Stresses Produced in an Elastic Half-Plane by Moving Loads along Its Surface

By

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Stresses and displacements induced in an elastic half-plane by arbitrary loads moving along its surface with constant speed were studied. Three different cases characterized by the speed of the moving loads, *i.e.* subsonic, transsonic and supersonic cases, were individually investigated. The complete solutions are obtained for each cases and numerical results for some simple examples are presented.

1. Introduction

The response of an elastic half-plane to moving loads along its surface is a problem of considerable importance in the design of blast-resistant and aseismatic structures. Problems of this type have the essential features of stress propagation. The problem consists of three different cases characterized by the velocity of the moving loads:

a. The load is moving more slowly than either the longitudinal or the transversal wave velocity in the elastic medium (subsonic case),
b. The load velocity is between the two wave velocities (transsonic case),
c. The load velocity is greater than either wave velocities (supersonic case).

The problem of the subsonic case in the state of elastic plane strain was first investigated by I.N. Sneddon[1] by the aid of Fourier transform technique. Using the same assumptions, J. Cole and J. Huth[2] have obtained the stresses and displacements of the each cases in the elastic half-plane induced by a concentrated moving load with constant velocity. J.W. Miles[3] has obtained the formal integral solution for the response of an elastic half-space to a radially symmetric pressure with variable velocity. He also has developed an assymptotic approximation. No numerical results were obtained in his paper.

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The authors of this paper studied the response of an elastic half-plane to an arbitrary load moving with constant velocity along its surface and presented some numerical results of simple examples.

2. Mathematical Formulation

We shall consider the response of an elastic half-plane (plane strain) to moving loads $p(x)$ and $q(x)$ with constant velocity C. The general boundary conditions are shown in Fig. 1. With respect to the fixed coordinates $(\bar{x}, \bar{y})$ in the medium, the stress-strain relations expressed in terms of displacements are represented

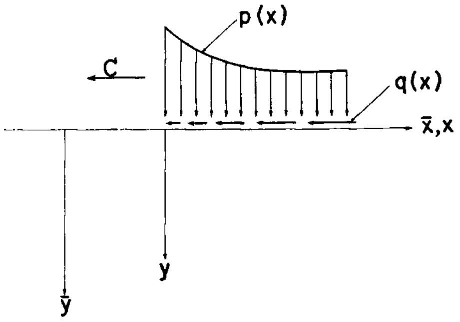


Fig. 1. Representation of boundary conditions.

$$\left.\begin{aligned} \bar{\sigma}_x &= \lambda\left(\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}\right)+2\mu\frac{\partial \bar{u}}{\partial \bar{x}}, \\ \bar{\sigma}_y &= \lambda\left(\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}\right)+2\mu\frac{\partial \bar{v}}{\partial \bar{y}}, \\ \bar{\tau}_{xy} &= \mu\left(\frac{\partial \bar{v}}{\partial \bar{x}}+\frac{\partial \bar{u}}{\partial \bar{y}}\right) \qquad , \end{aligned}\right\} \tag{2·1}$$

where $\bar{\sigma}_x$, $\bar{\sigma}_y$ and $\bar{\tau}_{xy}$ are the normal and shear stresses, and $\bar{u}, \bar{v}$ are displacements in $\bar{x}$- and $\bar{y}$-directions. λ and μ are Lamé's constants which are expressed in terms of elastic modulus E and Poisson's ratio ν as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

The equations of motion are expressed disregarding the body forces

$$\frac{\partial \bar{\sigma}_x}{\partial \bar{x}}+\frac{\partial \bar{\tau}_{xy}}{\partial \bar{y}} = \rho\frac{\partial^2 \bar{u}}{\partial t^2}, \quad \frac{\partial \bar{\tau}_{xy}}{\partial \bar{x}}+\frac{\partial \bar{\sigma}_y}{\partial \bar{y}} = \rho\frac{\partial^2 \bar{v}}{\partial t^2}, \tag{2·2}$$

where ρ is the density of the medium. Substituting Eqs. (2·1) into Eqs. (2·2), we obtain the equations of motion expressed in terms of displacements

$$\left.\begin{aligned}\lambda\left(\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{y}}\right)+\mu\left(2\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\right)&=\rho\frac{\partial^2\bar{u}}{\partial t^2},\\ \mu\left(\frac{\partial^2\bar{v}}{\partial\bar{x}^2}+\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{y}}+2\frac{\partial^2\bar{v}}{\partial\bar{y}^2}\right)+\lambda\left(\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\bar{v}}{\partial\bar{y}^2}\right)&=\rho\frac{\partial^2\bar{v}}{\partial t^2}.\end{aligned}\right\}\tag{2·3}$$

After differentiating the 1st equation by $\bar{x}$ and the 2nd by $\bar{y}$, or the 1st by $\bar{y}$ and the 2nd by $\bar{x}$, then adding the first two equations, or substracting the 4th from the 3rd, we obtain expressions

$$\left.\begin{aligned}(\lambda+2\mu)\nabla^2\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}\right)&=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}\right),\\ \mu\nabla^2\left(\frac{\partial\bar{u}}{\partial\bar{y}}-\frac{\partial\bar{v}}{\partial\bar{x}}\right)&=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial\bar{u}}{\partial\bar{y}}-\frac{\partial\bar{v}}{\partial\bar{x}}\right),\end{aligned}\right\}\tag{2·4}$$

where $\nabla^2=\dfrac{\partial^2}{\partial\bar{x}^2}+\dfrac{\partial^2}{\partial\bar{y}^2}.$

If we introduce a dilatational potential ϕ and a shear potential ψ such that the displacements are defined as

$$\bar{u}=\frac{\partial\phi}{\partial\bar{x}}-\frac{\partial\psi}{\partial\bar{y}},\quad \bar{v}=\frac{\partial\phi}{\partial\bar{y}}+\frac{\partial\psi}{\partial\bar{x}},\tag{2·5}$$

then the Eqs. (2·1) and (2·4) are expressed as follows.

$$\left.\begin{aligned}\bar{\sigma}_x&=\lambda\nabla^2\phi+2\mu\left(\frac{\partial^2\phi}{\partial\bar{x}^2}-\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}}\right),\\ \bar{\sigma}_y&=\lambda\nabla^2\phi+2\mu\left(\frac{\partial^2\phi}{\partial\bar{y}^2}+\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}}\right),\\ \bar{\tau}_{xy}&=\mu\left(2\frac{\partial^2\phi}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\psi}{\partial\bar{x}^2}-\frac{\partial^2\psi}{\partial\bar{y}^2}\right).\end{aligned}\right\}\tag{2·6}$$

and

$$\left.\begin{aligned}\nabla^2\phi&=\frac{1}{C_L^2}\frac{\partial^2\phi}{\partial t^2},\\ \nabla^2\psi&=\frac{1}{C_T^2}\frac{\partial^2\psi}{\partial t^2},\end{aligned}\right\}\tag{2·7}$$

where $C_L^2=\dfrac{\lambda+2\mu}{\rho},\quad C_T^2=\dfrac{\mu}{\rho}.$

Eqs. (2·7) are well-known wave equations; the 1st expresses the longitudinal (dilatational) wave and the 2nd transversal (shear) wave. C_L and C_T are the longitudinal and transversal wave velocities, respectively.

Now the assumption is made that the load is moving steadily for such a long time as to produce a steady response with respect to moving coordinates (x, y) with the same velocity C as the moving load.

Application of the Galilean transformation

$$x=\bar{x}+Ct,\quad y=\bar{y},\tag{2·8}$$

to Eqs. (2·5), (2·7) and (2·6) furnishes

$$u = \frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial x}, \tag{2·9}$$

$$\left.\begin{aligned} \nabla^2\phi &= M_L^2\frac{\partial^2\phi}{\partial x^2}, \quad \nabla^2\psi = M_T^2\frac{\partial^2\psi}{\partial t^2}, \\ M_L &= \frac{C}{C_L}, \quad M_T = \frac{C}{C_T}. \end{aligned}\right\} \tag{2·10}$$

and

$$\left.\begin{aligned} \sigma_x &= \lambda M_L^2\frac{\partial^2\phi}{\partial x^2}+2\mu\left(\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\psi}{\partial x\,\partial y}\right), \\ \sigma_y &= \lambda M_L^2\frac{\partial^2\phi}{\partial x^2}+2\mu\left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\psi}{\partial x\,\partial y}\right), \\ \tau_{xy} &= 2\mu\left(\frac{\partial^2\phi}{\partial x\,\partial y}+\frac{\partial^2\psi}{\partial x^2}\right)-\mu M_T^2\frac{\partial^2\psi}{\partial x^2}. \end{aligned}\right\} \tag{2·11}$$

Eqs. (2·11) are rearranged by using the relation

$$\frac{\lambda}{\mu} = \left(\frac{C_L}{C_T}\right)^2-2 = \left(\frac{M_T}{M_L}\right)^2-2$$

and Eqs. (2·10) as

$$\left.\begin{aligned} \frac{\sigma_x}{\mu} &= (M_T^2-2M_L^2+2)\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial^2\psi}{\partial x\,\partial y}, \\ \frac{\sigma_y}{\mu} &= -(2-M_T^2)\frac{\partial^2\phi}{\partial x^2}+2\frac{\partial^2\psi}{\partial x\,\partial y}, \\ \frac{\tau_{xy}}{\mu} &= 2\frac{\partial^2\phi}{\partial x\,\partial y}+(2-M_T^2)\frac{\partial^2\psi}{\partial x^2}. \end{aligned}\right\} \tag{2·11)'$$

The general boundary conditions are expressed

$$\sigma_y = -p(x), \quad \tau_{xy}=q(x) \quad \text{at} \quad y=0, \tag{2·12a}$$

$$\lim_{x\to\pm\infty}[u, v;\ \sigma_x, \sigma_y, \tau_{xy}] = 0. \tag{2·12b}$$

These conditions are not always sufficient in the cases of subsonic and transsonic. Additional conditions will be mentioned individually.

