Sub-Optimal Relay Control Systems for Non-Linear Plants with Chattering-Less Switching Line

By

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It is well-known that the time-optimum or the minimum integrated-square-error control system for a second order linear plant is realized as a relay control system with a suitable switching line.

In the present paper, first, it will be shown that such a relay system suffers so-called chattering phenomenon if there exists a soft-saturation type non-linearity on the velocity, however small it is. Second, it will be shown that the sub-optimal system with chattering-less switching line is easily obtained by introducing the method of instantaneous linearization. Two examples are given to demonstrate the effectiveness of the present method.

List of Principal Symbols

$f(x_2)$: non-linear feedback gain in the controlled system

$F_{s.l.}(x_1, x_2)$ and $F_{i.l.}(x_1, x_2)$: switching functions derived by the method of simple linearization and of instantaneous linearization, respectively

J : integrated-square-error type performance index

k and $k(t)$: linearized feedback gains by the method of simple linearization and of instantaneous linearization, respectively

t : real time

u : control signal

u^* : optimum control policy

$u^*_{s.l.}$ and $u^*_{i.l.}$: approximately optimum control policies obtained by the method of simple linearization and of instantaneous linearization, respectively

U : magnitude constraint on u

$x_i(\sigma)$: i-th state variable

$\left(\frac{dx_2}{dx_1}\right)_{s.l.}$ and $\left(\frac{dx_2}{dx_1}\right)_{i.l.}$: tangents of the switching line derived by the method of simple linearization and of instantaneous linearization, respectively

$\left(\frac{dx_2}{dx_1}\right)_t$: tangent of the trajectory on the state plane

σ : dummy time variable

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1. Introduction

In the past decade, interest in problems of system optimization and of optimum control has been awakened. This interest has created an increasing need for methods and techniques for the design of optimum systems. It is, however, very difficult to obtain analytically closed-form mathematical expressions of the optimum control policy for non-linear controlled systems except in very special and simple cases. On the other hand, notably in linear stationary control problems, the solution can be extremely simple. We should notice that approximate linearization methods play a very important role on constructing the quasi-optimum system, and that a variety of linearization methods has been developed each with a restricted range of application[1].

The next problem is that many troubles occur in the system constructed quasi-optimally by various linearization methods because the quasi-optimum system is essentially not optimum. One of these troubles is the chattering phenomenon, or the highest frequency vibration of relay, which is different from the limit cycle phenomena.

In this paper, it will be shown that, in constructing a quasi-optimum relay control system for the non-linear plant which has soft-saturation property on the velocity, the occurrence of chattering phenomenon is inevitable if we use the quasi-optimum control policy based on the method of simple linearization.

We must, then, develop a linearization method to overcome these troubles, holding onto its simplicity. In order to circumvent the chattering phenomenon, we introduce the method of instantaneous linearization[2),3)] which was pioneered by J.D. Pearson, and it is shown that this method provides us a useful approach in constructing the chattering-less sub-optimal system.

2. Second Order Controlled Plant with Soft-Saturation Type Non-linearity on the Velocity

In this section, we specify the controlled system to study and its properties.

A simplified model of the plant with a non-linear feedback element is shown in Fig. 1, where x_1, x_2 and u denote position, velocity and control signal, respectively. The equations governing the motion of this model are written as

$$\frac{dx_1}{d\sigma} = x_2 \,, \tag{2.1$_1$}$$

$$\frac{dx_2}{d\sigma} = -f(x_2) + u \,, \tag{2.1$_2$}$$

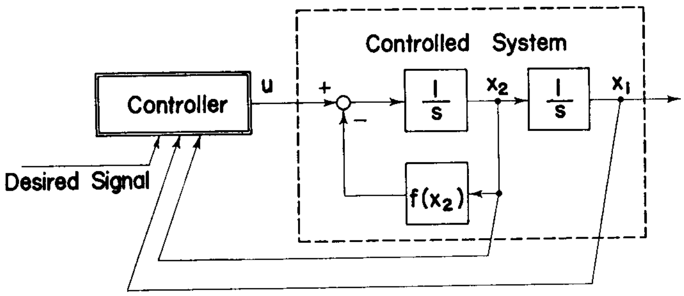

Fig. 1. Block diagram of the control system.

