Analysis of the Variation of Unconfined Ground-Water Level Due to the Change of Water Level in a Surface Reservoir

By

Shin-ichiro MATSUO* and Iichiro KOHNO*

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In this paper, an analytical method is proposed for calculating the variation of unconfined ground-water level in steady state due to the change of water level in a lake or a river and presents some examples which deal with the variation of ground-water level in the southeast alluvial plain of Lake Biwa.

In most alluvial plains, the compositions of the strata are so complicated that the depth, the permeability and the ground-water flow can not be surveyed continuously. Then even if some pumping tests and some observations of ground-water level are carried out, it is very difficult or almost impossible to comprehend the behaviors and to calculate the variation accurately by means of a normal treatment.

Authors propose to adopt the conception "*quasi-depth*" which means the permeable capacity into this analytical method. Only the "*quasi-depth*" is variable and the other elements can be treated as constant factors in analyzing the variation of ground-water level numerically and graphically. The most advantageous point for using the "*quasi-depth*" in this method is that it is not necessary to measure the depth and the permeability continuously except for the depth at any selected point along a longitudinal section.

1. Introduction

The problems on the water supply, especially on pumping up and the utilization of ground-water, and the troubles on the drawdown or the rise of ground-water level due to puming up or excavation, have been prevalent throughout industrial regions and other places in Japan. But the solution to these subjects has not been found although the radical counterplan is in great demand because it is difficult to comprehend the behaviors of the ground-water in an extended region.

Most of the studies and the investigations on ground-water have been carried out in comparatively limited regions, so that the beneficial result is not good in extended regions. Until now authors have made progress in their studies[1),2),3,4)]. The common aim of their studies is to investigate and to understand the behaviors of the ground-water in an extended region and to set up an analytical method systematically.

* Department of Civil Engineering.

In this report a part of these studies is determined as an analytical method for culculating the variation of ground-water level.

2. Analysis Method

(1) The Fundamental Theory

In general, the hydraulic gradient of ground-water is so small that the flow may be laminar to be represented by Darcy's law. Therefore Eq. (1) for ground-

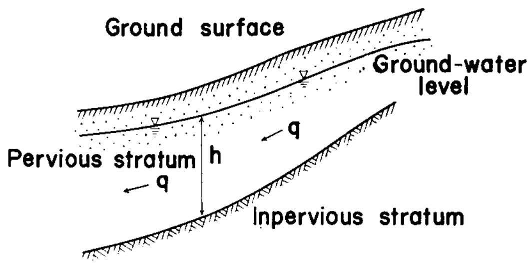

Fig. 1. Ground-water flow.

water flow per unit width is written.

$$q = k \cdot i \cdot h \qquad \cdots\cdots\cdots\cdots(1)$$

where q is the quantity of ground-water flow in the cross section of unit width, k is the coefficient of permeability, i is the hydraulic gradient, h is the depth of pervious aquifer, and the Dupuit-Forchheimer's assumptions hold, for convencience the direction of ground-water flow is made to correspond to the negative one of the hydraulic gradient.

Authors built the hypothesis (2):

$$(q/k) = constant \qquad \cdots\cdots\cdots\cdots(2)$$

This hypothesis is discussed in **4. (1)**.

Combining Eqs. (1) and (2) gives: $i \cdot h = constant$ and i is inversely proportionate to h, in steady flow. So that h at any selected point A in the longitudinal section and i being measured, h can be calculated from the relation: $(q/k)_a = i \cdot h$. The "h" which is arrived at in the above hypothesis (2) is distinguished from the actual depth of aquifer, termed "*quasi-depth*" which is denoted by the simbol H:

$$H = (q/k)_a \cdot \frac{1}{i} \qquad \cdots\cdots\cdots\cdots(3)$$

The relationship between the variation of the quasi-depth H and that of the hydraulic gradient i is written as follows:

$$i \pm \Delta i = (q/k)_a \cdot \frac{1}{H \mp \Delta H} \qquad \cdots\cdots\cdots\cdots(4)$$

It being imposssible to determine ΔH actually, Δh is substituted for ΔH and Eq. (5) is given

$$i \pm \Delta i = (q/k)_a \cdot \frac{1}{H \mp \Delta h} \qquad \cdots\cdots\cdots\cdots(5)$$

The difference between both results from Eqs. (4) and (5) is discussed in **4**. (**1**).