The character of the solutions will change in accordance with moving velocity of the load C so that the three cases, *i. e.*, b) $M_L<1$, $M_T<1$, subsonic, b) $M_L<1$, $M_T>1$, transsonic, c) $M_L>1$, $M_T>1$, supersonic, must be considered individually.

3. Solutions

a) Subsonic case: $M_L<1$, $M_T<1$.

Substituting the expressions

$$M_L = (1-\alpha_L^2)^{1/2}, \quad M_T = (1-\alpha_T^2)^{1/2}$$

(α_L, α_T can be assumed positive without loss of generality) into Eqs. (2·10), we obtain the Laplace type equations

$$\alpha_L^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,\quad \alpha_T^2\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0. \tag{3·1}$$

The solutions of Eqs. (3·1) in terms of dilatational and shear potentials ϕ and ψ, respectively, with boundary conditions (2·12a), (2·12b) and

$$\lim_{y\to\infty}[u, v;\ \sigma_x, \sigma_y, \tau_{xy}]=0, \tag{3·2}$$

lead to the stress and strain components expressed by Eqs. (2·9) and (2·10), respectively. The boundary conditions (2·12) and (3·2) are expressed in terms of ϕ and ψ such that

$$\lim_{\substack{x\to\pm\infty\\ y\to\infty}}\left[\frac{\partial^2\phi}{\partial x^2}, \frac{\partial^2\phi}{\partial y^2}, \frac{\partial\phi}{\partial x\partial y}, \frac{\partial^2\psi}{\partial x\partial y}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial x}, \phi, \psi\right]=0. \tag{3·3}$$

(Strictly speaking $\lim_{\substack{x\to\pm\infty\\ y\to\infty}}[\phi,\psi]=\text{const.}$, but this constant can be put equal to zero without affecting the values of displacements and stresses.)

Let $\bar{\phi}(\xi, y)$ denote Fourier transform according to

$$\bar{\phi}(\xi, y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x, y)e^{i\xi x}dx,$$

the inverse of this transform

$$\phi(x, y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{\phi}(\xi, y)e^{-ix\xi}d\xi.$$

Applying Fourier transform to Eqs. (3·1) with boundary conditions (3·3), differential equations with parameter ξ are derived

$$\left.\begin{aligned}\frac{d^2\bar{\phi}(\xi, y)}{dy^2}-\alpha_L^2\xi^2\bar{\phi}(\xi, y)=0,\\ \frac{d^2\bar{\Psi}(\xi, y)}{dy^2}-\alpha_T^2\xi^2\bar{\Psi}(\xi, y)=0.\end{aligned}\right\} \tag{3·4}$$

The solutions of these equations are

$$\left.\begin{aligned}\bar{\phi}(\xi, y)&=A_1(\xi)e^{-\alpha_L|\xi|y}+B_1(\xi)e^{\alpha_L|\xi|y},\\ \bar{\Psi}(\xi, y)&=A_2(\xi)e^{-\alpha_T|\xi|y}+B_2(\xi)e^{\alpha_T|\xi|y}.\end{aligned}\right\} \tag{3·5}$$

The conditions (3·2) can be rewritten as

$$\lim_{y\to\infty}[\bar{\phi}_{yy}, \bar{\phi}_y, \bar{\Psi}_y, \bar{\phi}, \bar{\Psi}]=0. \tag{3·2$'$}$$

$B_1(\xi)$ and $B_2(\xi)$ must be equal to zero from these conditions. The expressions (3·5) are reduced to simpler forms

$$\bar{\phi}(\xi, y) = A_1(\xi)e^{-\alpha_L|\xi|y},\ \bar{\psi}(\xi, y) = A_2(\xi)e^{-\alpha_T|\xi|y} \tag{3·5$'$}$$

Substituting Eqs. (3·5) into the transformed equations of Eqs. (2·11)$'$ and using the boundary conditions (2·12a) at $y=0$, we obtain a system of equations with respect to unknowns $A_1(\xi)$ and $A_2(\xi)$

$$\left.\begin{aligned}(2-M_T^2)\xi^2A_1(\xi)+2i\alpha_T\xi|\xi|A_2(\xi) &= -\frac{\bar{p}(\xi)}{\mu},\\ 2i\alpha_L\xi|\xi|A_1(\xi)-(2-M_T^2)\xi^2A_2(\xi) &= \frac{\bar{q}(\xi)}{\mu},\end{aligned}\right\} \tag{3·6}$$

where

$$\left.\begin{aligned}\bar{p}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}p(x)e^{i\xi x}dx,\\ \bar{q}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}q(x)e^{i\xi x}dx.\end{aligned}\right\} \tag{3·7}$$

$A_1(\xi)$ and $A_2(\xi)$ are solved to

$$\left.\begin{aligned}A_1(\xi) &= \frac{-1}{[(2-M_T^2)^2-4\alpha_L\alpha_T]\mu}\cdot\frac{1}{\xi^4}\cdot[(2-M_T^2)\xi^2\bar{p}(\xi)-2i\alpha_T\xi|\xi|\bar{q}(\xi)],\\ A_2(\xi) &= \frac{-1}{[(2-M_T^2)^2-4\alpha_L\alpha_T]\mu}\cdot\frac{1}{\xi^4}\cdot[2i\alpha_L\xi|\xi|\bar{p}(\xi)+(2-M_T^2)\xi^2\bar{q}(\xi)].\end{aligned}\right\} \tag{3·8}$$

The inverse transforms of Eqs. (3·5)$'$ furnish displacement potentials ϕ and ψ

$$\left.\begin{aligned}\phi(x, y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A_1(\xi)e^{-\alpha_L|\xi|y}e^{-ix\xi}d\xi,\\ \psi(x, y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A_2(\xi)e^{-\alpha_T|\xi|y}e^{-ix\xi}d\xi.\end{aligned}\right\} \tag{3·9}$$

The displacements and stresses are obtained by putting Eqs. (3·9) into Eqs. (2·9) and (2·11)$'$, respectively,

$$\left.\begin{aligned}u &= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{iA_1(\xi)\xi e^{-\alpha_L|\xi|y}-\alpha_T A_2(\xi)|\xi|e^{-\alpha_T|\xi|y}\}e^{-ix\xi}d\xi,\\ v &= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{\alpha_L A_1(\xi)|\xi|e^{-\alpha_L|\xi|y}+iA_2(\xi)\xi e^{-\alpha_T|\xi|y}\}e^{-ix\xi}d\xi,\end{aligned}\right\} \tag{3·10}$$

$$\left.\begin{aligned}\frac{\sigma_x}{\mu} &= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{(M_T^2-2M_L^2+2)\xi^2A_1(\xi)e^{-\alpha_L|\xi|y}\\ &\qquad +2i\alpha_T\xi|\xi|A_2(\xi)e^{-\alpha_T|\xi|y}\}e^{-ix\xi}d\xi,\\ \frac{\sigma_y}{\mu} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{(2-M_T^2)\xi^2A_1(\xi)e^{-\alpha_L|\xi|y}\\ &\qquad +2i\alpha_T\xi|\xi|A_2(\xi)e^{-\alpha_T|\xi|y}\}e^{-ix\xi}d\xi,\\ \frac{\tau_{xy}}{\mu} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{2i\alpha_L\xi|\xi|A_1(\xi)^{-\alpha_L|\xi|y}\\ &\qquad -(2-M_T^2)\xi^2A_2(\xi)e^{-\alpha_T|\xi|y}\}e^{-ix\xi}d\xi.\end{aligned}\right\} \tag{3·11}$$

b). Transsonic case: $M_L<1$, $M_T>1$.

Using new parameters β_L and β_T (both are assumed to be positive) such that

$$M_L = (1-\beta_L)^{1/2}, \quad M_T = (1+\beta_T)^{1/2},$$

the Eqs. (2.10) are rearranged into

$$\beta_L^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0, \quad \beta_T^2\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2} = 0. \tag{3·12}$$

As the 1st equation is the same as that of Eqs. (1·1) with regard to dilatational potential, the similar solution to Eqs. (3·5)′ may be expected under the same conditions as (3·3) and (3·2)′. The transformed solution is expressed with unknown coefficient $B_1(\xi)$ as

$$\bar{\phi}(\xi, y) = B_1(\xi)e^{-\beta_L|\xi|y}. \tag{3·13}$$

As the 2nd equation of Eqs. (3·12) with regard to shear potential implies the supersonic case concerning shear wave, some of the conditions for subsonic case must be modified. Conditions (2·12) remain unmodified. As the consequence, the conditions (3·3) with respect to x at infinity remain valid. Conditions with respect to y at infinity cannot be determined, since the disturbances excited along the surface of the half-plane will propagate to the infinity of y retaining the original shape. The shear wave, however, is restricted to only the backward running wave, so the solution is obtainable irrespective of conditions of y at infinity.