with initial conditions

$$x_1(0) = x_1^0 , \qquad (2.2)_1$$

$$x_2(0) = x_2^0 . \qquad (2.2)_2$$

Next, we assume that the non-linear function $f(x_2)$ in Eq. $(2.1)_2$ has the following properties;

(I) $f(x_2)$ is a one-valued first differentiable function and satisfies

$$f(0) = 0 , \qquad (2.3)_1$$

$$f(x_2)\, x_2 \geq 0 . \qquad (2.3)_2$$

(II) For all x_2, the inequality

$$\{f'(x_2)\, x_2 - f(x_2)\}\, x_2 \geq 0 \qquad (2.4)$$

is satisfied. $\left(" \, ' \, " \text{ denotes } \frac{d}{dx_2} .\right)$

The intuitive representation for the property (II) is given as follows;

For all x_2, the graph $x_2 \sim f(x_2)$ exists between the x_2-axis and the line which connects the origin with the point $(x_2, f(x_2))$. (as shown in Fig. 2)

The system shows soft-saturation on the velocity if $f(x_2)$ satisfies the above assumptions. As an example, let us assume that $f(x_2) = x_2 + cx_2^3$ $(c>0)$. Fig. 3 illustrates the trajectories on the state plane where $x_1^0 = 1$, $x_2^0 = 0$, $u = -1$ and c is the parameter. We may see that the term cx_2^3 causes the system to have soft-saturation property on the velocity.

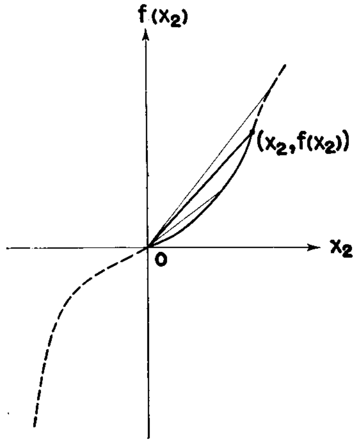

Fig. 2. Illustration of the property (II) of $f(x_2)$.

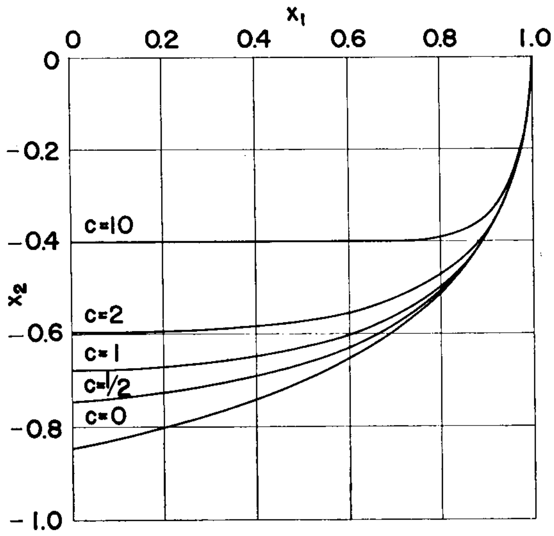

Fig. 3. Trajectories for $f(x_2)=x_2+cx_2^3$.

3. The Method of Simple Linearization and the Occurrence of Chattering Phenomenon

We consider the time-optimum control problem under the constraint

$$|u| \leq U \tag{3.1}$$

for the controlled system assumed in the previous section. Unfortunately, it is very difficult to determine the optimum policy in analytical form because of its non-linearity. Approximation method is, therefore, of importance to overcome the difficulty. The first and the most simple approximation method is as follows; Eq. (2.1) is rewritten as such a linear system that

$$\frac{dx_1}{d\sigma} = x_2\,, \tag{3.2$_1$}$$

$$\frac{dx_2}{d\sigma} = -kx_2+u\,, \tag{3.2$_2$}$$

where

$$k = \left.\frac{df(x_2)}{dx_2}\right|_{x_2=0} \tag{3.3}$$

in the vicinity of the origin $(x_1, x_2)=(0, 0)$. In the neighborhood of the origin, Eqs. (3.2) and (3.3) can be expected to approximate sufficiently Eq. (2.1). The optimum control signal $u^*_{s.l.}$ for this simply linearized system is already well-known and the result shows that $u^*_{s.l.}$ is reduced to the bang-bang control. Then, the switching line is given by[4)]

$$x_1 = -\frac{x_2}{k} + \frac{U}{k^2}\frac{x_2}{|x_2|}\ln\left(1+\frac{k}{U}|x_2|\right). \tag{3.4}$$