(2) The Analysis Procedure

The procedure of analysis is explained as an example for calculting the drawdown of ground-water level due to that of water level in lake. As for the rise it is dealt with as well as for the drawdown.

i) Selecting a longitudinal section of unit width of which the direction is parallel to that of the ground-water flow, and drawing the profile of the goundwater level before the fall down as in **Fig. 2**, it is convenient to take advantage of the map of ground-water level.

ii) The hydraulic gradient i from the ground-water level on the profile in **Fig. 2** is calculated and the relationship between i and x is plotted, where x is the

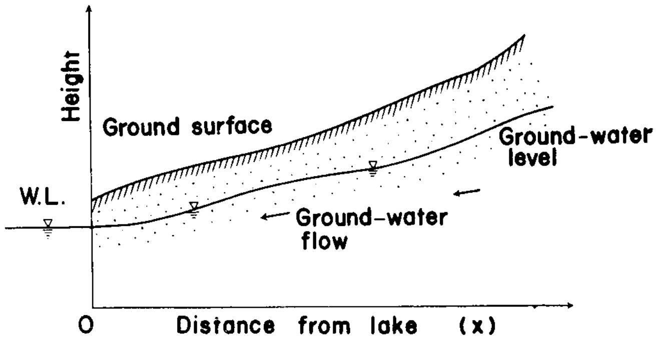

Fig. 2. Procedure i).

distance from lake as in **Fig. 3**.

iii) Calculating the quasi-depth H.

a) Measuring i, h at a selected point A in the longitudinal section and calculating: $(q/k)_a = i_a \cdot h_a$.

b) Substituting $(q/k)_a$ into Eq. (3), getting H.

c) Drawing the profile of the aquifer (drawing the quasi-surface of inpervious aquifer in Fig. 2), as shown in **Fig. 4**.

iv) Analizing the drawdown of ground-water level,

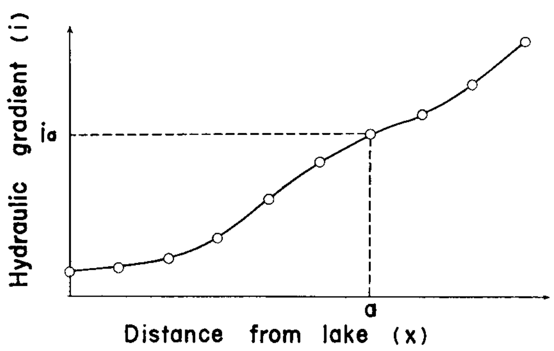

Fig. 3. Procedure ii).

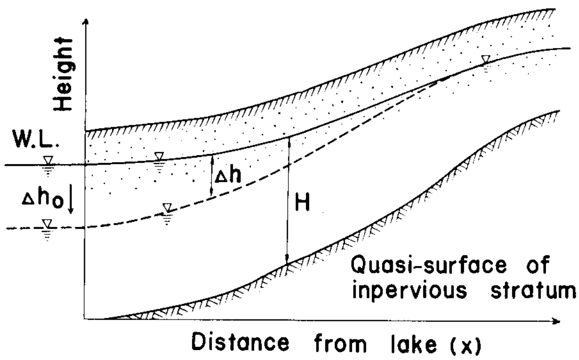

Fig. 4. Procedure iv).

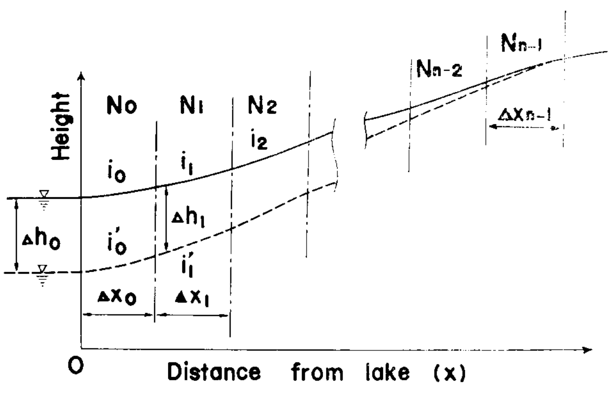

Fig. 5. Procedure v).

a) As shown in **Fig. 5**, the longitudinal section is divided to adequate small ones (N_0, N_1, ... N_{n-1}).
The way of this division is discussed in **2.** (**3**).

b) By using Eq. (6) repeatedly from the origin where h is known, pursuing the phreatic surface of ground-water level which has fallen:

$$i_j' = (q/k)_a \cdot \frac{1}{H_j - \Delta h_j} \qquad \cdots\cdots\cdots\cdots (6)$$

where i_j' is the average of the hydraulic gradient in the section N_j which increases as the ground-water level falls, H_j is the average of the quasi-depth in the section N_j and Δh_j is the amount of the drawdown at the initial point in the section N_j.

For example, when the drawdown Δh_0 is at $x=0$, the influence progresses to upstream and the fallen ground-water level may be shown with the broken line as in **Fig. 4.** The hydraulic gradient in the section N_0 is written: $i_0'=(q/k)_a \cdot 1/(H_0 - \Delta h_0)$, where h_0 is treated as the average of the drawdown in the section N_0. With the gradient i_0', we can draw the profile of ground-water level in N_0 and to get Δh_1 which is the drawdown at the initial point of the section N_1. This similar process goes on toward the upstream to the point where the drawdown is within the allowable error, while it is possible to understand the relationship: $i' \geqq i$ from Eqs. (3) and (6).

(3) The Way of Division

For examining the accumulation of the error which is due to the numerical analysis with Eq. (6), Eq. (7) is used in liew to Eq. (1)

$$q = k(f-G)\frac{df}{dx} \qquad \cdots\cdots (7)$$

where f is the ground-water level, G is the level of the quasi-surface of inpervious aquifer, hence $(f-G)$ is the quasi-depth.

