_n(x)dx \tag{3·4$'$}$$
$$(n=2,3,\cdots)\,,$$

$$\left.\begin{aligned}&g_n''(\pm h)+\nu(f_n'\cdot f_n')g_n(\pm h) = -\int_{-1}^{+1}\{\phi_{yy}(x,\pm h)-\nu\phi_{xx}(x,\pm h)\}f_n(x)dx\,,\\&g_n'''(\pm h)-(2+\nu)(f_n'\cdot f_n')g_n'(\pm h) = -\int_{-1}^{+1}\{\phi_{yyy}(x,\pm h)+(2+\nu)\phi_{xxy}(x,\pm h)\}f_n(x)dx,\\&\qquad\qquad\qquad\qquad(n=2,3,\cdots)\,.\end{aligned}\right\} \tag{3·5$'$}$$

The unknowns $g_n(y)$ are determined by Eq. (3·4)′ with boundary conditions (3·5)′.

In the present problem, the geometry of the specimen and the boundary conditions are symmetrical with respect to x axis. Thus, $g_n(y)$ can be conveniently expressed as

$$g_n(y) = A_n \cos\beta_n y \cosh\alpha_n y + B_n \sin\beta_n y \sinh\alpha_n y + G_n(y)\,, \tag{3·6}$$

where α_n and β_n are the real and the imaginary part of the Eigenvalues of Eq. (3·4)′ and $G_n(y)$ is a particular solution of Eq. (3·4)′. Considering the two states of Fig. 3, we have ${}^*p_x = -{}^\circ p_x = -\nu^\circ\sigma_y$ and

$$\phi = -\frac{\nu^\circ\sigma_y}{2}y^2\,. \tag{3·7}$$

Substitution Eqs. (3·6) and (3·7) into Eq. (3·5)′ furnishes a system of equations with unknowns A_n and B_n.

$$\begin{aligned}&[(\alpha_n^2-\beta_n^2)\cos\beta_n h\cosh\alpha_n h-2\alpha_n\beta_n\sin\beta_n h\sinh\alpha_n h+\nu(f_n'\cdot f_n')\cos\beta_n h\cosh\alpha_n h]A_n\\&+[(\alpha_n^2-\beta_n^2)\sin\beta_n h\sinh\alpha_n h+2\alpha_n\beta_n\cos\beta_n h\cosh\alpha_n h+\nu(f_n'\cdot f_n')\sin\beta_n h\sinh\alpha_n h]B_n\\&=L_n\,,\end{aligned} \tag{3·8a}$$

$$\begin{aligned}&[\beta_n\{(\beta_n^2-3\alpha_n^2)+(2+\nu)(f_n'\cdot f_n')\}\sin\beta_n h\cosh\alpha_n h\\&\qquad+\alpha_n\{(\alpha_n^2-3\beta_n^2)-(2+\nu)(f_n'\cdot f_n')\}\cos\beta_n h\sinh\alpha_n h]A_n\\&+[\alpha_n\{(\alpha_n^2-3\beta_n^2)-(2+\nu)(f_n'\cdot f_n')\}\sin\beta_n h\cosh\alpha_n h\\&\qquad-\beta_n\{(\beta_n^2-3\alpha_n^2)+(2+\nu)(f_n'\cdot f_n')\}\cos\beta_n h\sinh\alpha_n h]B_n\\&=0\end{aligned} \tag{3·8b}$$

$$(n=2, 4, 6, 8)$$

where

$$L_n = -\int_{-1}^{+1}\phi_{yy}(x, h)\cdot f_n(x)\,dx = \nu^\circ\sigma_y\int_{-1}^{+1}f_n(x)\,dx\,.$$

5. Stresses in Specimens

Unknown constants A_n and B_n (n up to 8) are calculated for $\nu=0.2$ and listed in Table 1. Stresses are shown in Figs. 4, 5, 6 and 7, in which compressive stresses are taken to be positive.

Stresses along $y=h$, are expected to become smoother as more terms of $f_n(x)$ are considered. Theoretically σ_x will be uniform on $y=h$ except the corner point (singular point). At the point the magnitude of σ_x approaches to infinity as proceeded along $y=h$ whereas that remains zero as proceeded along $x=1$. Distribution of τ_{xy} is approximately triangular. Near the singular point τ_{xy} behaves in the similar manner as σ_x.