If we define the switching function $F_{s.l.}(x_1, x_2)$ to be

$$F_{s.l.}(x_1, x_2) = x_1 + \frac{x_2}{k} - \frac{U}{k^2}\frac{x_2}{|x_2|}\ln\left(1+\frac{k}{U}|x_2|\right), \tag{3.5}$$

then, $u^*_{s.l.}$ is given by

$$u^*_{s.l.} = -U\, sgn\, F_{s.l.}(x_1, x_2). \tag{3.6}$$

Although it is expected that the original non-linear system behaves well if the control policy of Eq. (3.6) is utilized, in the vicinity of the origin, the system suffers one of the most undesirable phenomena in relay systems —chattering— however small the non-linearity and the initial deviations are. The proof is shown as below only in the region $x_2<0$ because the consideration in the region $x_2>0$ reaches at the same results.

The tangent of the switching line on the state plane for $x_2<0$ is given by

$$\begin{aligned}\left(\frac{dx_2}{dx_1}\right)_{s.l.} &= \frac{-kx_2+U}{x_2} \\ &= \frac{-f'(0)x_2+U}{x_2},\end{aligned} \tag{3.7}$$

and the tangent of the trajectory with $u=U$ for $x_2<0$ is given from Eq. (2.1) by

$$\left(\frac{dx_2}{dx_1}\right)_t = \frac{-f(x)+U}{x_2}. \tag{3.8}$$

Now, the property (I) of $f(x_2)$ assumed in section 2 yields

$$f'(0) \geq 0,$$

and for all $x_2<0$

$$f(x_2) \leq 0.$$

Using these relations and $U>0$, we obtain that

$$\left(\frac{dx_2}{dx_1}\right)_t \leq 0 \quad \text{and} \quad \left(\frac{dx_2}{dx_1}\right)_{s.l.} \leq 0 \tag{3.9}$$

from the definitions (3.7) and (3.8). Moreover, since the property (II) of $f(x_2)$ yields

$$f(x)-f'(0)x_2 < 0 \tag{3.10}$$

for all $x_2<0$, we can perform the following derivation

$$\left|\left(\frac{dx_2}{dx_1}\right)_{s.l.}\right| - \left|\left(\frac{dx_2}{dx_1}\right)_t\right| = \left(\frac{dx_2}{dx_1}\right)_t - \left(\frac{dx_2}{dx_1}\right)_{s.l.}$$

$$= -\frac{f(x_2)-f'(0)x_2}{x_2} < 0\,, \tag{3.11}$$

i.e.,

$$\left|\left(\frac{dx_2}{dx_1}\right)_{s.l.}\right| < \left|\left(\frac{dx_2}{dx_1}\right)_t\right|. \tag{3.12}$$

Ineqs. (3.9) and (3.12) indicate that the so-called endpoint phenomena[5] may occur on the switching line given by Eq. (3.4). In reality, these endpoint phenomena give rise to the chattering in relay due to the existing small time-delay in the switching element. (as shown in Fig. 4)

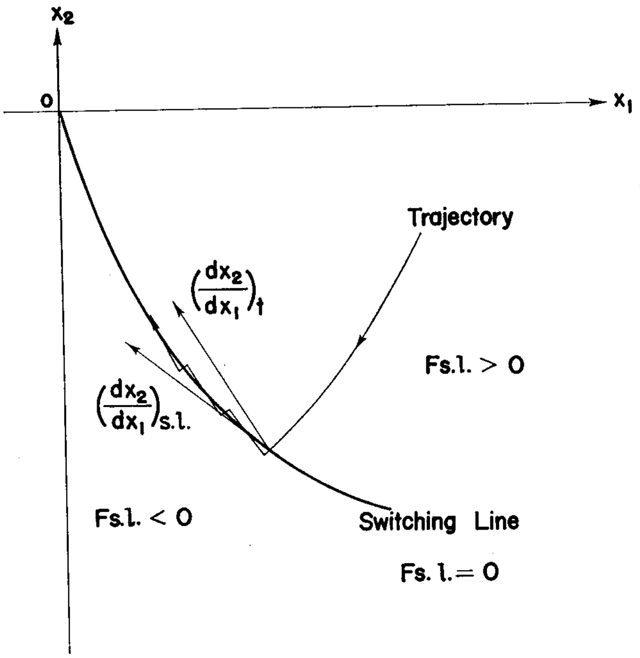

Fig. 4. Sketch of chattering phenomenon.