The approximate value of ground-water level which is calculated by the numerical analysis with Eq. (6), is denoted by F.

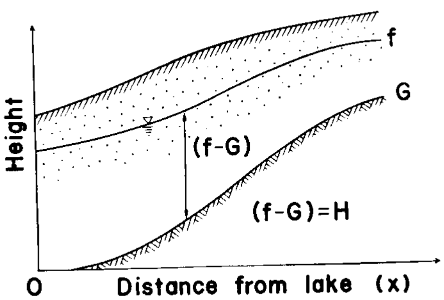

Fig. 6. Examination.

$$F_{j+1} - F_j = (q/k)_a \cdot \frac{\Delta x_j}{F_j - G_j} \qquad \cdots\cdots\cdots\cdots (8)$$

Using Taylor's expansion in Eq. (7) gives:

$$f_{j+1}-f_j = (q/k)_a \cdot \frac{\Delta x_j}{f_j-G_j}+\frac{(\Delta x_j)^2}{2!}\cdot\frac{d^2f}{dx^2}\Big|_j+\frac{(\Delta x_j)^3}{3!}\cdot\frac{d^3f}{dx^3}\Big|_j+\cdots \qquad \cdots\cdots(9)$$

Putting $\varepsilon=F-f$ from Eqs. (8) and (9), gives Eq. (10) which is the equation with respect to the difference between both values of the ground-water level calculated from Eqs. (8) and (9).

$$\varepsilon_{j+1} = \varepsilon_j\left\{1-(q/k)_a\cdot\frac{\Delta x_j}{(f_j-G_j)(F_j-G_j)}\right\}-(\Delta x_j)^2\left\{\frac{1}{2!}\cdot\frac{d^2f}{dx^2}\Big|_j+\frac{\Delta x_j}{3!}\cdot\frac{d^3f}{dx^3}\Big|_j+\cdots\right\} \qquad \cdots\cdots\cdots\cdots(10)$$

Eq. (10) can be written as follows:

$$\varepsilon_{j+1} = \varepsilon_j X_j+(\Delta x_j)^2 Y_j \qquad \cdots\cdots\cdots\cdots(11)$$

where,

$$X_j = 1-(q/k)_a\cdot\frac{\Delta x_j}{(f_j-G_j)(F_j-G_j)}$$

$$Y_j = \frac{1}{2!}\frac{d^2f}{dx^2}\Big|_j+\frac{\Delta x_j}{3!}\cdot\frac{d^3f}{dx^3}\Big|_j+\cdots$$

Assuming that X, $(\Delta x)^2\cdot Y$ are constant:

$$\varepsilon_j = \varepsilon_0 X^j-(\Delta x)^2 Y\cdot\frac{1-X^j}{1-X} \qquad \cdots\cdots\cdots\cdots(12)$$

If $(\Delta x)^2\cdot Y$ is negligibly small and X is selected satisfying the relation:

$$1>X>-1 \qquad \cdots\cdots\cdots\cdots(13)$$

which has the effect that the error at j-phase extinguishes one at $(j+1)$-phase, Eq. (14) is given:

$$|\varepsilon_{j+1}|<|\varepsilon_j| \qquad \cdots\cdots\cdots\cdots(14)$$

Then the accumulation of the error should not occur.

Combining with Eqs. (11) and (13) gives:

$$\Delta x<\frac{2(f-G)^2}{(q/k)_a} \quad \text{or} \quad \frac{2H^2}{(q/k)_a} \qquad \cdots\cdots\cdots\cdots(15)$$

Assuming that i, H are constant, Eq. (15) is simplified:

$$\Delta x<2H/i \qquad \cdots\cdots\cdots\cdots(16)$$

The larger the hydraulic gradient is and the smaller the quasi-depth is, the smaller the divided section should be determined. But the right term in Eq. (16) is generally calculated so large that it is found not to be severely limited. While

$(\Delta x)^2 \cdot Y$ should be small enough to be able to neglect the difference between both values of the ground-water levels which are calculated with the divisions of (Δx) and $(\Delta x/2)$.

For example as shown in **Fig. 7**,

(a) Case of (Δx) division

(b) Case of $(\Delta x/2)$ division

(c) Case of $(\Delta x/4)$ division

The difference between the cases of (a) and (b) being comparatively large, the difference between (b) and (c) is small enough to be neglected. Then $(\Delta x/2)$ division should be selected.

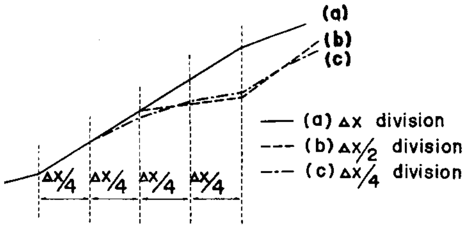

Fig. 7. Division.

3. Example

In **Fig. 8** are shown the map of ground-water level in the alluvial plain in the southeast side of Lake Biwa and two longitudinal sections (I, II) where this analysis is carried out.