The assumed function of the backward running wave

$$\psi = \Psi(x-\beta_T y)$$

and the conditions (2·12a) lead to

$$\left.\begin{aligned} \frac{p(x)}{\mu} &= (2-M_T^2)\frac{\partial^2\phi}{\partial x^2}+2\beta_T\Psi''(x), \\ \frac{q(x)}{\mu} &= 2\frac{\partial^2\phi}{\partial x\partial y}+(2-M_T^2)\Psi''(x), \end{aligned}\right\} \tag{3·15}$$

where prime (′) implies the differentiation with respect to $(x-\beta_T y)$. From the above equations

$$\frac{1}{\mu}[(2-M_T^2)p(x)-2\beta_T q(x)] = (2-M_T^2)\frac{\partial^2\phi}{\partial x^2}-4\beta_T\frac{\partial^2\phi}{\partial x\partial y} \tag{3·16}$$

is obtained. The transformed expression of Eq. (3·16) with conditions

$\lim_{x\to\infty}\left[\frac{\partial\phi}{\partial x}, \phi\right] = 0$ is

$$\frac{1}{\mu}[(2-M_T^2)\bar{p}(\xi)-2\beta_T\bar{q}(\xi)] = -\xi^2(2-M_T^2)^2\bar{\phi}(\xi)+4i\beta_T\xi\bar{\phi}_y(\xi). \tag{3·17}$$

The unknown $B_1(\xi)$ is determined from Eq. (3·17) with Eq. (3·13) as

$$B_1(\xi) = \frac{-[(2-M_T^2)\bar{p}(\xi)-2\beta_T\bar{q}(\xi)]}{\mu[(2-M_T^2)^2\xi^2+4i\beta_L\beta_T\xi|\xi|]}. \tag{3·18}$$

Finally the dilatational potential is obtained

$$\phi(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B_1(\xi)e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi. \tag{3·19}$$

Substituting Eq. (3·19) into Eq. (3·15), we have the alternative expressions

$$\left.\begin{aligned} (2-M_T^2)\Psi''(x-\beta_T y) &= \frac{q(x-\beta_T y)}{\mu}-\frac{2i\beta_L}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B_1(\xi)\xi|\xi|e^{-i(x-\beta_T y)\xi}d\xi, \\ &\text{or} \\ 2\beta_T\Psi''(x-\beta_T y) &= \frac{p(x-\beta_T y)}{\mu}+\frac{(2-M_T^2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B_1(\xi)\xi^2e^{-i(x-\beta_T y)\xi}d\xi. \end{aligned}\right\} \tag{3·20}$$

$\Psi'(x-\beta_T y)$ can be easily obtained by the direct integration of Eqs. (3·15), although it may be obtained by integrating the above expressions. As the integral constants can be put equal to zero (they at most introduce constant displacements), the expressions

$$\int\frac{p(x)}{\mu}dx = (2-M_T^2)\frac{\partial\phi}{\partial x}+2\beta_T\Psi'(x),$$

$$\int\frac{q(x)}{\mu}dx = 2\frac{\partial\phi}{\partial y}+(2-M_T^2)\Psi'(x)$$

are obtained. These expressions with ϕ furnish the alternative expressions of $\Psi'(x-\beta_T y)$

$$\left.\begin{aligned} 2\beta_T\Psi'(x-\beta_T y) &= \int\frac{p(x-\beta_T y)}{\mu}d(x-\beta_T y)+\frac{i(2-M_T^2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B_1(\xi)\xi e^{-i(x-\beta_T y)\xi}d\xi, \\ &\text{or} \\ (2-M_T^2)\Psi'(x-\beta_T y) &= \int\frac{q(x-\beta_T y)}{\mu}d(x-\beta_T y)+\frac{2\beta_L}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B_1(\xi)|\xi|e^{-i(x-\beta_T y)\xi}d\xi. \end{aligned}\right\} \tag{3·21}$$

$\Psi(x-\beta_T y)$ is not shown here, since it has no contribution to displacements and stresses.

Displacements and stresses are determined from Eqs. (2·9) and (2·11)' with Eqs. (3·19), (3·20) and (3·21) as follows.

$$\begin{aligned} u &= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B_1(\xi)\xi e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\ &\quad +\frac{\beta_T}{(2-M_T^2)}\int\frac{q(x-\beta_T y)}{\mu}d(x-\beta_T y)+\frac{2\beta_L\beta_T}{\sqrt{2\pi}(2-M_T^2)}\int_{-\infty}^{\infty} B_1(\xi)|\xi|e^{-i(x-\beta_T y)\xi}d\xi, \\ &\text{or} \end{aligned}$$

$$\left.\begin{aligned}
u &= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B_1(\xi)\xi e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad +\frac{1}{2}\int\frac{p(x-\beta_T y)}{\mu}d(x-\beta_T y)+\frac{i(2-M_T^2)}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)e^{-i(x-\beta_T y)\xi}d\xi\,, \\
v &= \frac{-\beta_T}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)|\xi|e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad +\frac{1}{(2-M_T^2)}\int\frac{q(x-\beta_T y)}{\mu}d(x-\beta_T y)+\frac{2\beta_L}{\sqrt{2\pi}(2-M_T^2)}\int_{-\infty}^{\infty}B_1(\xi)|\xi|e^{-i(x-\beta_T y)\xi}d\xi\,, \\
&\text{or} \\
v &= \frac{-\beta_L}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)|\xi|e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad +\frac{1}{2\beta_T}\int\frac{p(x-\beta_T y)}{\mu}d(x-\beta_T y)+\frac{i(2-M_T^2)}{2\sqrt{2\pi}\,\beta_T}\int_{-\infty}^{\infty}B_1(\xi)\xi e^{-i(x-\beta_T y)\xi}d\xi\,,
\end{aligned}\right\} \tag{3·22}$$

$$\left.\begin{aligned}
\frac{\sigma_x}{\mu} &= \frac{-1}{\sqrt{2\pi}}(M_T^2-2M_L^2+2)\int_{-\infty}^{\infty}B_1(\xi)\xi^2e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad +\frac{2q(x-\beta_T y)\beta_T}{\mu(2-M_T^2)}-\frac{4i\beta_L\beta_T}{\sqrt{2\pi}(2-M_T^2)}\int_{-\infty}^{\infty}B_1(\xi)\xi|\xi|e^{-i(x-\beta_T y)\xi}d\xi\,, \\
&\text{or} \\
\frac{\sigma_x}{\mu} &= \frac{-1}{\sqrt{2\pi}}(M_T^2-2M_L^2+2)\int_{-\infty}^{\infty}B_1(\xi)\xi^2e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad +\frac{p(x-\beta_T y)}{\mu}+\frac{(2-M_T^2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi^2e^{-i(x-\beta_T y)\xi}d\xi\,, \\
\frac{\sigma_y}{\mu} &= \frac{(2-M_T^2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi^2e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad -\frac{2\beta_T q(x-\beta_T y)}{2-M_T^2}+\frac{4i\beta_L\beta_T}{\sqrt{2\pi}(2-M_T^2)}\int_{-\infty}^{\infty}B_1(\xi)\xi|\xi|e^{-i(x-\beta_T y)\xi}d\xi\,, \\
&\text{or} \\
\frac{\sigma_y}{\mu} &= \frac{(2-M_T^2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi^2e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad -\frac{p(x-\beta_T y)}{\mu}-\frac{(2-M_T^2)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi^2e^{-i(x-\beta_T y)\xi}d\xi\,, \\
\frac{\tau_{xy}}{\mu} &= \frac{2i\beta_L}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi|\xi|e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad +\frac{q(x-\beta_T y)}{\mu}-\frac{2i\beta_L}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi|\xi|e^{-i(x-\beta_T y)\xi}d\xi\,, \\
&\text{or} \\
\frac{\tau_{xy}}{\mu} &= \frac{2i\beta_L}{\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi|\xi|e^{-\beta_L|\xi|y}e^{-ix\xi}d\xi \\
&\quad +\frac{(2-M_T^2)}{2\beta_T\mu}p(x-\beta_T y)+\frac{(2-M_T^2)^2}{2\beta_T\sqrt{2\pi}}\int_{-\infty}^{\infty}B_1(\xi)\xi^2e^{-i(x-\beta_T y)\xi}d\xi\,.
\end{aligned}\right\} \tag{3·23}$$

c). Supersonic case: $M_L>1$, $M_T>1$.

The wave equations (2·10) are rearranged with positive parameters γ_L, γ_T such that $M_L=(1+\gamma_L)^{1/2}$, $M_T=(1+\gamma_T)^{1/2}$ into

$$\gamma_L^2\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}=0, \qquad \gamma_T^2\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}=0. \tag{3·24}$$

$$\phi=\Phi(x-\gamma_L y), \qquad \psi=\Psi(x-\gamma_T y) \tag{3·25}$$

are obviously the solutions of Eqs. (3·24), and their forms will be determined by boundary conditions (2·12a) and (2·12b).