It is seen that the approximation method based on the simple linearization of the non-linearity is of no use in our situation. Then, it is desirable to develop another approximation method which does not give rise to the chattering phenomenon.

4. The Method of Instantaneous Linearization and the Extinction of the Chattering Phenomenon

Eq. (3.11) indicates that the chattering phenomenon rises from approximating the feedback gain $f(x_2)$ smaller than its real value. Now, let us introduce the method of instantaneous linearization which constructs the sub-optimal system. (From now on, we will use this term without validity.) From the properties of $f(x_2)$, we may easily see that this method is such as to approximate the feedback gain larger than its real value.

Since the instantaneously linearized system is given by

$$\frac{dx_1}{d\sigma} = x_2 , \tag{4.1)$_1$}$$

$$\frac{dx_2}{d\sigma} = -k(t)x_2+u , \tag{4.2)$_2$}$$

where $k(t)$ represents the instantaneously linearized gain written as

$$k(t) = \left.\frac{f(x_2)}{x_2}\right|_{x_2=x_2(t)} , \tag{4.2}$$

the sub-optimal control policy $u^*_{i.l.}$ is obtained to be

$$u^*_{i.l.} = -U\, sgn\, F_{i.l.}\,(x_1, x_2) , \tag{4.3}$$

where the sub-optimal switching function $F_{i.l.}(x_1, x_2)$ is derived as

$$F_{i.l.}(x_1, x_2) = x_1+\frac{x_2^2}{f(x_2)}-\frac{Ux_2^2}{f^2(x_2)}\frac{x_2}{|x_2|}\ln\left(1+\frac{f(x_2)}{Ux_2}|x_2|\right) \tag{4.4}$$

by substituting Eq. (4.2) into Eq. (3.5). (The general derivation is given in Appendix.) The reciprocal of the tangent of this sub-optimal switching line is

$$\left(\frac{dx_1}{dx_2}\right)_{i.l.} = \frac{x_2}{U-f(x_2)}\left[2\left\{\frac{\ln(1-\hat{f})}{\hat{f}}+1\right\}\left(\frac{1}{\hat{f}}-1\right)\left(\frac{\hat{f}'}{\hat{f}}x_2-1\right)+\frac{\hat{f}'}{\hat{f}}x_2\right], \tag{4.5}$$

where

$$\hat{f} = \frac{f(x)}{U} \quad \text{and} \quad \hat{f}' = \frac{1}{U}\frac{d}{dx_2}f(x_2) . \tag{4.6}$$

From now on, we consider only the region $x_2<0$ for simplicity. (It is easily seen how to derive the same results also in the region $x_2>0$.) Since the tangent of the trajectory given by Eq. (3.8) is negative for all $x_2<0$, two chattering-less cases (a) and (b) may be considered as illustrated in Fig. 5. The case (a) is represented

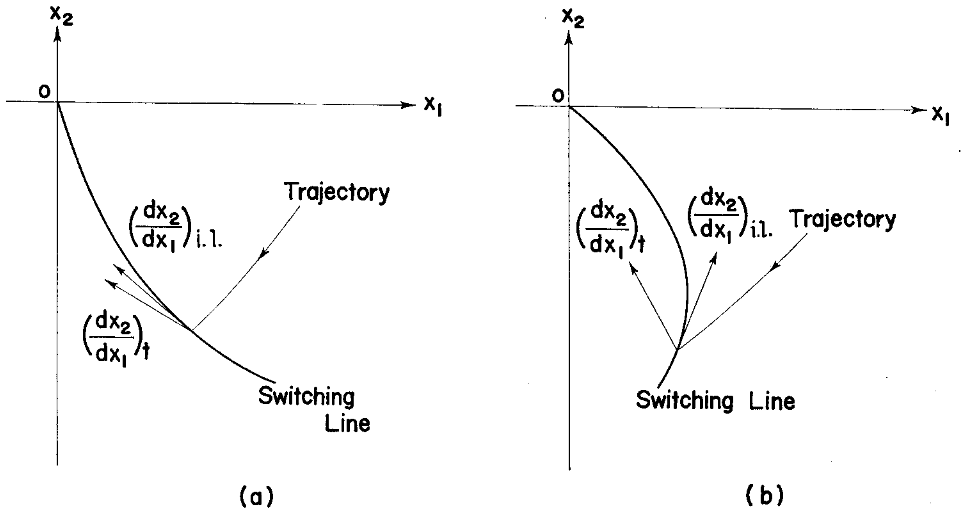

Fig. 5. Sketches of two chattering-less cases.

