The Numerical Experiments on the Alternating Direction Implicit Method Using Interlacing Scanning for Elliptic Partial Differential Equation

By

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In this paper, the authors introduce a new variant of alternating direction implicit methods. The alternating direction implicit methods (the ADI methods) are proposed to solve elliptic and parabolic partial differential equations in the paper of Peaceman and Rachford, in 1955. The numerical solution of elliptic partial differential equations by finite difference generally leads to the problem of matrix equation $Ax=K$. The matrix A is split into two matrices H and V in which their nonzero entries appear at the position corresponding row and column directional mesh points, respectively. The ADI methods consist of the alternating computation implicitly about the row and column directional matrix equations.

Our variant ADI method is such a method that each row and column directional computations proceed on every other line interlacingly, in the closed mesh region. We prove the convergence of this variant ADI method (the interlacing ADI method). The average rate of convergence of the interlacing ADI method is approximated almost twice that of the normal ADI method. Numerical experiments on model problems show less iteration times than ones of the normal ADI methods for model problems.

1. Introduction

The alternating direction implicit methods constitute a powerful technique for solving elliptic and parabolic partial differential equations. The computation scheme of basic ADI method consists of row and column directional computations in which each computation make use of Jacobi like iteration matrix. Then each directional computation proceeds line by line using direct elimination method. In this method, the newest approximate values obtained in the horizontal and vertical computation are stored to use in the vertical and horizontal computation, respectively.

We supposed that the convergence speed of the ADI methods may be increased using some variant method in which each directional computation makes use of

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Gauss-seidel like iteration matrix. For this purpose we adopt the double interlacing scheme on row and column directional computations in which at first, eliminations proceed on every other line using the old values, and next, elimination proceeds on every rest line using a part of the newest values.

This variant method proves its convergence whenever row and column operator matrices are real symmetric and positive definite. The average rate of convergence of this interlacing ADI method is approximated as twice that of one of the normal ADI method.

As a numerical experiment, we applied this variant ADI method for some model problem with various regions for Laplace's equation which boundary values are all zero and initial values all unity.

The numerical examples show the relation of required numbers of iterations between normal ADI method and the interlacing ADI method. As convergence parameters, we used Peaceman-Rachford parameters, Wachspress parameters and optimum parameter which are given in the reference 1).

2. Matrix Problem

The numerical solution of elliptic differential equations by finite difference leads to the problem of solving large systems of algebraic equations. For simplicity, we consider the partial differential equation of the unknown function $u(x, y)$,

$$-u_{xx}(x,y)-u_{yy}(x,y)+\sigma u(x,y) = S(x,y)\ , \qquad (x,y)\in R \tag{2.1}$$

in the unit square R: $0<x, y<1$ of the $x-y$ plane, where $\sigma\geq 0$. And $S(x, y)$ is known function in R. For boundary conditions, we assume that

$$u(x,y) = \gamma(x,y)\ , \qquad (x,y)\in\Gamma\ , \tag{2.2}$$

where $\gamma(x,y)$ is a prescribed function on the boundary Γ of R. If a uniform mesh of length h in each coordinate direction is imposed on R, then the partial differential equation (2.1) is approximated, using central differences, by

$$(\boldsymbol{H}+\boldsymbol{V}+\boldsymbol{\Sigma})\boldsymbol{u} = \boldsymbol{K}\ , \tag{2.3}$$

where, the three matrices $\boldsymbol{H}$, $\boldsymbol{V}$ and $\boldsymbol{\Sigma}$ arise respectively, as central difference approximations to the first three terms of the differential equation (2.1).

For the spatial mesh point (x,y), $[\boldsymbol{Hu}]$ (x_0, y_0), $[\boldsymbol{Vu}]$ (x_0,y_0) and $[\boldsymbol{\Sigma u}]$ (x_0,y_0) denote the following as the components of the vector $\boldsymbol{Hu}$, $\boldsymbol{Vu}$ and $\boldsymbol{\Sigma u}$,

$$\left.\begin{aligned}[\boldsymbol{Hu}](x_0,y_0) &\equiv -u(x_0-h,y_0)+2u(x_0,y_0)-u(x_0+h,y_0)\ ,\\ [\boldsymbol{Vu}](x_0,y_0) &\equiv -u(x_0,y_0-h)+2u(x_0,y_0)-u(x_0,y_0+h)\ ,\\ [\boldsymbol{\Sigma u}](x_0,y_0) &\equiv \sigma h^2 u(x_0,y_0)\ .\end{aligned}\right\} \tag{2.4}$$

the following properties of matrices $\boldsymbol{H}$, $\boldsymbol{V}$ and $\boldsymbol{\Sigma}$ are readily verified. They are all real symmetric $n \times n$ matrices. The matrix $\boldsymbol{\Sigma}$ is a non-negative diagonal matrix, and is thus nonnegative definite. The matrices $\boldsymbol{H}$ and $\boldsymbol{V}$ each have no more than three nonzero entries per row, and both $\boldsymbol{H}$ and $\boldsymbol{V}$ are diagonally dominant matrices with positive diagonal entries and nonpositive off-diagonal entries, and thus *stieltjes* matrices.