Substituting Eqs. (3·25) into conditions (2·12a), we have the relations

$$\left.\begin{aligned} \frac{p(x)}{\mu}&=(2-M_T^2)\Phi''(x)+2\gamma_T\Psi''(x),\\ \frac{q(x)}{\mu}&=-2\gamma_L\Phi''(x)+(2-M_T^2)\Psi''(x). \end{aligned}\right\} \tag{3·26}$$

These are solved

$$\left.\begin{aligned} \Phi''(x)&=\frac{(2-M_T^2)p(x)-2\gamma_T q(x)}{\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]},\\ \Psi''(x)&=\frac{2\gamma_L p(x)+(2-M_T^2)q(x)}{\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]}. \end{aligned}\right\} \tag{3·27}$$

Therefore, the expressions for $\Phi''(x-\gamma_L y)$ and $\Psi''(x-\gamma_T y)$ are

$$\left.\begin{aligned} \Phi''(x-\gamma_L y)&=\frac{(2-M_T^2)p(x-\gamma_L y)-2\gamma_T q(x-\gamma_T y)}{\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]},\\ \Psi''(x-\gamma_T y)&=\frac{2\gamma_L p(x-\gamma_T y)+(2-M_T^2)q(x-\gamma_T y)}{\mu'[(2-M_T^2)^2+4\gamma_L\gamma_T]}. \end{aligned}\right\} \tag{3·28}$$

Applying Fourier transform to Eqs. (3·27) with conditions $\lim_{x\to\pm\infty}[\Phi'(x), \Psi'(x)]=0$ which are equivalent to conditions (2·12b), we have the expressions

$$\left.\begin{aligned} \overline{\Phi'}(\xi)&=\frac{i[(2-M_T^2)\bar{p}(\xi)-2\gamma_T\bar{q}(\xi)]}{\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]\xi},\\ \overline{\Psi'}(\xi)&=\frac{i[2\gamma_L\bar{p}(\xi)+(2-M_T^2)\bar{q}(\xi)]}{\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]\xi}. \end{aligned}\right\} \tag{3.29}$$

The inverse transforms of the above expressions are

$$\left.\begin{aligned} \Phi'(x)&=\frac{i}{\sqrt{2\pi}\cdot\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]}\int_{-\infty}^{\infty}\left[(2-M_T^2)\frac{\bar{p}(\xi)}{\xi}-2\gamma_T\frac{\bar{q}(\xi)}{\xi}\right]e^{-ix\xi}d\xi,\\ \Psi'(x)&=\frac{i}{\sqrt{2\pi}\cdot\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]}\int_{-\infty}^{\infty}\left[2\gamma_L\frac{\bar{p}(\xi)}{\xi}+(2-M_T^2)\frac{\bar{q}(\xi)}{\xi}\right]e^{-ix\xi}d\xi. \end{aligned}\right\} \tag{3·30}$$

Therefore ϕ' and ψ' are obtained in the forms

$$\left.\begin{aligned}\phi' &= \Phi'(x-\gamma_L y)\\ &= \frac{i}{\sqrt{2\pi}\cdot\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]}\int_{-\infty}^{\infty}\left[(2-M_T^2)\frac{\bar{p}(\xi)}{\xi}-2\gamma_T\frac{\bar{q}(\xi)}{\xi}\right]\cdot e^{-i(x-\gamma_L y)\xi}d\xi\,,\\ \psi' &= \Psi'(x-\gamma_T y)\\ &= \frac{i}{\sqrt{2\pi}\cdot\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]}\int_{-\infty}^{\infty}\left[2\gamma_L\frac{\bar{p}(\xi)}{\xi}+(2-M_T^2)\frac{\bar{q}(\xi)}{\xi}\right]\cdot e^{-i(x-\gamma_T y)\xi}d\xi\,.\end{aligned}\right\}\quad(3\cdot31)$$

Finally the displacements and stresses are determined from Eqs. (2·9) and (2·11)′ with Eqs. (3·31) and (3.28) as follows.

$$\left.\begin{aligned}u &= \frac{i}{\sqrt{2\pi}\cdot\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]}\int_{-\infty}^{\infty}\left[\left\{(2-M_T^2)\frac{\bar{p}(\xi)}{\xi}-2\gamma_T\frac{\bar{q}(\xi)}{\xi}\right\}e^{-i(x-\gamma_L y)\xi}\right.\\ &\quad\left.+\left\{2\gamma_L\gamma_T\frac{\bar{p}(\xi)}{\xi}+\gamma_T(2-M_T^2)\frac{\bar{q}(\xi)}{\xi}\right\}e^{-i(x-\gamma_T y)\xi}\right]d\xi\,,\\ v &= \frac{-i}{\sqrt{2\pi}\cdot\mu[(2-M_T^2)^2+4\gamma_L\gamma_T]}\int_{-\infty}^{\infty}\left[\left\{\gamma_L(2-M_T^2)\frac{\bar{p}(\xi)}{\xi}-2\gamma_L\gamma_T\frac{\bar{q}(\xi)}{\xi}\right\}e^{-i(x-\gamma_L y)\xi}\right.\\ &\quad\left.-\left\{2\gamma_L\frac{\bar{p}(\xi)}{\xi}+(2-M_T^2)\frac{\bar{q}(\xi)}{\xi}\right\}e^{-i(x-\gamma_T y)\xi}\right]d\xi\,,\end{aligned}\right\}\quad(3\cdot32)$$

$$\left.\begin{aligned}\sigma_x &= \frac{1}{(2-M_T^2)^2+4\gamma_L\gamma_T}[(M_T^2-2M_L^2+2)\{(2-M_T^2)p(x-\gamma_L y)-2\gamma_T q(x-\gamma_L y)\\ &\quad+2\gamma_T\{2\gamma_L p(x-\gamma_T y)+(2-M_T^2)q(x-\gamma_T y)\}]\,,\\ \sigma_y &= \frac{-1}{(2-M_T^2)^2+4\gamma_L\gamma_T}[(2-M_T^2)\{(2-M_T^2)p(x-\gamma_L y)-2\gamma_T q(x-\gamma_L y)\}\\ &\quad+2\gamma_T\{2\gamma_L p(x-\gamma_T y)+(2-M_T^2)q(x-\gamma_T y)\}]\,,\\ \tau_{xy} &= \frac{-1}{(2-M_T^2)+4\gamma_L\gamma_T}[2\gamma_L\{(2-M_T^2)p(x-\gamma_L y)-2\gamma_T q(x-\gamma_L y)\}\\ &\quad-(2-M_T^2)\{2\gamma_L p(x-\gamma_T y)+(2-M_T^2)q(x-\gamma_T y)\}]\,.\end{aligned}\right\}\quad(3\cdot33)$$

4. Examples

A). A concentrated vertical load P.

As the simplest example, we consider the response of an elastic half-plane induced by a concentrated vertical load moving with constant velocity along its surface. The same problem has been solved by J. Cole and J. Huth.[2)]

The general boundary conditions of this problem are easily visualized in Fig. 1 by putting

$$p(x) = p\delta(x), \quad q(x) = 0 \qquad (A\cdot1)$$

where $\delta(x)$ means the Dirac's delta function.

a). Subsonic case.

From Eqs. (3·7) with Eqs. (A·1), $\bar{p}(\xi)$ and $\bar{q}(\xi)$ are expressed.

$$\left.\begin{aligned} \bar{p}(\xi) &= \frac{P}{\sqrt{2\pi}}, \\ \bar{q}(\xi) &= 0. \end{aligned}\right\} \tag{A·2}$$

Eqs. (3·8) and (A·2) furnish $A_1(\xi)$ and $A_2(\xi)$

$$\left.\begin{aligned} A_1(\xi) &= \frac{-K_1 P}{\sqrt{2\pi}\mu}\cdot\frac{1}{\xi^2}, \\ A_2(\xi) &= \frac{-iK_2 P}{\sqrt{2\pi}\mu}\cdot\frac{|\xi|}{\xi^3}, \end{aligned}\right\} \tag{A·3}$$

where

$$K_1 = \frac{2-M_T^2}{(2-M_T^2)^2-4\alpha_L\alpha_T}, \quad K_2 = \frac{2\alpha_L}{(2-M_T^2)^2-4\alpha_L\alpha_T}.$$

Substituting $A_1(\xi)$ and $A_2(\xi)$ into Eqs. (3·10) and (3·11), we obtain displacements and stresses as follows.

$$\left.\begin{aligned} \frac{\pi\mu u}{P} &= K_1\tan^{-1}\left(\frac{x}{\alpha_L y}\right) - \alpha_T K_2 \tan^{-1}\left(\frac{x}{\alpha_L y}\right), \\ \frac{\pi\mu v}{P} &= -\alpha_L K_1 \ln(x^2+\alpha_L^2 y^2) + K_2 \ln(x^2+\alpha_T^2 y^2), \end{aligned}\right\} \tag{A·4}$$

$$\left.\begin{aligned} \frac{\pi\sigma_x}{P} &= (M_T^2-2M_L^2+2)K_1\frac{\alpha_L y}{x^2+\alpha_L^2 y^2} - 2\alpha_T K_2 \frac{\alpha_T y}{x^2+\alpha_T^2 y^2}, \\ \frac{\pi\sigma_y}{P} &= -(2-M_T^2)K_1\frac{\alpha_L y}{x^2+\alpha_L^2 y^2} + 2\alpha_T K_2 \frac{\alpha_T y}{x^2+\alpha_T^2 y^2}, \\ \frac{\pi\tau_{xy}}{P} &= -2\alpha_L K_1\left[\frac{x}{x^2+\alpha_L^2 y^2} - \frac{x}{x^2+\alpha_T^2 y^2}\right]. \end{aligned}\right\} \tag{A·5}$$

b) Transsonic case.