Assuming that the water level in Lake Biwa has fallen 3.0 m, then the ground-water level in the plain varies. The analytical result in the section (I) is shown in

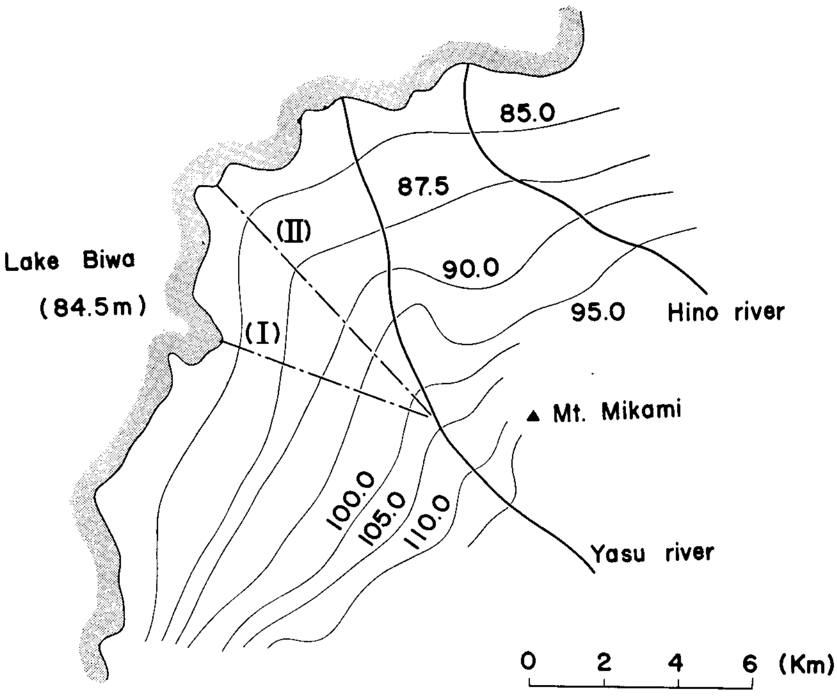

Fig. 8. The map of ground-water level and the longitudinal section (I, II).

Table 1. Analysis table (Section I).

x (m)	i (1/1000)	$(q/k)_a$ (m)	$H=q/k\cdot\frac{1}{i}$ (m)	Δh (m)	$H-\Delta h$ (m)	$i'=q/k\cdot\frac{1}{H-\Delta h}$ (1/1000)	$\Delta h/2$ (m)	$H-\Delta h/2$ (m)	$i'=q/k\cdot\frac{1}{H-\Delta h/2}$ (1/1000)	$2\Delta h$ (m)	$H-2\Delta h$ (m)	$i'=q/k\cdot\frac{1}{H-2\Delta h}$ (1/1000)
0	0.38	$\left(\frac{1000}{4.16}\right)$	109.5	3.0	106.5	0.39	1.5	108.0	0.39	6.0	103.5	0.40
0.25	0.40		104.0	3.0	101.0	0.41	1.5	102.5	0.41	6.0	98.0	0.42
0.50	0.50	↓	83.2	3.0	80.2	0.52	1.5	8.17	0.51	6.0	77.2	0.53
0.75	0.58		71.7	3.0	68.7	0.60	1.5	70.7	0.58	6.0	65.7	0.63
1.00	0.70		59.4	3.0	56.4	0.73	1.5	57.9	0.72	6.0	53.4	0.77
1.25	0.84		49.5	2.8	46.7	0.89	1.5	47.9	0.86	6.0	43.9	0.94
1.50	1.04		40.0	2.7	37.3	1.11	1.4	38.6	1.07	6.0	34.4	1.20
1.75	1.28		32.5	2.7	29.8	1.39	1.4	31.1	1.33	5.6	27.1	1.53
2.00	1.60		26.0	2.5	23.4	1.77	1.4	25.6	1.68	5.6	20.8	2.00
2.25	1.96		21.2	2.5	18.7	2.22	1.4	19.8	2.09	5.4	16.2	2.56
2.50	2.32		17.9	2.5	15.4	2.70	1.4	16.7	2.50	5.2	13.1	3.17
2.75	2.73		15.2	2.4	12.8	3.25	1.4	13.8	3.01	5.0	10.6	3.92
3.00	3.10		13.4	2.4	11.0	3.78	1.4	12.0	3.46	4.8	9.0	4.62
3.25	3.50		11.9	2.3	9.6	4.33	1.4	10.5	3.96	4.6	7.9	5.26
3.50	3.76		11.1	2.0	9.1	4.57	1.3	8.9	4.24	4.4	7.9	5.26
3.75	4.03		10.3	1.8	8.5	4.89	1.3	9.0	4.62	4.0	7.9	5.26
(4.00)	4.16		10.0	1.6	8.4	4.95	1.2	8.8	4.72	3.2	8.1	5.13
4.25	4.12		10.1	1.3	8.8	4.72	1.2	8.9	4.67	2.4	8.7	4.78
4.50	3.90		10.7	1.0	9.7	4.28	1.1	9.6	4.35	1.9	9.5	4.37
4.75	3.58		11.6	1.0	10.6	3.92	1.1	10.5	3.96	1.4	10.5	3.96
5.00	3.28		12.7	0.8	11.9	3.49	0.8	11.9	3.55	1.2	11.9	3.49
5.25	3.00		13.9	0.5	13.4	3.10	0.6	13.3	3.20	1.0	13.7	3.03
5.50	2.76		15.1				0.4	14.7	2.88	0.8		
5.75	3.30		12.6				0.2	12.4	3.43	0.2		
6.00	5.40		7.7									