by the notations both

$$\left(\frac{dx_2}{dx_1}\right)_{i.l.} \leq 0 \quad \text{and} \quad \left(\frac{dx_2}{dx_1}\right)_t \geq \left(\frac{dx_2}{dx_1}\right)_{i.l.}, \tag{4.7}$$

and the case (b) is

$$\left(\frac{dx_2}{dx_1}\right)_{i.l.} < 0 . \tag{4.8}$$

We obtain for the case (a)

$$-1 \leq \left(\frac{\hat{f}'}{\hat{f}} x_2 - 1\right)\left[2\left\{\frac{\ln(1-\hat{f})}{\hat{f}} + 1\right\}\left(\frac{1}{\hat{f}} - 1\right) + 1\right] \leq 0 \tag{4.9}$$

by substituting Eqs. (3.8) and (4.5) into Eq. (4.7), and for the case (b)

$$\left(\frac{\hat{f}'}{\hat{f}} x_2 - 1\right)\left[2\left\{\frac{\ln(1-\hat{f})}{\hat{f}} + 1\right\}\left(\frac{1}{\hat{f}} - 1\right) + 1\right] < -1 \tag{4.10}$$

by substituting Eq. (4.5) into Eq. (4.8). Hence the chattering-less condition is obtained to be

$$(\hat{f} - \hat{f}'x_2)[2\{\ln(1-\hat{f}) + \hat{f}\}(1-\hat{f}) + \hat{f}^2] \leq 0 \tag{4.11}$$

from Eqs. (4.9) and (4.10).

Meanwhile, the properties (I) and (II) of $f(x_2)$ are rewritten for all $x_2 < 0$ as

$$\hat{f} < 0 , \tag{4.12}$$

and

$$\hat{f}-\hat{f}'x_2 \geq 0 \tag{4.13}$$

from the definition of $\hat{f}$, and introducting the notation $P(\hat{f})$ as the contents of [·] in Eq. (4.11), we get

$$P(0) = 0 \quad \text{and} \quad \frac{dP(\hat{f})}{d\hat{f}} = -2\{\ln(1-\hat{f})+\hat{f}\}.$$

Since Eq. (4.12) guarantees $\frac{dP(\hat{f})}{d\hat{f}}$ to be positive, for all $x_2<0$, then we have

$$P(\hat{f}) < 0. \tag{4.14}$$

We have just seen that Eqs. (4.13) and (4.14) bring about the condition (4.11). In other words, the chattering phenomenon never occurs in the sub-optimal system constructed by the method of instantaneous linearization so far as $f(x_2)$ holds its properties (I) and (II).

5. Example-1

In order to illustrate the results obtained in previous sections, the following example is considered. Non-linear function $f(x_2)$ is chosen as

$$f(x_2) = x_2+x_2^3, \tag{5.1}$$

which obviously satisfies the required properties (I) and (II). Then, the controlled system is described by

$$\frac{dx_1}{d\sigma} = x_2, \tag{5.2$_1$}$$

$$\frac{dx_2}{d\sigma} = -(1+x_2^2)x_2+u, \tag{5.2$_2$}$$

corresponding to Eq. (2.1). The control signal $u^*_{s.l.}$ is given by Eq. (3.6) where Eq. (3.5) is rewritten as

$$F_{s.l.}(x_1, x_2) = x_1+x_2-\frac{x_2}{|x_2|}\ln(1+|x_2|), \tag{5.3}$$

and the sub-optimal control signal $u^*_{i.l.}$ is given by Eq. (4.3) where Eq. (4.4) is rewritten as

$$F_{i.l.}(x_1, x_2) = x_1+\frac{x_2}{1+x_2^2}-\frac{1}{(1+x_2^2)^2}\frac{x_2}{|x_2|}\ln\{1+(1+x_2^2)x_2\}. \tag{5.4}$$