Substitution $\bar{p}(\xi)$ and $\bar{q}(\xi)$ into Eq. (3·18) leads to

$$\left.\begin{aligned} B_1(\xi) &= \frac{-P}{\sqrt{2\pi}\,\mu}(2-M_T^2)\left[T_1\frac{1}{\xi^2} - iT_2\frac{|\xi|}{\xi^3}\right], \\ T_1 &= \frac{(2-M_T^2)^2}{(2-M_T^2)^4+16\beta_L^2\beta_T^2}, \\ T_2 &= \frac{4\beta_L\beta_T}{(2-M_T^2)^4+16\beta_L^2\beta_T^2}. \end{aligned}\right\} \tag{A·6}$$

Eqs. (3·22) and (3·33) with Eq. (A·6) furnish displacements and stresses.

$$\left.\begin{aligned} \frac{\pi\mu u}{P} &= (2-M_T^2)\left[2\pi\tan^{-1}\left(\frac{x}{\beta_L y}\right) - T_2\ln(x^2+\beta_L^2 y^2)\right] \\ &\quad + 2\beta_L\beta_T\left[T_1\ln(x-\beta_T y) + \pi T_2 H(x-\beta_T y)\right], \\ \frac{\pi\mu v}{P} &= -\frac{\beta_L}{2}(2-M_T^2)\left[T_1\ln(x^2+\beta_L^2 y^2) + 2T_2\tan^{-1}\left(\frac{x}{\beta_L y}\right)\right] \\ &\quad + 2\beta_L\left[T_1\ln(x-\beta_T y) + \pi T_2 H(x-\beta_T y)\right], \\ H(x-\beta_T y) &= \begin{cases} 1 & \text{for } x \geq \beta_T y \\ 0 & \text{for } x < \beta_T y. \end{cases} \end{aligned}\right\} \tag{A·7}$$

$$\left.\begin{aligned}\frac{\pi\sigma_x}{P}&=(M_T^2-2M_L^2+2)(2-M_T^2)(T_1\beta_L y+T_2x)\frac{1}{x^2+\beta_L^2y^2}\\&\quad-4\beta_L\beta_T\left[\frac{T_1}{x-\beta_T y}-\pi T_2\delta(x-\beta_T y)\right],\\\frac{\pi\sigma_y}{P}&=-(2-M_T^2)^2(T_1\beta_L y+T_2x)\frac{1}{x^2+\beta_L^2y^2}\\&\quad+4\beta_L\beta_T\left[\frac{T_1}{x-\beta_T y}-\pi T_2\delta(x-\beta_T y)\right],\\\frac{\pi\tau_{xy}}{P}&=-2\beta_L(2-M_T^2)(T_1x+T_2\beta_L y)\frac{1}{x^2+\beta_L^2y^2}\\&\quad-2\beta_L(2-M_T^2)\left[\frac{T_1}{x-\beta_T y}-\pi T_2\delta(x-\beta_T y)\right].\end{aligned}\right\}\qquad(A\cdot 8)$$

c). Supersonic case.

Displacements are obtained from Eqs. (3·32) with Eqs. (A·2) as

$$\left.\begin{aligned}\frac{\mu u}{P}&=S_1H(x-\gamma_L y)+\gamma_T S_2H(x-\gamma_T y),\\\frac{\mu v}{P}&=-\gamma_L S_1H(x-\gamma_L r)+S_2H(x-\gamma_T y),\\S_1&=\frac{2-M_T^2}{(2-M_T^2)^2+4\gamma_L\gamma_T},\\S_2&=\frac{2\gamma_L}{(2-M_T^2)^2+4\gamma_L\gamma_T}.\end{aligned}\right\}\qquad(A\cdot 9)$$

Stresses are obtained from Eqs. (3·33) with Eqs. (A·1)

$$\left.\begin{aligned}\frac{\sigma_x}{P}&=(M_T^2-2M_L^2+2)S_1\delta(x-\gamma_L y)+2\gamma_T S_2\delta(x-\gamma_T y),\\\frac{\sigma_y}{P}&=-(2-M_T^2)S_1\delta(x-\gamma_L y)-2\gamma_T S_2\delta(x-\gamma_T y),\\\frac{\tau_{xy}}{P}&=-2\gamma_L S_1\delta(x-\gamma_L y)+(2-M_T^2)S_2\delta(x-\gamma_T y).\end{aligned}\right\}\qquad(A\cdot 10)$$

Stresses are shown in Figs. 2, 3 and 4. Figs. 5, 6 and 7 shows the stresses as M_T approaches to 1. The results obtained here agree with those of J. J. Cole and Huth.[2)]

B). A concentrated horizontal load Q.

A concentrated horizontal load Q moving with constant velocity C is represented in Fig. 1 as

$$p(x)=0,\qquad q(x)=Q\,\delta(x).$$

a). Subsonic case.

Eqs. (3·8) and

$$\bar{p}(\xi)=0,\qquad \bar{q}(\xi)=\frac{Q}{\sqrt{2\pi}}\qquad(B\cdot 1)$$

furnish

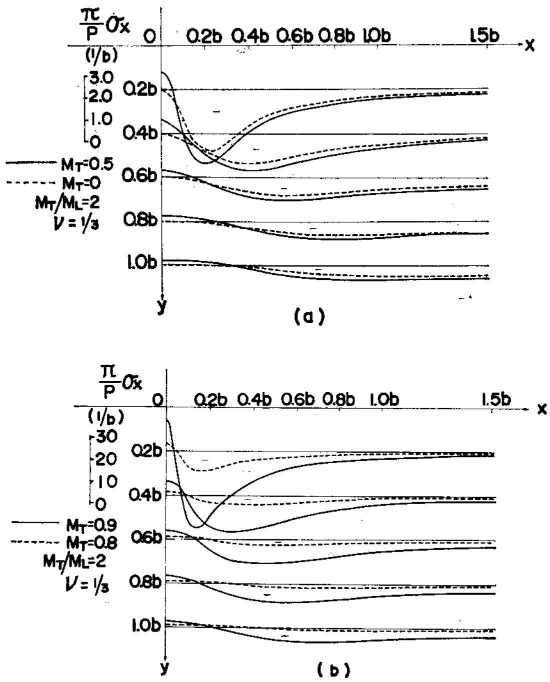


Fig. 2. Distribution of $\pi\sigma_x/P$.

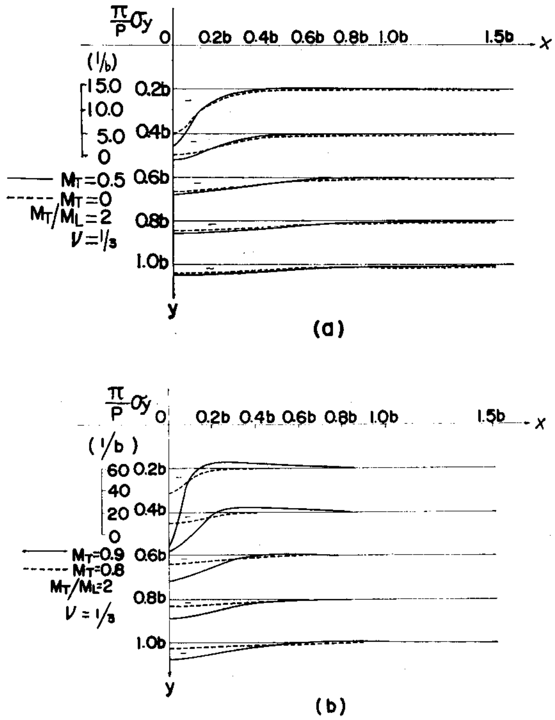


Fig. 3. Distribution of $\pi\sigma_y/P$.

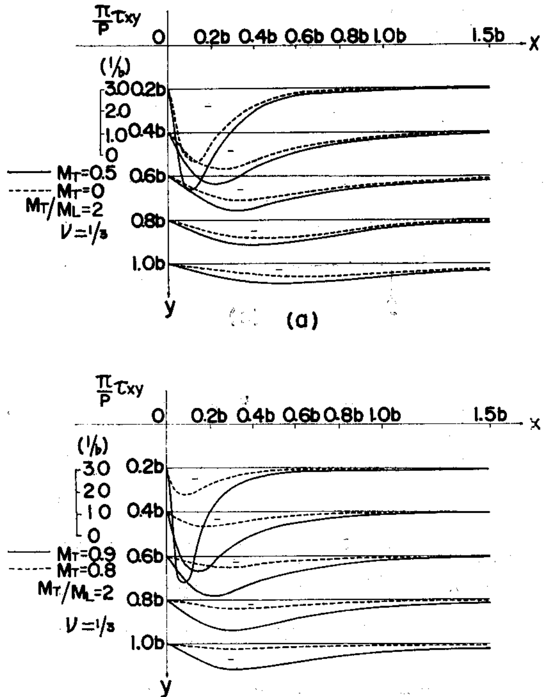


Fig. 4. Distribution of $\pi\tau_{xy}/P$.

$$\left.\begin{aligned} A_1(\xi) &= \frac{iQK_3}{\sqrt{2\pi}\mu}\cdot\frac{|\xi|}{\xi^3}, \\ A_2(\xi) &= -\frac{QK_1}{\sqrt{2\pi}\mu}\cdot\frac{1}{\xi^2}, \\ K_1 &= \frac{2-M_T^2}{(2-M_T^2)^2-4\alpha_L\alpha_T}, \\ K_2 &= \frac{2\alpha_T}{(2-M_T^2)^2-4\alpha_L\alpha_T}. \end{aligned}\right\} \tag{B·2}$$