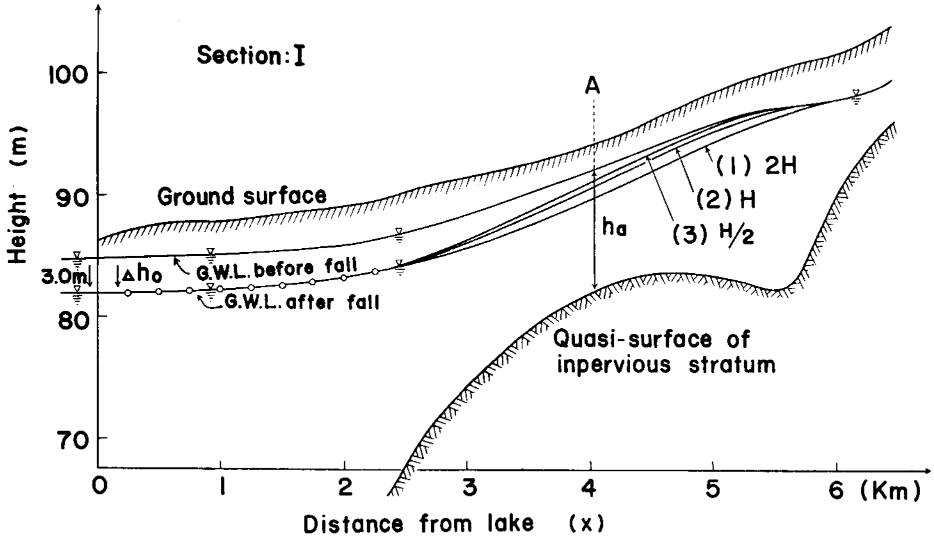

Fig. 9. Analysis of drawdown.

Fig. 9 and Table 1. In this analysis the results are examined by using the different values of the quasi-depth H, $H/2$ and $2H$. The aim of the comparison is to examine the error which is caused by the measured errors of i_a, h_a or $(q/k)_a$. The relationship between x, Δh are plotted in **Fig. 10** for the section (I) and in **Fig. 11** for the section (II), which may be termed drawdown curve. The region of the influence is about 5~6 Km from Lake Biwa for (I) and 6~7 Km for (II).

By this examination it can be seen that the smaller the value of $(q/k)_a$ measurement than the actual one, the smaller the drawdown and the range of influence are. But the error which may occur from the error in measuring i_a, h_a, is comparatively small from the macroscopic view point.

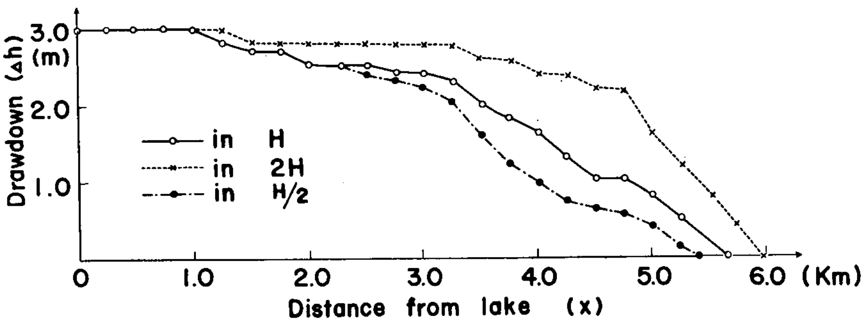

Fig. 10. Drawdown curve (Section I).

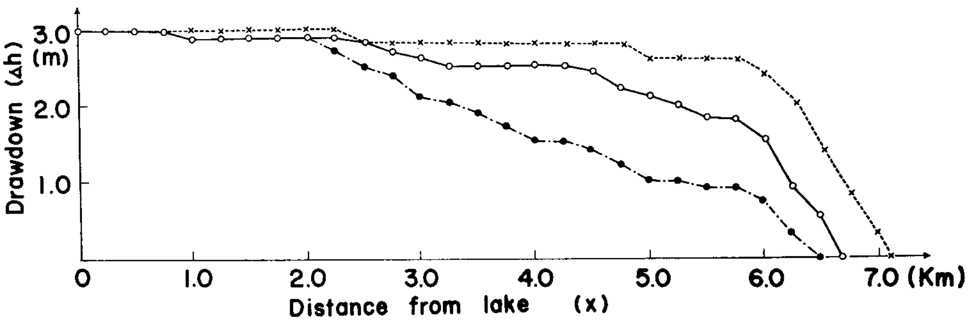

Fig. 11. Drowdown curve (Section II).