These properties of matrices $\boldsymbol{H}$, $\boldsymbol{V}$ and $\boldsymbol{\Sigma}$ hold for the approximation of more general type of elliptic differential equations. Then the solution of the matrix equation (2.3) is unique[2)].

3. Basic ADI Operators

The equation (2.3) is clearly equivalent, for any matrices $\boldsymbol{D}$ and $\boldsymbol{E}$, to each of the two vector equations

$$(\boldsymbol{H}+\boldsymbol{\Sigma}+\boldsymbol{D})\boldsymbol{u} = \boldsymbol{K}-(\boldsymbol{V}-\boldsymbol{D})\boldsymbol{u}\,, \tag{3.1}$$

$$(\boldsymbol{V}+\boldsymbol{\Sigma}+\boldsymbol{E})\boldsymbol{u} = \boldsymbol{K}-(\boldsymbol{H}-\boldsymbol{E})\boldsymbol{u}\,, \tag{3.2}$$

provided $(\boldsymbol{H}+\boldsymbol{\Sigma}+\boldsymbol{D})$ and $(\boldsymbol{V}+\boldsymbol{\Sigma}+\boldsymbol{E})$ are nonsingular. This was first observed by Peaceman and Rachford[3)] for the case $\boldsymbol{\Sigma}=0$, $\boldsymbol{D}=\boldsymbol{E}=\rho\boldsymbol{I}$ a scalar matrix. In this case, (3.1) and (3.2) reduce to

$$(\boldsymbol{H}+\rho\boldsymbol{I})\boldsymbol{u} = \boldsymbol{K}-(\boldsymbol{V}-\rho\boldsymbol{I})\boldsymbol{u}\,, \quad (\boldsymbol{V}+\rho\boldsymbol{I})\boldsymbol{u} = \boldsymbol{K}-(\boldsymbol{H}-\rho\boldsymbol{I})\boldsymbol{u}\,.$$

The generalization to $\boldsymbol{\Sigma} \neq 0$ and arbitrary $\boldsymbol{D}=\boldsymbol{E}$ was made by Wachspress and Habetler[4)].

For the case $\boldsymbol{\Sigma}=0$, $\boldsymbol{D}=\boldsymbol{E}=\rho\boldsymbol{I}$ which they considered, Peaceman and Rachford proposed solving (2.3) by choosing an appropriate sequence of positive numbers ρ_n, and calculating the sequence of vector $\boldsymbol{u}_n$, $\boldsymbol{u}_{n+1/2}$ defined from the sequence of matrices $\boldsymbol{D}_n=\boldsymbol{E}_n=\rho_n\boldsymbol{I}$, by the formulas

$$(\boldsymbol{H}+\boldsymbol{\Sigma}+\boldsymbol{D}_n)\boldsymbol{u}_{n+1/2} = \boldsymbol{K}-(\boldsymbol{V}-\boldsymbol{D}_n)\boldsymbol{u}_n\,, \tag{3.3}$$

$$(\boldsymbol{V}+\boldsymbol{\Sigma}+\boldsymbol{E}_n)\boldsymbol{u}_{n+1} = \boldsymbol{K}-(\boldsymbol{H}-\boldsymbol{E}_n)\boldsymbol{u}_{n+1/2}\,. \tag{3.4}$$

Provided the matrices which have to be inverted are similer to positive definite (hence nonsinguler) well-conditioned tridiagonal matrices under permutation matrices, each of equations (3.3) and (3.4) can be rapidly solved by Gaussian elimination. The aim is to choose the initial trial vector $\boldsymbol{u}_0$ and the matrices $\boldsymbol{D}_1$, $\boldsymbol{E}_1$, $\boldsymbol{D}_2$, $\boldsymbol{E}_2\cdots$ so as to make the sequence $\{\boldsymbol{u}_n\}$ converge rapidly.

Peaceman and Rachford considered the iteration of (3.3) and (3.4) when $\boldsymbol{D}_n$ and $\boldsymbol{E}_n$ are given by $\boldsymbol{D}_n=\rho_n\boldsymbol{I}$ and $\boldsymbol{E}_n=\tilde{\rho}_n\boldsymbol{I}$.