Displacements and stresses are obtained from Eqs. (3·10) and (3·11) with Eqs. (B·2) as

$$\left.\begin{aligned} \frac{\pi\mu u}{Q} &= \frac{1}{2}\,[\alpha_T K_1 \ln(x^2+\alpha_T^2 y^2)-K_3\ln(x^2+\alpha_L^2 y^2)]. \\ \frac{\pi\mu v}{Q} &= K_1\tan^{-1}\!\left(\frac{x}{\alpha_T y}\right)-\alpha_L K_3\tan^{-1}\!\left(\frac{x}{\alpha_L y}\right), \end{aligned}\right\} \tag{B·3}$$

$$\left.\begin{aligned} \frac{\pi\sigma_x}{Q} &= 2\alpha_T K_1\frac{x}{x^2+\alpha_T^2 y^2}-(M_T^2-2M_L^2+2)K_3\frac{x}{x^2+\alpha_L^2 y^2}, \\ \frac{\pi\sigma_y}{Q} &= -2\alpha_T K_1\frac{x}{x^2+\alpha_T^2 y^2}+(2-M_T^2)K_3\frac{x}{x^2+\alpha_L^2 y^2}, \\ \frac{\pi\tau_{xy}}{Q} &= (2-M_T^2)K_1\frac{\alpha_T y}{x^2+\alpha_T^2 y^2}-2K_3\alpha_L\frac{\alpha_L y}{x^2+\alpha_L^2 y^2}. \end{aligned}\right\} \tag{B·4}$$

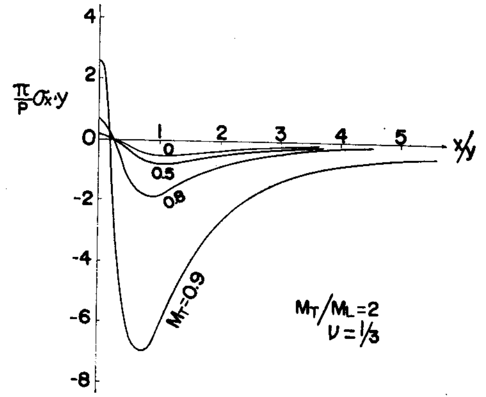


Fig. 5. $\pi\sigma_x \cdot y/P$ as M_T approaches to 1.

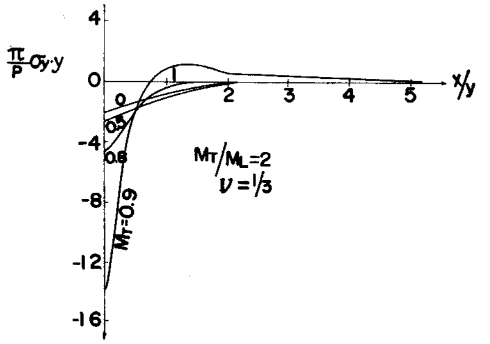


Fig. 6. $\pi\sigma_y \cdot y/P$ as M_T approaches to 1.

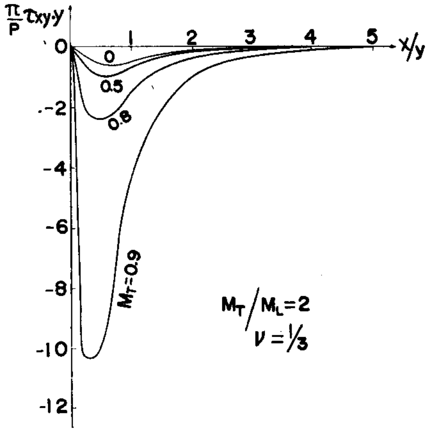


Fig. 7.
$\pi\tau_{xy} \cdot y/P$ as M_T approaches to 1.

b). Transsonic case

Substitution Eqs. (B·1) into Eq. (3·18) furnishes

$$\left.\begin{aligned} B_1(\xi) &= \frac{2\beta_T Q}{\sqrt{2\pi}\mu}\left[T_1\frac{1}{\xi} - i\,T_2\frac{|\xi|}{\xi^3}\right], \\ T_1 &= \frac{(2-M_T^2)^2}{(2-M_T^2)^4+16\beta_L^2\beta_T^2}, \\ T_2 &= \frac{4\beta_L\beta_T}{(2-M_T^2)^4+16\beta_L^2\beta_T^2}. \end{aligned}\right\} \quad (B\cdot 5)$$

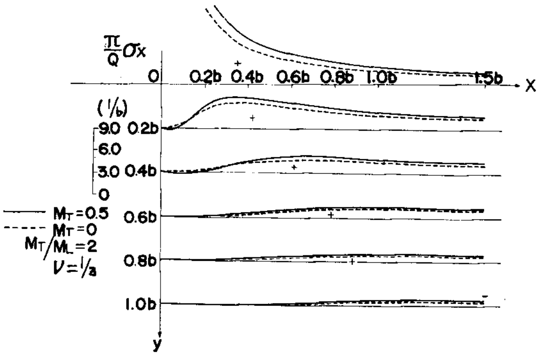
Displacements and stresses are obtained from Eqs. (3·22) and (3·23) with Eqs. (B·5) as follows.

$$\left.\begin{aligned} \frac{\pi\mu u}{Q} &= \beta_T\left[-2T_1\tan^{-1}\left(\frac{x}{\beta_L y}\right) + T_2\ln(x^2+\beta_L^2 y^2)\right] \\ &\quad + \beta_T(2-M_T^2)[\pi T_1 H(x-\beta_T y) - T_2\ln(x-\beta_T y)], \\ \frac{\pi\mu v}{Q} &= \beta_L\beta_T\left[T_1\ln(x^2+\beta_L^2 y^2) + 2T_2\tan^{-1}\left(\frac{x}{\beta_L y}\right)\right] \\ &\quad + (2-M_T^2)[\pi T_1 H(x-\beta_T y) - T_2\ln(x-\beta_T y)], \end{aligned}\right\} \quad (B\cdot 6)$$

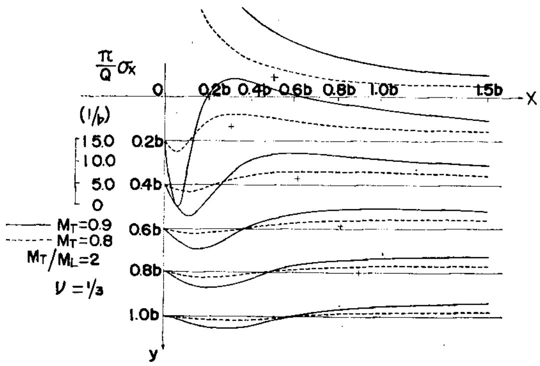
$$\left.\begin{aligned}
\frac{\pi\sigma_x}{Q} &= -2\beta_T(M_T^2-2M_L^2+2)\left[T_1\frac{\beta_L y}{x^2+\beta_L^2y^2}-T_2\frac{x}{x^2+\beta_L^2y^2}\right]\\
&\quad +2\beta_T(2-M_L^2)\left[\pi T_1\delta(x-\beta_T y)-\frac{T_2}{x-\beta_T y}\right],\\
\frac{\pi\sigma_y}{Q} &= 2\beta_T(2-M_T^2)\left[T_1\frac{\beta_L y}{x^2+\beta_L^2y^2}-T_2\frac{x}{x^2+\beta_L^2y^2}\right]\\
&\quad -2\beta_L(2-M_T^2)\left[\pi T_1\delta(x-\beta_T y)-\frac{T_2}{x-\beta_T y}\right],\\
\frac{\pi\tau_{xy}}{Q} &= 2\beta_L\beta_T\left[T_1\frac{x}{x^2+\beta_L^2y^2}+T_2\frac{\beta_L y}{x^2+\beta_L^2y^2}\right]\\
&\quad +(2-M_T^2)^2\left[\pi T_1\delta(x-\beta_T y)-\frac{T_2}{x-\beta_T y}\right].
\end{aligned}\right\} \qquad (B\cdot7)$$

c). Supersonic case

Displacements and stresses are obtained from Eqs. (3·32) and (3·33) with Eqs. (B·1).



(a)



(b)

Fig. 8. Distribution of $\pi\sigma_x/Q$

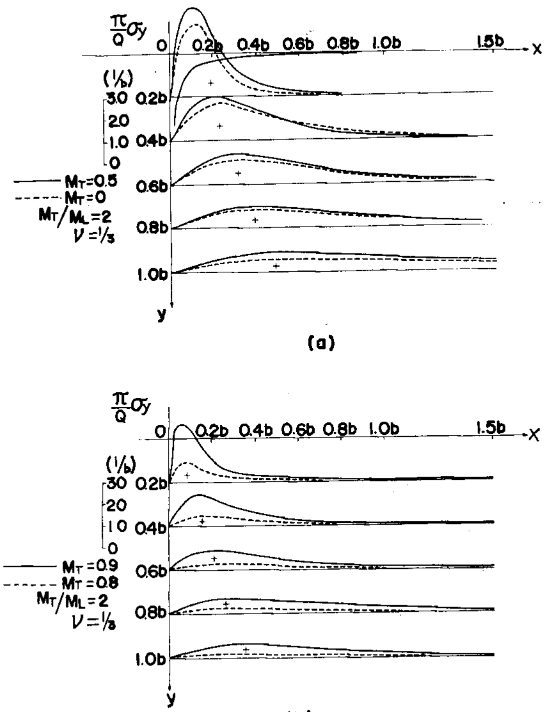


Fig. 9. Distribution of $\pi\sigma_x/Q$.

$$\left.\begin{aligned}
\frac{\mu u}{Q} &= 2\gamma_T S_1 H(x-\gamma_L y) - S_3 H(x-\gamma_L y)\,, \\
\frac{\mu v}{Q} &= S_1 H(x-\gamma_T y) + \gamma_L S_3 H(x-\gamma_L y)\,, \\
S_1 &= \frac{2-M_T^2}{(2-M_T^2)^2+4\gamma_L\gamma_T}\,, \quad S_1 = \frac{2\gamma_T}{(2-M_T^2)^2+4\gamma_L\gamma_T}\,,
\end{aligned}\right\} \tag{B·8}$$

$$\left.\begin{aligned}
\frac{\sigma_x}{Q} &= 2\gamma_T S_1 \delta(x-\gamma_T y) - (M_T^2-2M_L^2+2) S_3 \delta(x-\gamma_L y)\,, \\
\frac{\sigma_y}{Q} &= 2\gamma_T S_1 [\delta(x-\gamma_L y) - \delta(x-\gamma_T y)]\,, \\
\frac{\tau_{xy}}{Q} &= (2-M_T^2) S_1 \delta(x-\gamma_T y) + 2\gamma_L S_3 \delta(x-\gamma_L y)\,.
\end{aligned}\right\} \tag{B·9}$$

Stresses are shown in Figs. 8, 9 and 10. In Figs. 11, 12 and 13 the stress components as M_T approaches to 1 are shown.