The restraining effect of the end faces appears considerable in the distri-

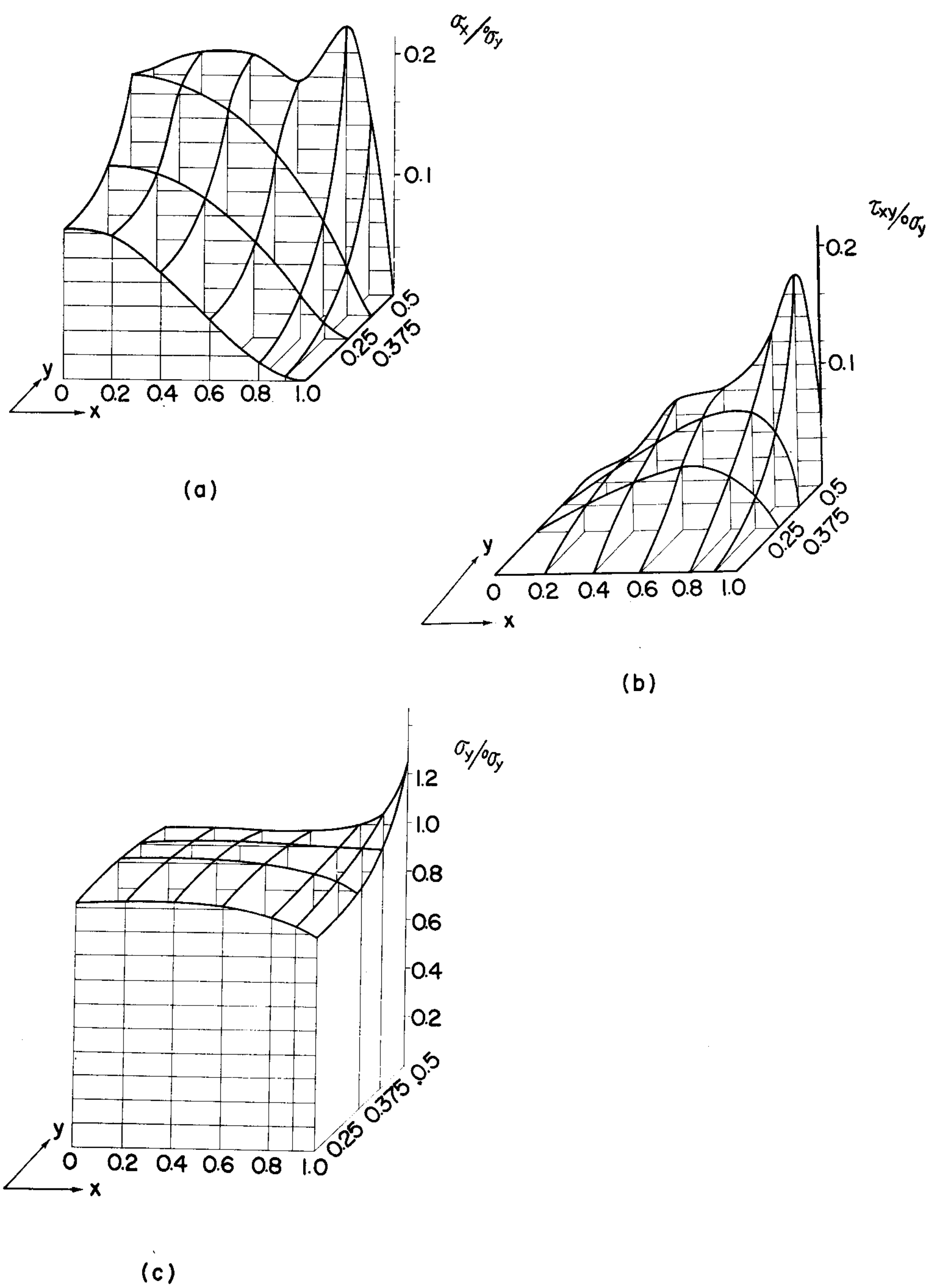

Fig. 4. Stresses in the specicimen with height/width=0.5.