4. Discussion

(1) The Discussion of the Hypothesis

This analytical method is based on the hypothesis (2): $(q/k)_a = constant$. But by this examination may be found to be able to adopt in the case; $(q/k) \neq constant$ too, within the variation of (q/k) being not so large.

i) When q, k are constant or the ratio (q/k) is constant, the relationship of Eq. (2) is formed actually and the quasi-depth H is equal to the depth of aquifer h.

ii) In most fields q, k are not constant throughout. However the permeability k seems to become larger in progressing upstream as the grain size of the pervious aquifer becomes larger, while q seems to be larger in upstream than in downstream as the width of the flow may expand downstream. Therefore the variation of the quasi-depth may be similar to that of the actual depth of aquifer.

iii) In general (q/k) is not always constant. Examining and comparing with both results: one is in the case of (q/k) being constant, and the other is in (q/k) being not constant but being a function with respect to x.

(a) (q/k) = const. : the hypothesis (2)

(b) $(q/k) = f(x)$: a function with respect to x.

In the section (I) in **Fig. 8**

(a) $(q/k)_a = \dfrac{4.16}{100}$

(b) $(q/k) = \dfrac{2.08}{100} + \dfrac{2.08}{400}x$ $\quad (x : \text{Km})$

In the section (II)

(a) $(q/k)_a = \dfrac{1.40}{100}$

(b) $(q/k) = \dfrac{0.70}{100} + \dfrac{0.70}{400}x$ $\quad (x : \text{Km})$

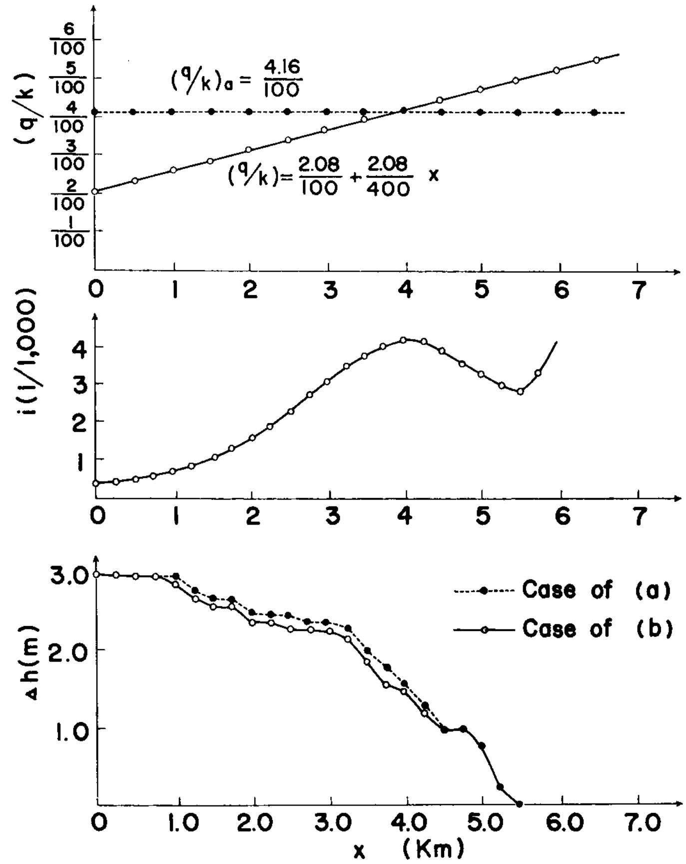

Fig. 12. Examination of the hypothesis (Section I).

In **Fig. 12** and in **Fig. 13** are shown the drawdown curves arrived at in the section (I) and in the section (II) respectively. The full line is for (a) and the broken line is for (b). The difference between both results in (a) and in (b) is negligibly small and the cause of the difference is for using Eq. (5) in lieu to Eq. (4).

It is found from the above examinations that the hypothesis (2) should not limit severely the range of application of this analytical method, while the quasi-depth may be said to be the function in which the variable elements of h, k, and q are included.

(3) The Relationship between the Quasi-depth, the Hydraulic Gradient and the Drawdown of Ground-water Level

For the sake of discussing the mutual relationship, the drawdown of ground-water level is considered in the simplified condition as shown in **Fig. 14**, where H