This defines the Paeceman-Rachford method:

$$\boldsymbol{u}_{n+1/2} = (\boldsymbol{H}+\boldsymbol{\Sigma}+\rho_n\boldsymbol{I})^{-1}[\boldsymbol{K}-(\boldsymbol{V}-\rho_n\boldsymbol{I})\boldsymbol{u}_n]\,, \tag{3.5}$$

$$\boldsymbol{u}_{n+1} = (\boldsymbol{V}+\boldsymbol{\Sigma}+\rho_n\boldsymbol{I})^{-1}[\boldsymbol{K}-(\boldsymbol{H}-\tilde{\rho}_n\boldsymbol{I})\boldsymbol{u}_{n+11/2}]\,. \tag{3.6}$$

The rate of convergence will strongly depend on the choice of the iteration parameters ρ_n, $\tilde{\rho}_n$.

An interesting variant of the Peaceman-Rachford method was suggested by Douglas and Rachford, again for the case $\boldsymbol{\Sigma}=0$. It can be defined for general $\boldsymbol{\Sigma}\geqq 0$ by

$$\boldsymbol{u}_{n+1/2} = (\boldsymbol{H}_1+\rho_n\boldsymbol{I})^{-1}[\boldsymbol{K}-(\boldsymbol{V}_1-\rho_n\boldsymbol{I})\boldsymbol{u}_n]\,, \tag{3.7}$$

$$\boldsymbol{u}_{n+1} = (\boldsymbol{V}_1+\rho_n\boldsymbol{I})^{-1}[\boldsymbol{V}_1\boldsymbol{u}_n-\rho_n\boldsymbol{u}_{n+1/2}]\,, \tag{3.8}$$

where $\boldsymbol{H}_1$ and $\boldsymbol{V}_1$ are defined as $\boldsymbol{H}-(1/2)\boldsymbol{\Sigma}$ and $\boldsymbol{V}-(1/2)\boldsymbol{\Sigma}$, respectively. This amounts to setting $\boldsymbol{D}_n=\boldsymbol{E}_n=\rho_n\boldsymbol{I}-(1/2)\boldsymbol{\Sigma}$ in (3.3) and (3.4), and making some elementary manipulations. Hence (3.7) and (3.8) are also equivalent to (2.7), if $\boldsymbol{u}_n=\boldsymbol{u}_{n+1/2}=\boldsymbol{u}_{n+1}$.

4. Interlacing Scanning ADI Operators

The basic ADI method proposed solving (2.3) by choosing an appropriate sequence of positive numbers ρ_n for the case $\boldsymbol{D}=\boldsymbol{E}=\rho_n\boldsymbol{I}$. Therefore, we do not use the newest value of $\boldsymbol{u}_{n+1/2}$ or $\boldsymbol{u}_{n+1}$, when solving equations (3.3) and (3.4) by Gaussian elimination. We supposed that the convergence rate of such a variant method will be able to increase, in which we use a part of the newest values of $\boldsymbol{u}_{n+1/2}$ and $\boldsymbol{u}_{n+1}$. So we now propose such a variant method in which a line Gaussian ellimination will proceed line by line, after that, by another line by line to solve the equation (3.3) and (3.4), respectively.

This variant method defines the next scheme, that we call the interlacing ADI method.

$$(\boldsymbol{H}_1+\rho\boldsymbol{L})\boldsymbol{u}_{n+1/2} = \boldsymbol{K}-(\boldsymbol{V}_1-\rho\boldsymbol{L})\boldsymbol{u}_n\,, \tag{4.1}$$

$$(\boldsymbol{V}_1+\rho\boldsymbol{W})\boldsymbol{u}_{n+1} = \boldsymbol{K}-(\boldsymbol{H}_1-\rho\boldsymbol{W})\boldsymbol{u}_{n+1/2}\,, \tag{4.2}$$

where

$$\boldsymbol{D}=\rho\boldsymbol{L}\,,\qquad \boldsymbol{E}=\rho\boldsymbol{W}\,, \tag{4.3}$$

$$\boldsymbol{L}\boldsymbol{u} = \begin{pmatrix}\boldsymbol{I} & 0\\ \boldsymbol{C} & \boldsymbol{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\end{pmatrix},$$

$$\boldsymbol{C} \equiv \begin{pmatrix} -(1/2)\boldsymbol{I} & -(1/2)\boldsymbol{I} & 0\cdots\cdots 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & 0\\ 0\cdots\cdots 0 & & -(1/2)\boldsymbol{I} & -(1/2)\boldsymbol{I}\end{pmatrix}. \tag{4.4}$$

The matrices $\boldsymbol{H}_1$ and $\boldsymbol{V}_1$ are defined by $\boldsymbol{H}_1=\boldsymbol{H}+(1/2)\boldsymbol{\Sigma}$ and $\boldsymbol{V}_1=\boldsymbol{V}+(1/2)\boldsymbol{\Sigma}$, respectively. In the equation (4.4), $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ express the vectors consisting of odd row and even row, respectively.

The matrix $\boldsymbol{W}$ is reduced to the same form as matrix $\boldsymbol{L}$ when matrix $\boldsymbol{V}_1$ is reduced to the same form as matrix $\boldsymbol{H}_1$ by permutation of changing indices.