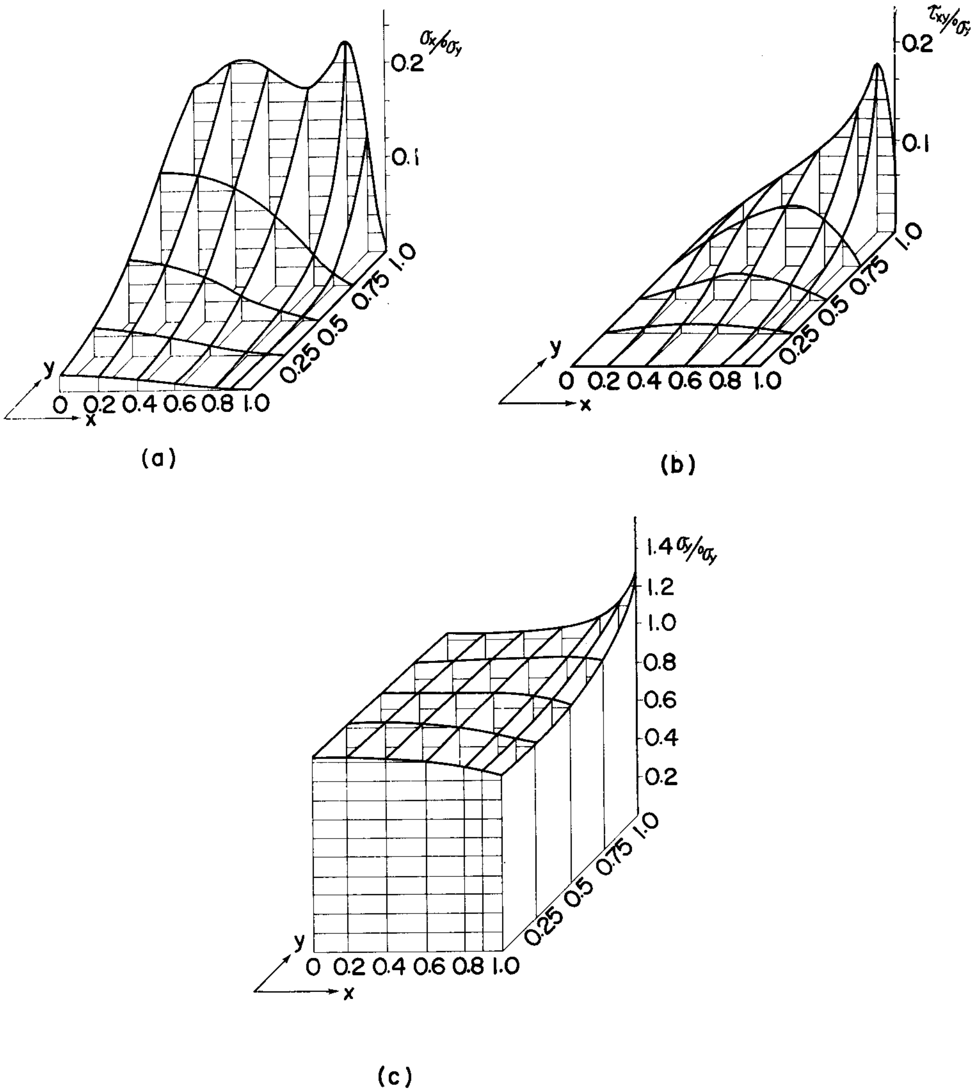

Fig. 5. Stresses in the specimen with height/width=1.0

butions of σ_x and τ_{xy} in short specimens, whereas the distribution of σ_y is almost uniform over the entire specimen. The effect of the end friction is negligibly small in the mid-region when the height-width ratio is more than unity. If we want to assure the uniform compression zone in the mid-height such that the height of the zone is at least equal to the width of the specimen, the total height of the specimen must be more than two times as the width.