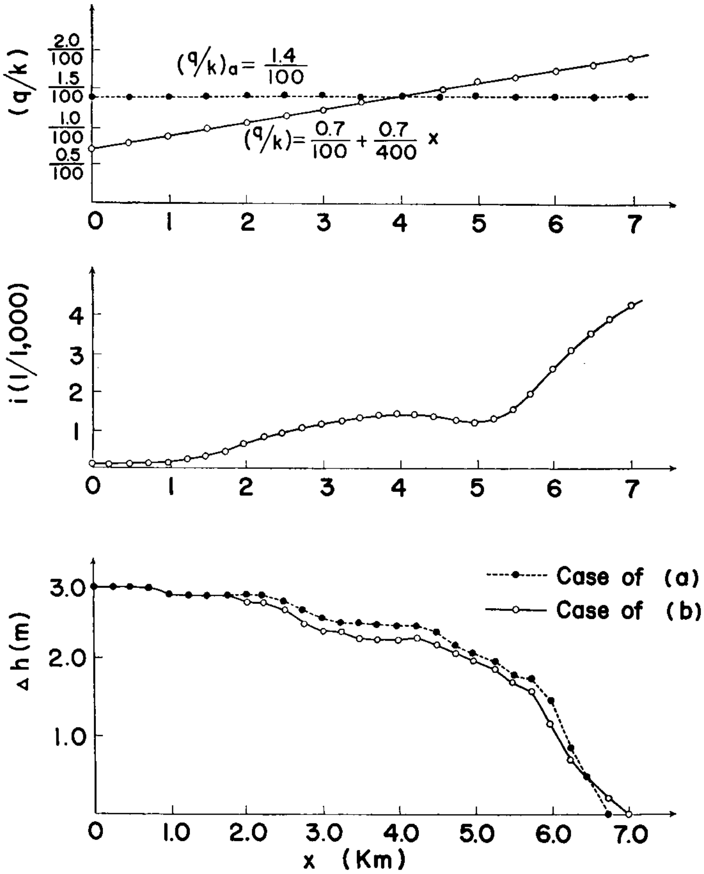

Fig. 13. Examination of the hypothesis (Section II).

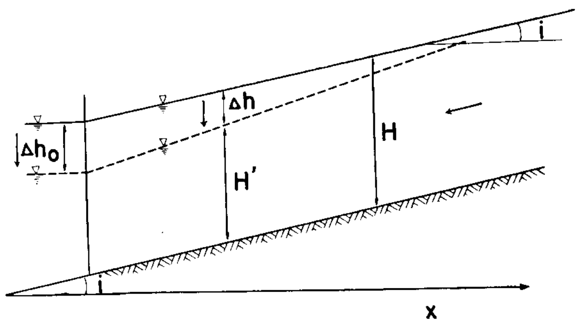

Fig. 14. Model of ground-water flow.

and i are constant before the fall and H' is the quasi-depth after the fall. The relation is given as follows:

$$\left.\begin{aligned} &\Delta h-\Delta h_0+H\ ln\frac{\Delta h_0}{\Delta h}=i\cdot x \\ \text{or} \\ &H_0'-H+H\ ln\frac{H-H_0'}{H-H'}=i\cdot x \end{aligned}\right\} \cdots\cdots\cdots\cdots(17)$$

Fig. 15 indicates the correlation between x and $(\Delta h/\Delta h_0)$ with (h_0/H) as a parameter. In this figure it can be seen that the difference of (h_0/H) being comparatively small, the effect on the drawdown of the ground-water level is small.

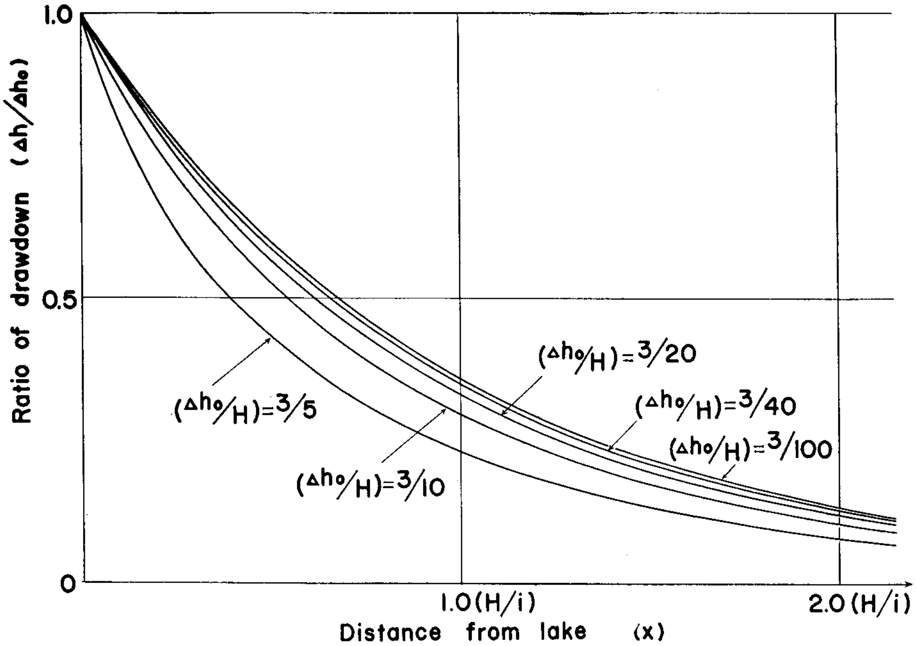

Fig. 15. Drowdown curve for model.

Then Δh becomes nearly α-times as Δh_0 increased α-times: the drawdown Δh_0 for the different Δh_0 can be guessed by the method of multiplication. The larger (h_0/H) increases, the above linear relationship cannot be satisfied. The scale of the distance from lake x being (H/i), the region of the influence becomes α-times as H is α-times or i is one-α'th times, if the other factors are constant. **In Fig. 16** are shown the drawdown curves when $H=20$ m, $i=1/100$, ⓐ $\Delta h_0=3.0$ m and ⓑ $\Delta h_0=1.5$ m.