In these switching functions, we assume that $U=1$. Fig. 6 which illustrates the magnitude relations among $\left(\frac{dx_2}{dx_1}\right)_{s.l.}$, $\left(\frac{dx_2}{dx_1}\right)_{i.l.}$ and $\left(\frac{dx_2}{dx_1}\right)_t$, guarantees the results

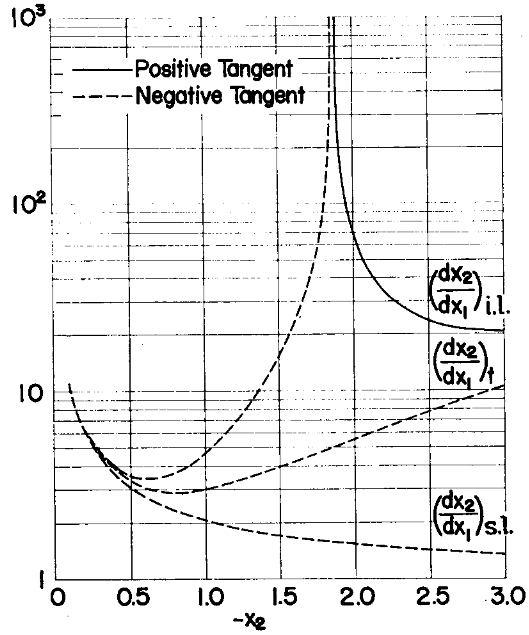

Fig. 6. Tangents of switching line for time-optimum control and of trajectroy.

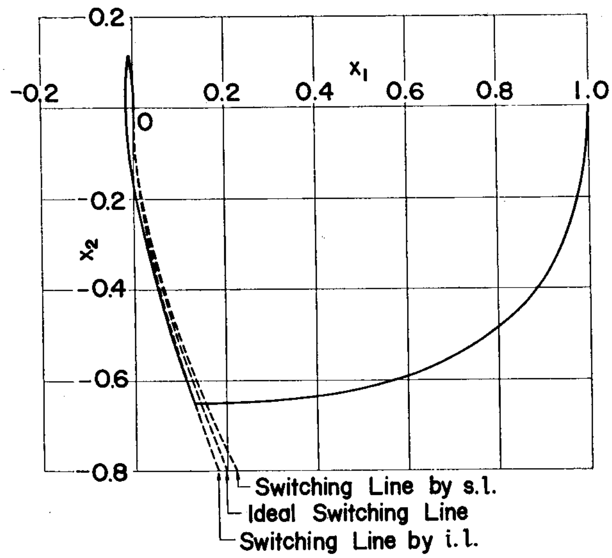

Fig. 7. Trajectory by sub-time-optimal control and switching lines.

represented analytically in sections 3 and 4 for this example. Moreover, Fig. 7 illustrates the trajectory with initial conditions $x_1^0=1$ and $x_2^0=0$ by the sub-optimal control, and three switching lines. The curve of x_1 illustrated in Fig. 8 suggests

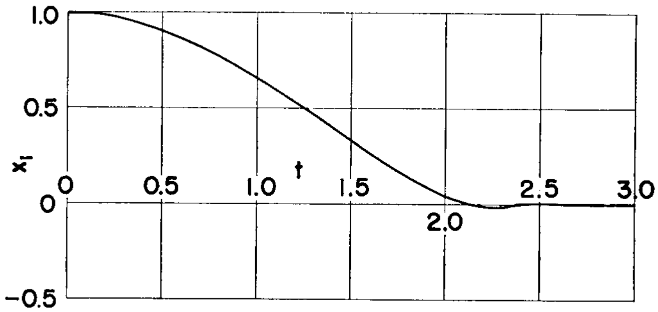

Fig. 8. Response of x_1 by sub-time-optimal control.

that we may adopt the sub-optimal control as the optimum control in the practical sense. In fact, examining the absolute values of the ratio of the change in overshoot to the initial value with respect to the numbers of switching, we obtain such data as 0.76 per cent for once switching and 0.80×10^{-2} per cent for twice switching. Thus, we may expect to settle the overshoot into the predetermined tolerance by once or twice switching.

6. Example-2: Application to the Design of System with Integrated-Square-Error Type Performance Index

The purpose of this section is to illustrate numerically that we may obtain the same consequence in optimization for integrated-square-error type performance index.