C). A distributed vertical load of constant intensity.

The response of the elastic half-plane to a distributed vertical load $p(x)$ of constant intensity over a length $2a$ will be considered. The solution may be constructed from the results of the example A) by superposition. Stresses

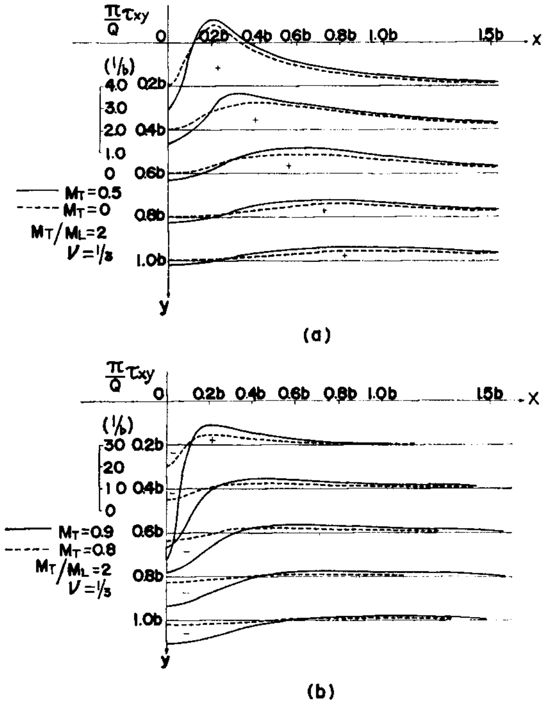


Fig. 10. Distribution of $\pi\tau_{xy}/Q$.

obtained here, however, are derived directly from the general solutions.

It may be convenient to shift the origin of the moving co-ordinates (x, y) to the right by a in Fig. 1 so that the distributed load may be expressed as

$$p(x) = p_0\{H(x+a)-H(x-a)\}, \quad q(x) = 0\,, \tag{C·1}$$

where p_0 is the intensity of the distributed load. Eqs. (3·7) corresponding to Eqs. (C·1) are

$$\bar{p}(\xi) = \frac{2p_0}{\sqrt{2\pi}}\cdot\frac{\sin(a\xi)}{\xi}, \quad \bar{q}(\xi) = 0\,. \tag{C·2}$$

a). Subsonic case.

$A_1(\xi)$ and $A_2(\xi)$ are determined from Eqs. (3·8) with Eqs. (C·2)

$$\left.\begin{aligned} A_1(\xi) &= -\frac{2p_0K_1}{\sqrt{2\pi}\mu}\cdot\frac{\sin(a\xi)}{\xi^3}, \\ A_2(\xi) &= -\frac{2ip_0K_2}{\sqrt{2\pi}\mu}\cdot\frac{|\xi|\sin(a\xi)}{\xi^4}, \\ K_1 &= \frac{2-M_T^2}{(2-M_T^2)^2-4\alpha_L\alpha_T}, \quad K_2 = \frac{2\alpha_L}{(2-M_T^2)^2-4\alpha_L\alpha_T}\,. \end{aligned}\right\} \tag{C·3}$$

Stresses are expressed Eqs. (3·11) and (C·3) as follows.

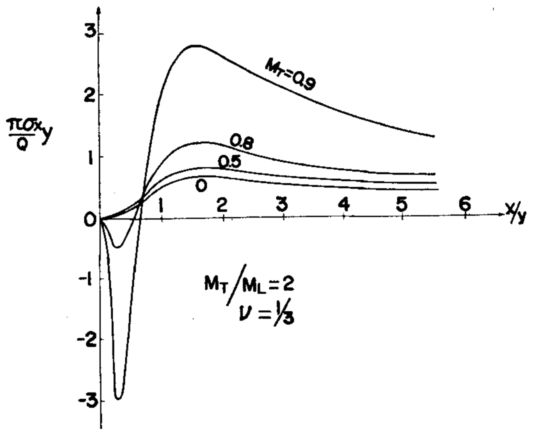


Fig. 11. $\pi\sigma_x\cdot y/Q$ as M_T approaches to 1.

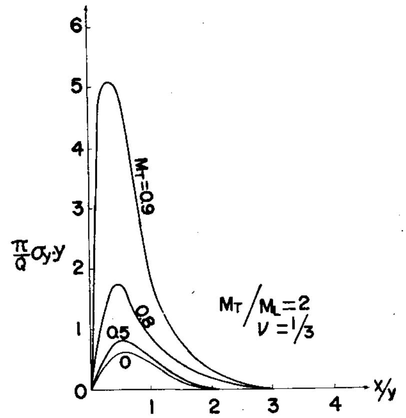


Fig. 12. $\pi\sigma_y\cdot y/Q$ as M_T approaches to 1.

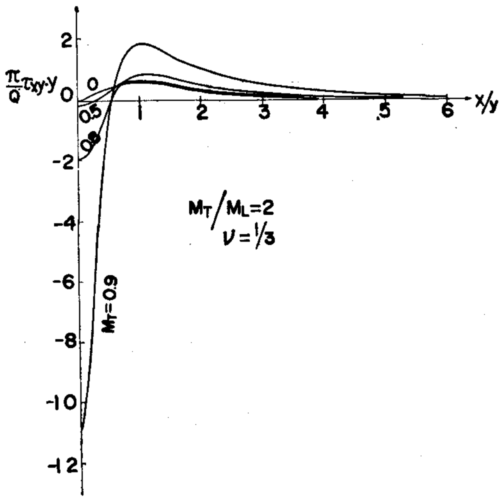


Fig. 13. $\pi\tau_{xy}\cdot y/Q$ as M_T approaches to 1.

$$\left.\begin{aligned}
\frac{\pi\sigma_x}{p_0} &= (M_T^2-2M_L^2+2)K_1\tan^{-1}\frac{2a\alpha_L y}{x^2+\alpha_L^2y^2-a^2}-2\alpha_T K_2\tan^{-1}\frac{2a\alpha_T y}{x^2+\alpha_T^2y^2-a^2},\\
\frac{\pi\sigma_y}{p_0} &= -(2-M_T^2)K_1\tan^{-1}\frac{2a\alpha_L y}{x^2+\alpha_L^2y^2-a^2}+2\alpha_T K_2\tan^{-1}\frac{2a\alpha_T y}{x^2+\alpha_T^2y^2-a^2},\\
\frac{\pi\tau_{xy}}{p_0} &= -\alpha_L K_1\ln\frac{\alpha_L^2y^2+(x+a)^2}{\alpha_L^2y^2+(x-a)^2}+\frac{(2-M_T^2)}{2}\ln\frac{\alpha_T^2y^2+(x+a)^2}{\alpha_T^2y^2+(x-a)^2}.
\end{aligned}\right\}\quad (C\cdot 4)$$

b). Transsonic case.