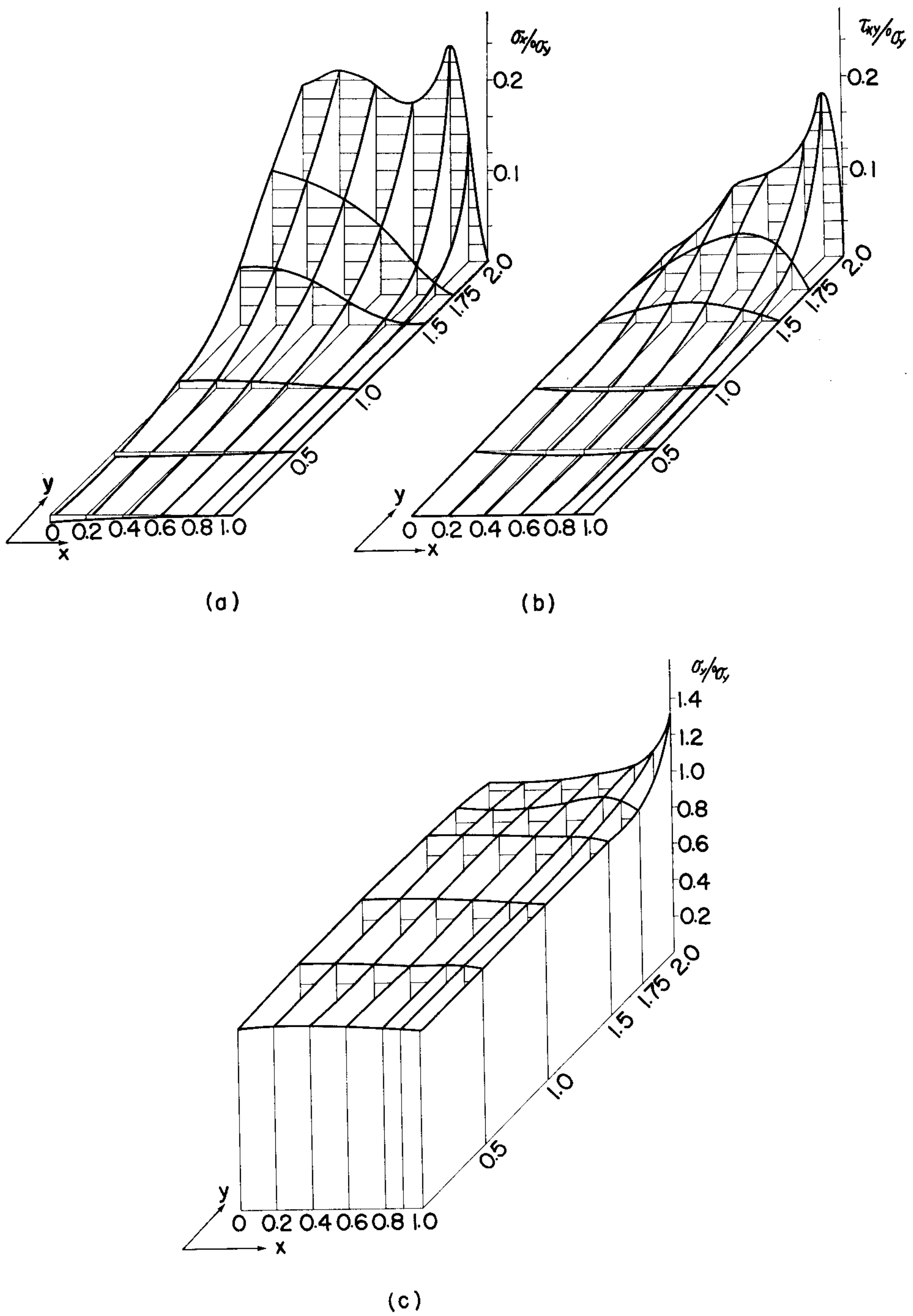

Fig. 6. Stresses in the specimen with height/width = 2.0.