Let us assume the performance index to be

$$J=\int_0^\infty x_1^2(\sigma)d\sigma\ , \tag{6.1}$$

and the controlled system to be the same one in section 2, where u should be determined to minimize the value of J. In such a case, the computational solution for the switching function of quasi-optimum control by the method of simple linearization has been obtained[6] as

$$F_{s.l.}(x_1, x_2) = x_1+\frac{0.445}{U}x_2|x_2|\Big(1-0.702\frac{k}{U}|x_2| +0.426\frac{k^2}{U^2}|x_2^2|-0.115\frac{k^3}{U^3}|x_2^3|\Big) \tag{6.2}$$

with parameter k. The substitution of Eq. (4.2) into Eq. (6.2) yields the sub-optimal switching function

$$F_{i.l.}(x_1, x_2) = x_1 + \frac{0.445}{U} x_2 |x_2| \left(1 - 0.701 \frac{f(x_2)}{U} \frac{x_2}{|x_2|} + 0.426 \frac{f^2(x_2)}{U^2} - 0.115 \frac{f^3(x_2)}{U^3} \frac{x_2}{|x_2|}\right). \tag{6.3}$$

By these equations, we cannot analytically show whether the chattering phenomenon occurs or not, and it is not worth doing so because Eqs. (6.2) and (6.3) are approximately obtained. The numerical investigation is, therefore, available.

Now, we assume that non-linear feedback gain $f(x_2)$ is given by Eq. (5.1) and that $U=1$. Fig. 9 illustrates the relations among $\left(\frac{dx_2}{dx_1}\right)_{s.l.}$, $\left(\frac{dx_2}{dx_1}\right)_{i.l.}$ which is derived from the switching function described by Eqs. (6.2) and (6.3), respectively, and $\left(\frac{dx_2}{dx_1}\right)_t$. As shown in Fig. 9, for $-1.9<x_2<-0.6$, the chattering phenomenon does not occur by sub-optimal control while it does by the quasi-optimum control obtained by the method of simple linearization. Fig. 10 illustrates the trajectory on the state plane by the sub-optimal control and, for reference, the switching lines

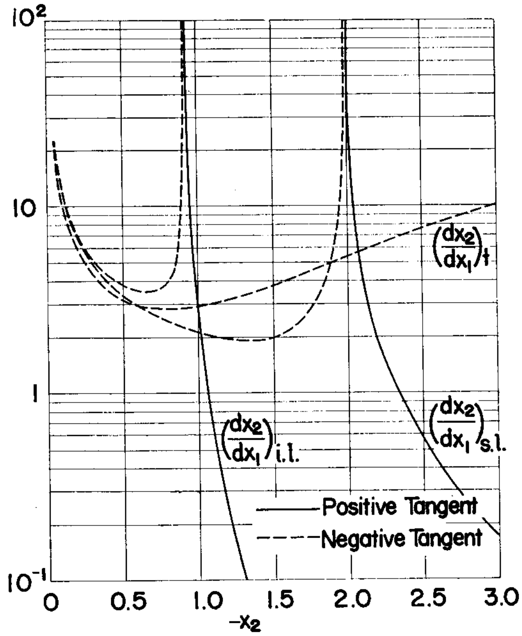

Fig. 9. Tangents of switching lines for integrated-square-error type performance and of trajectory.

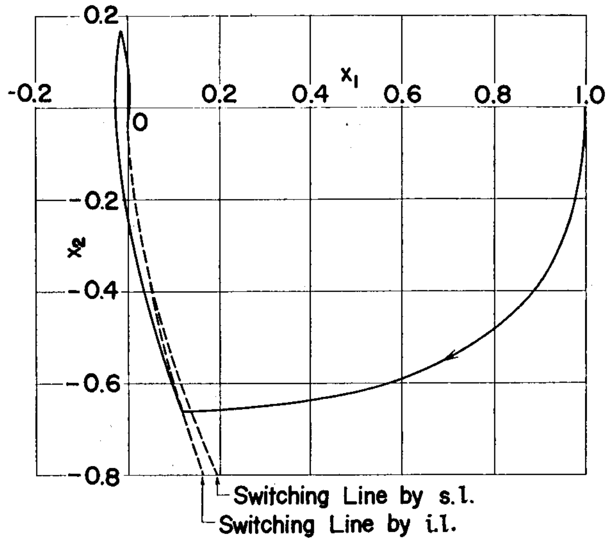

Fig. 10. Trajectory by sub-optimal control for integrated-square-error type performance and switching lines.

obtained by the method of simple linearization and of instantaneous linearization, respectively. Moreover, if we set such initial conditions as $x_1^0<0.5$ and $x_2^0=0$ in order that the chattering phenomenon does not occur by both methods, then we obtain the result that the difference by these methods between the values of $\int_0^T x_1^2(\sigma)d\sigma$ at sufficiently large T is less than 0.02 per cent. We may, therefore, adopt the method of instantaneous linearization in this case from the practical point of view.