Eq. (3·18) with Eqs. (C·2) yields

$$\left.\begin{aligned} B_1(\xi) &= \frac{-2p_0}{\sqrt{2\pi}\,\mu}(2-M_T^2)\Big[T_1\frac{1}{\xi^3}-iT_2\frac{|\xi|}{\xi^4}\Big]\sin(a\xi),\\ T_1 &= \frac{(2-M_T^2)^2}{(2-M_T^2)^4+16\beta_L^2\beta_T^2},\\ T_2 &= \frac{4\beta_L\beta_T}{(2-M_T^2)^4+16\beta_L^2\beta_T^2}. \end{aligned}\right\} \tag{C·5}$$

Eqs. (3·23) with Eqs. (C·5) lead to

$$\left.\begin{aligned} \frac{\pi\sigma_x}{p_0} &= (2-M_T^2)(M_T^2-2M_L^2+2)\Big[T_1\tan^{-1}\Big(\frac{2a\beta_L y}{x^2+\beta_L^2y^2-a^2}\Big)-\frac{T_2}{2}\ln\frac{\beta_L^2y^2+(x+a)^2}{\beta_L^2y^2+(x-a)^2}\Big]\\ &\quad+4\beta_L\beta_T\Big[T_1\ln\Big|\frac{x+a-\beta_T y}{x-a-\beta_T y}\Big|+2T_2F(x-\beta_T y)\Big],\\ \frac{\pi\sigma_y}{p_0} &= -(2-M_T^2)\Big[T_1\tan^{-1}\Big(\frac{2a\beta_L y}{x^2+\beta_L^2y^2-a^2}\Big)-\frac{T_2}{2}\ln\frac{\beta_L^2y^2+(x+a)^2}{\beta_L^2y^2+(x-a)^2}\Big]\\ &\quad-4\beta_L\beta_T\Big[T_1\ln\Big|\frac{x+a-\beta_T y}{x-a-\beta_T y}\Big|+2T_2F(x-\beta_T y)\Big],\\ \frac{\pi\tau_{xy}}{p_0} &= -\beta_L(2-M_T^2)\Big[T_1\ln\frac{\beta_L^2y^2+(x+a)^2}{\beta_L^2y^2+(x-a)^2}+2T_2\tan^{-1}\Big(\frac{2a\beta_L y}{x^2+\beta_L^2y^2-a^2}\Big)\\ &\quad-2T_1\ln\Big|\frac{x+a-\beta_T y}{x-a-\beta_T y}\Big|-4T_2F(x-\beta_T y)\Big],\\ F(x-\beta_T y) &= \begin{cases} 0 & \text{for}\quad 0<x-\beta_T y<a\\ \dfrac{\pi}{4} & \text{for}\quad x-\beta_T y=a\\ \dfrac{\pi}{2} & \text{for}\quad a<x-\beta_T y<\infty. \end{cases} \end{aligned}\right\} \tag{C·6}$$

c). Supersonic case.

Eqs. (3·33) with Eqs. (C·1) provide stresses as follows.

$$\left.\begin{aligned} \frac{\sigma_x}{p_0} &= (M_T^2-2M_L^2+2)S_1\{H(x+a-\gamma_L y)-H(x-a-\gamma_L y)\}\\ &\quad+2\gamma_T S_2\{H(x+a-\gamma_T y)-H(x-a-\gamma_T y)\},\\ \frac{\sigma_y}{p_0} &= -(2-M_T^2)S_1\{H(x+a-\gamma_L y)-H(x-a-\gamma_L y)\}\\ &\quad-2\gamma_T S_2\{H(x+a-\gamma_T y)-H(x-a-\gamma_T y)\},\\ \frac{\tau_{xy}}{p_0} &= -2\gamma_L S_1[\{H(x+a-\gamma_L y)-H(x-a-\gamma_L y)\\ &\quad-\{H(x+a-\gamma_T y)-H(x-a-\gamma_T y)\}],\\ S_1 &= \frac{2-M_T^2}{(2-M_T^2)^2+4\gamma_L\gamma_T},\\ S_2 &= \frac{2\gamma_L}{(2-M_T^2)^2+4\gamma_L\gamma_T}. \end{aligned}\right\} \tag{C·7}$$

D). A distributed horizontal load of constant intensity.

Although the superposition of the results of the example B) may provide the solution of this example, the direct calculated results from the general solutions are shown here.

The origin of the moving co-ordinates (x, y) in Fig. 1 is shifted a to the right for the sake of convenience so that the load may be expressed

$$p(x) = 0,\ q(x) = q_0\{H(x+a) - H(x-a)\}, \tag{D·1}$$

where q_0 is the intensity of the load.
$\bar{p}(\xi)$ and $\bar{q}(\xi)$ are expressed as

$$\bar{p}(\xi) = 0,\ \bar{q}(\xi) = \frac{2q_0}{\sqrt{2\pi}} \cdot \frac{\sin(a\xi)}{\xi}. \tag{D·2}$$

a). Subsonic case.

Eqs. (3.8) with Eqs. (D·2) yield

$$\left.\begin{aligned} A_1(\xi) &= \frac{2iq_0K_3}{\sqrt{2\pi}\,\mu} \cdot \frac{|\xi|}{\xi^4} \sin(a\xi), \\ A_2(\xi) &= \frac{-2q_0K_1}{\sqrt{2\pi}\,\mu} \cdot \frac{\sin(a\xi)}{\xi^3}, \\ K_1 &= \frac{2-M_T^2}{(2-M_T^2)^2 - 4\alpha_L\alpha_T}, \\ K_3 &= \frac{2\alpha_T}{(2-M_T^2)^2 - 4\alpha_L\alpha_T}. \end{aligned}\right\} \tag{D·3}$$

Eqs. (3·11) with Eqs. (D·3) furnish

$$\left.\begin{aligned} \frac{\pi\sigma_x}{q_0} &= -\frac{K_3}{2}(M_T^2 - 2M_L^2 + 2)\ln\frac{\alpha_L^2y^2 + (x+a)^2}{\alpha_L^2y^2 + (x-a)^2} \\ &\quad + \alpha_TK_1 \ln\frac{\alpha_T^2y^2 + (x+a)^2}{\alpha_T^2y^2 + (x-a)^2}, \\ \frac{\pi\sigma_y}{q_0} &= \frac{K_3}{2}(2-M_T^2)\ln\frac{\alpha_L^2y^2 + (x+a)^2}{\alpha_L^2y^2 + (x-a)^2} \\ &\quad - \alpha_TK_1 \ln\frac{\alpha_T^2y^2 + (x+a)^2}{\alpha_T^2y^2 + (x-a)^2}, \\ \frac{\pi\tau_{xy}}{q_0} &= -2\alpha_LK_3 \tan^{-1}\frac{2a\alpha_Ly}{x^2 - a^2 + \alpha_L^2y^2} \\ &\quad + (2-M_T^2)K_1 \tan^{-1}\frac{2a\alpha_Ty}{x^2 - a^2 + \alpha_T^2y^2}. \end{aligned}\right\} \tag{D·4}$$

b). Transsonic case.

Eq. (3·18) with Eqs. (D·2) provide

$$\left.\begin{aligned} B_1(\xi) &= \frac{4\beta_Tq_0}{\sqrt{2\pi}\,\mu}\left[T_1\frac{1}{\xi^3} - iT_2\frac{|\xi|}{\xi^4}\right]\sin(a\xi), \\ T_1 &= \frac{(2-M_T^2)^2}{(2-M_T^2)^4 + 16\beta_L^2\beta_T^2}, \\ T_1 &= \frac{4\beta_L\beta_T}{(2-M_T^2)^4 + 16\beta_L^2\beta_T^2}. \end{aligned}\right\} \tag{D·5}$$

Substituting Eqs. (D·5) into Eqs. (3·23), we obtain stresses

$$\left.\begin{aligned}
\frac{\pi\sigma_x}{q_0} &= -(M_T^2-2M_L^2+2)\left[2T_1\tan^{-1}\frac{2a\beta_L y}{x^2-a^2+\beta_L^2 y^2}-T_2\ln\frac{\beta_L^2 y^2+(x+a)^2}{\beta_L^2 y^2+(x-a)^2}\right]\\
&\quad +2\beta_T(2-M_T^2)\left[2T_1F(x-\beta_T y)-T_2\ln\left|\frac{x+a-\beta_T y}{x-a-\beta_T y}\right|\right],\\
\frac{\pi\sigma_y}{q_0} &= \beta_T(2-M_T^2)\left[2T_1\tan^{-1}\frac{2a\beta_L y}{x^2-a^2+\beta_L^2 y^2}-T_2\ln\frac{\beta_L^2 y^2+(x-a)^2}{\beta_L^2 y^2+(x-a)^2}\right]\\
&\quad -2\beta_T(2-M_T^2)\left[2T_1F(x-\beta_T y)-T_2\ln\left|\frac{x+a-\beta_T y}{x-a-\beta_T y}\right|\right],\\
\frac{\pi\tau_{xy}}{q_0} &= \beta_L\beta_T\left[T_1\ln\frac{\beta_L^2 y^2+(x+a)^2}{\beta_L^2 y^2+(x-a)^2}+2T_2\tan^{-1}\frac{2a\beta_L y}{x^2-a^2+\beta_L^2 y^2}\right]\\
&\quad +(2-M_T^2)^2\left[2T_1F(x-\beta_T y)-T_2\ln\left|\frac{x+a-\beta_T y}{x-a-\beta_T y}\right|\right].
\end{aligned}\right\} \qquad (D\cdot 6)$$

c). Supersonic case.

Eqs. (3·33) with Eqs. (D·1) yield

$$\left.\begin{aligned}
\frac{\sigma_x}{q_0} &= -(M_T^2-2M_L^2+2)S_3\{H(x+a-\gamma_L y)-H(x-a-\gamma_L y)\}\\
&\quad +2\gamma_T S_1\{H(x+a-\gamma_T y)-H(x-a-\gamma_T)\},\\
\frac{\sigma_y}{q_0} &= 2\gamma_T S_1[\{H(x+a-\gamma_L y)-H(x-a-\gamma_L y)\}\\
&\quad -\{H(x+a-\gamma_T y)-H(x-a-\gamma_T y)\}],\\
\frac{\tau_{xy}}{q_0} &= 2\gamma_L S_3\{H(x+a-\gamma_L y)-H(x-a-\gamma_L y)\}\\
&\quad +(2-M_T^2)S_1\{H(x+a-\gamma_T y)-H(x-a-\gamma_T y)\}],\\
S_1 &= \frac{2-M_T^2}{(2-M_T^2)^2+4\gamma_L\gamma_T},\quad S_3=\frac{2\gamma_T}{(2-M_T^2)^2+4\gamma_L\gamma_T}.
\end{aligned}\right\} \qquad (D\cdot 7)$$

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