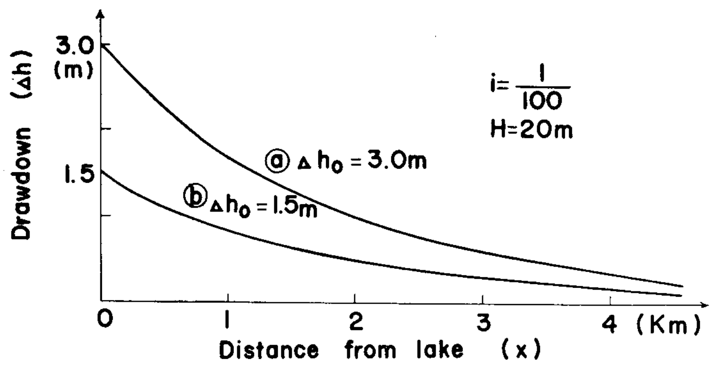

Fig. 16. Drowdown curves.

As above, the larger the quasi-depth of aquifer is and the samller the hydraulic gradient is, the larger the influence of the fall is.

(4) The Relationship between $(q/k)_a$ and the Drawdown

One of the most advatageous merits in this analytical method is to calculate the drawdown and the rise of ground-water level by measuring the depth at any selected point and the hydraulic gradient throughout the longitudinal section. The degree of the error in the analyzed result, which is dependent on the error in measuring $(q/k)_a$ or i_a, h_a, has been examined already in **3**. $(q/k)_a$ being measured larger than the actual one, the drawdown or the rise of the ground-water level and the region of influence are calculated with exaggeration. But the difference is rather small from the macroscopic viewpoint.

Now, being gotten the measured data of the depth of aquifer at two points or at more, (q/k) is possible to be treated as a variable function. But the significance is rather small as having been considered in **3**. It is more desirable that each datum is used independently in the analysis, each result of the drawdown is compared mutually and an adequate profile of the ground-water level after the fall should be determined.

5. Conclusion

In this paper an analytical method is reported for calculating the variation of unconfined ground-water level.

The quasi-depth which is introduced in this method, is defined as: "a conception of the permeable capacity of aquifer, which includes the variable elements of the depth, the permeability of aquifer and the quantity of ground-water flow".

The merits of this method are as follows:

(1) Only the hydraulic gradient throughout the longitudinal section and the depth at one selected point, are necessary to be measured and the other, that is the depth, the permeability, the quantity of the flow, are not necessary but they can be treated as constants.

(2) The depth where the flow does not exsit in the pervious stratum should not be included in the quasi-depth, although it is almost impossible to distinguish the part where the flow exists from that where the flow does not by the boring.

(3) The range of the utilization is not limited by the boundary condition because numerical and drawing techniques are adopted.

(4) The procedure being simple, a large amount of calculation is not necessary.

(5) The accuracy of calculation according to that of the surveyed data can be selected freely.

(6) In most fields it is not necessary to construct new boring for there may be several boring data and the results arrived at with several boring data can be compared with each other, for one result is calculated with a datum.

The remaining problems are as follows:

(7) When the direction and the quantity of the flow are changed by the variation of ground-water level and both factors being unknown, this method can not be used.

(8) This method treats in steady state but not in unsteady state. Then can be arrived at the profile and the drawdown or the rise after a long enough period. From the standpoint that the variation is undesirable, this method is conservative, involving the safety factor because the most dangerous condition is arrived at.

Until now the study and the investigation on the ground-water in extended regions seem to be delayed only by the experience, because the measurment of the permeability and the depth can not be carried out. In this report the quantitative treatment by introducing a conception "quasi-depth" and by using numerical, drawing techniques to the analysis of the variation of ground-water lelvel, is discussed aiming to grasp the behaviors of ground-water in extended regions scientifically and systematically.

References

1) S. Matsuo and I. Kohno: Study on the Ground-water arround Lake Biwa, A report to the Kinki District Office, the Ministry of Construction, pp 91～293 (1962)

2) S. Matsuo and I. Kohno: Study on the Ground-water arround Lake Biwa (II), A report to the Kinki District Office, the Ministry of Construction, pp 46～124 (1962)

3) S. Matsuo and I. Kohno: Study on the Exploitation of Ground-water in Nara Basin, A report to Nara Prefectural Office, pp 46～124 (1964)

4) S. Matsuo and I. Kohno: Study on the Exploitation of Ground-water in Yamato Hill, A report to Nara Prefectural Office, pp 19～133 (1965)
5) S. Matsuo and I. Kohno: Analysis of the Variation of Ground-water Level, Preprint of the Annual Conference of Kansai District of J.S.C.E. pp 73～74 (1962)
6) S. Matsuo and I. Kohno: On the Underground Coffer Dam, Preprint of the 19th Annual Conference of J.S.C.E. pp 24-1～3 (1966)