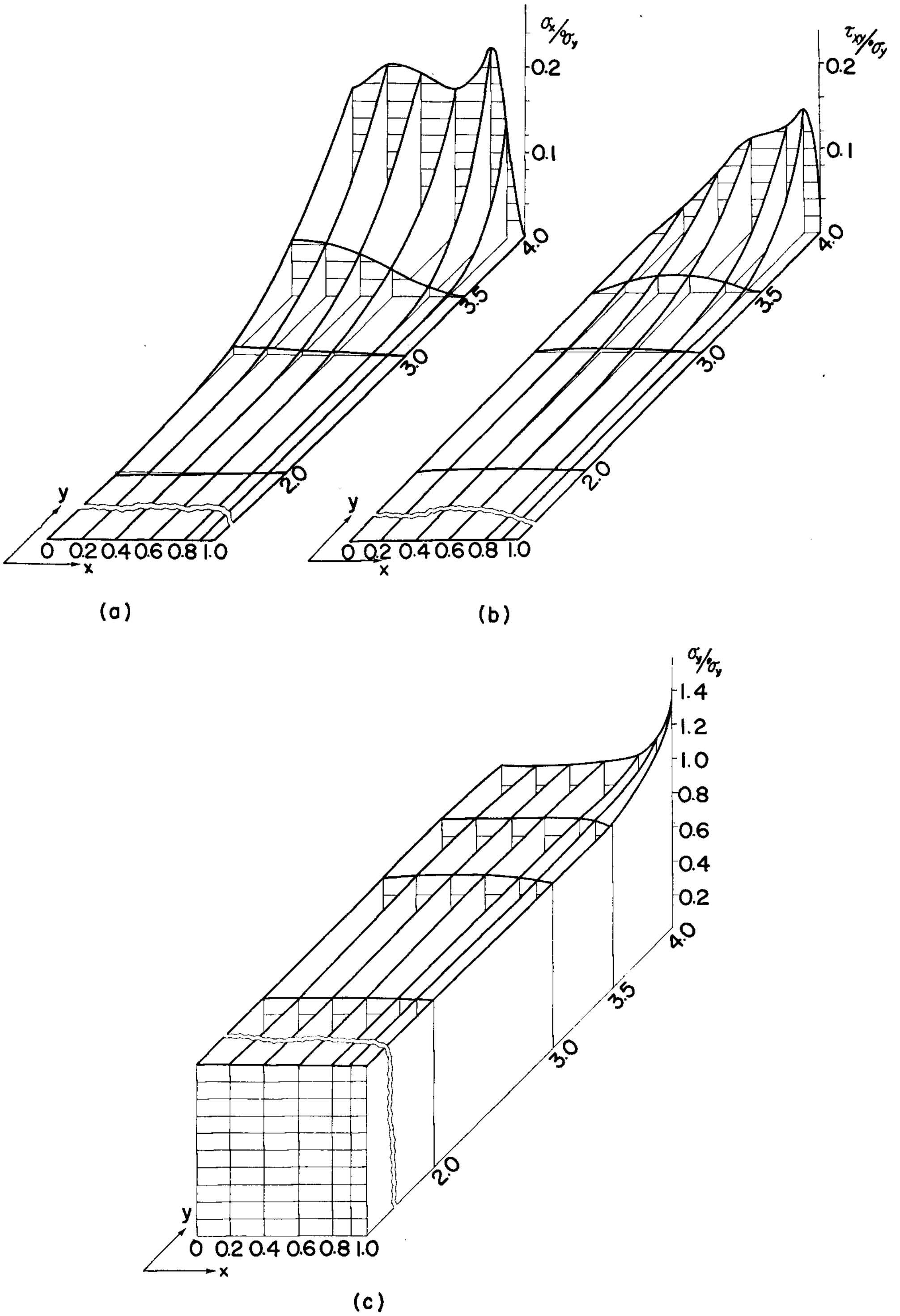

Fig. 7. Stresses in the specimen with height/width=4.0.

Table 1. Constants A_n and B_n $\times \nu^0 \sigma_y$

	n \ h	0.5	1.0	2.0	4.0
A_n	2	-0.040043	-0.034988	$-0.0^{2}64068$	$0.0^{4}86009$
	4	$-0.0^{2}18899$	$-0.0^{3}19351$	$0.0^{6}66166$	$-0.0^{10}27616$
	6	$-0.0^{3}10968$	$-0.0^{7}27056$	$-0.0^{10}55421$	$-0.0^{17}41922$
	8	$-0.0^{5}38632$	$0.0^{8}91128$	$0.0^{13}17538$	$-0.0^{24}31345$
B_n	2	0.142912	0.038587	$-0.0^{2}19598$	$-0.0^{4}53793$
	4	$0.0^{2}20217$	$-0.0^{4}40709$	$-0.0^{6}78766$	$-0.0^{11}30640$
	6	$0.0^{4}16657$	$-0.0^{5}14175$	$0.0^{9}22261$	$0.0^{17}43214$
	8	$-0.0^{5}33730$	$0.0^{8}15150$	$-0.0^{13}24207$	$0.0^{26}58944$

References

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