7. Conclusions

The utilities of the method of instantaneous linearization have been shown in constructing the chattering-less approximately optimum system which cannot be realized by the method of simple linearization for a specified non-linear controlled system. Unfortunately, we have no analytical guarantee for the optimality of our sub-optimal control policy, but we should notice that in many examples of this method the resulting trajectories do not materially differ from optimum ones for the same criterion. Moreover, we have much interest in its simplicity in determining the control policy. Included in this is that, by allowing for the variations of the controlled system, a realizable adaptive type of controller can be easily devised.

In this paper, we have only considered the specified controlled system, but it may be expected that the method of instantaneous linearization is valid in optimization of other systems with soft-saturation type non-linearity on the velocity or of higher order and further complex systems.

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Appendix: The Method of Instantaneous Linearization

In this appendix, the basic concenpt of the method of instantaneous linearization is given.

Let the dynamic equation of the controlled system be

$$\frac{d\boldsymbol{x}(\sigma)}{d\sigma} = A(\boldsymbol{x}(\sigma))\boldsymbol{x}(\sigma)+B(\boldsymbol{x}(\sigma))\boldsymbol{u}\,, \tag{A.1}$$

where $\boldsymbol{x}(\sigma)$ is a column vector of state variables, $A(\boldsymbol{x}(\sigma))$ and $B(\boldsymbol{x}(\sigma))$ are coefficient matrices, $\boldsymbol{u}$ is a column vector of control variables, and σ is the independent variable (often dummy time). Moreover, let the initial conditions be

$$\boldsymbol{x}(t_0) = \boldsymbol{x}^0\,, \tag{A.2}$$

where $\sigma = t_0$ is the initial time of control interval. The problem of optimum control is to determine the control policy which will minimize or maximize the predetermined performance index, subject to certain physical constraints, for example,

$$\boldsymbol{u} \in \mathcal{U}\,, \tag{A.3}$$

where $\mathcal{U}$ is an admissible control region.

Now, we assume that the optimum control policy can be obtained as $\boldsymbol{u}^*(\boldsymbol{x}(\sigma);\sigma)$. The current optimum control signal is given by

$$\boldsymbol{u}^*(\boldsymbol{x}(t);t) = \boldsymbol{u}^*(\boldsymbol{x}(\sigma);\sigma)\,|_{\sigma=t}\,, \tag{A.4}$$

where t is real time. Since Eq. (A.1), in general, is non-linear, it is very difficult to determine the analytic form of $\boldsymbol{u}^*(\boldsymbol{x}(\sigma);\sigma)$.

To determine the sub-optimal control policy by the method of instantaneous linearization, Eq. (A.1) is rewritten as

$$\frac{d\boldsymbol{x}(\sigma)}{d\sigma} = A(\boldsymbol{x}(t))\boldsymbol{x}(\sigma)+B(\boldsymbol{x}(t))\boldsymbol{u}\,. \tag{A.5}$$

It is relatively easy to obtain the quasi-optimum control policy $\boldsymbol{u}^*(\boldsymbol{x}(\sigma), \boldsymbol{x}(t); \sigma, t)$ for the instantaneously linearized equation (A.5). Therefore, we may regard

$$\boldsymbol{u}^*_{i.l.}(\boldsymbol{x}(t); t) = \boldsymbol{u}^*(\boldsymbol{x}(\sigma), \boldsymbol{x}(t); \sigma, t)|_{\sigma=t}, \quad \text{(A.6)}$$

as the sub-optimal control policy for the original problem. This method is shown to be very effective in the case where the performance index is of quadratic form and where the matrices $A(\boldsymbol{x}(\sigma))$ and $B(\boldsymbol{x}(\sigma))$ have not large variations along the change of $\boldsymbol{x}(\sigma)$[7),8)].

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