In the same way as basic ADI method, we propose to solve (2.3), choosing an appropriate sequence of positive numbers ρ_n, and calculating the sequence of vector $\boldsymbol{u}_n$ and $\boldsymbol{u}_{n+1/2}$ defined from the sequencese of matrices $\rho_n\boldsymbol{L}$ and $\rho_n\boldsymbol{W}$, by the formulas

$$(\boldsymbol{H}_1+\rho_n\boldsymbol{L})\boldsymbol{u}_{n+1/2} = \boldsymbol{K}-(\boldsymbol{V}_1-\rho_n\boldsymbol{L})\boldsymbol{u}_n\,, \tag{4.5}$$

$$(\boldsymbol{V}_1+\rho_n\boldsymbol{W})\boldsymbol{u}_{n+1} = \boldsymbol{K}-(\boldsymbol{H}_1-\rho_n\boldsymbol{W})\boldsymbol{u}_{n+1/2}\,. \tag{4.6}$$

5. Convergence of Interlacing ADI Method

We combine the two equations (4.5) and (4.6) in the form

$$\boldsymbol{u}_{n+1} = \boldsymbol{T}_{\rho_n}\boldsymbol{u}_n+\boldsymbol{g}_{\rho_n}(\boldsymbol{K})\,, \qquad n\geqq 1\,, \tag{5.1}$$

where

$$\boldsymbol{T}_{\rho_n} \equiv (\boldsymbol{V}_1+\rho_n\boldsymbol{W})^{-1}(\rho_n\boldsymbol{W}-\boldsymbol{H}_1)(\boldsymbol{H}_1+\rho_n\boldsymbol{L})^{-1}(\rho_n\boldsymbol{L}-\boldsymbol{V}_1) \tag{5.2}$$

and

$$\boldsymbol{g}_{\rho_n}(\boldsymbol{K}) \equiv (\boldsymbol{V}_1+\rho_n\boldsymbol{W})^{-1}\{(\rho_n\boldsymbol{W}-\boldsymbol{H}_1)(\boldsymbol{H}_1+\rho_n\boldsymbol{L})^{-1}+\boldsymbol{I}\}\boldsymbol{K}\,. \tag{5.3}$$

If $\varepsilon^{(n)}=\boldsymbol{u}^{(n)}-\boldsymbol{u}$ is the error vector associated with vector iterate $\boldsymbol{u}^{(n)}$ then $\varepsilon^{(n+1)}=\boldsymbol{T}_{\rho_{n+1}}\varepsilon^{(n)}$, and in general

$$\varepsilon^{(n)} = (\prod_{j=1}^{n}\boldsymbol{T}_{\rho_j})\varepsilon^{(0)}, \qquad n\geqq 1\,, \tag{5.4}$$

where

$$\prod_{j=1}^{n}\boldsymbol{T}_{\rho_j} = \boldsymbol{T}_{\rho_n}\cdot\boldsymbol{T}_{\rho_{n-1}}\cdots\cdots\boldsymbol{T}_1\,.$$

Now we call the matrix T the error reduction matrix. To indicate the convergence property of $(\prod_{j=1}^{n}\boldsymbol{T}_{\rho_j})$, we first consider the simple case in which all the constants ρ_j are equal to the fixed constant $\rho>0$. For fixed $\boldsymbol{L}$, $\boldsymbol{W}$, the interlacing ADI method have the form equation (5.1). This method is a stationary iterative method in the terminology of Forysthe and Wasow[5].

For such methods, it is well known that the asymptotic rate of convergence is determined by spectral radius $\Lambda(\boldsymbol{T}_\rho)$ of the associated error reduction matrix $\boldsymbol{T}$. A stationary iterative method is convergent if and only if its spectral radius is less than unity.

In applying the convergence criterion $\Lambda(\boldsymbol{T}_\rho)<1$ to the interlacing ADI method, it is convenient to use the following well known result[6].

Lemma 5.1. For the norm $||\boldsymbol{x}||=(\boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x})^{1/2}$, $\boldsymbol{Q}$ any real positive definite matrix, if, for a fixed real matrix $\boldsymbol{M}$, $||\boldsymbol{Mx}||\leq r||\boldsymbol{x}||$ for all real $\boldsymbol{x}$, then $\Lambda(\boldsymbol{M})\leqq r$. Next we will prove antoher lemma, which is similer to the algebraic content of a theorem of Wachspress and Habetler[3].

Lemma 5.2. Let $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{S}$ be positive definite real matrices, with $\boldsymbol{S}$ symmetric. If the norms of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are equal, then $\boldsymbol{Y}=(\boldsymbol{Q}-\boldsymbol{S})(\boldsymbol{P}+\boldsymbol{S})^{-1}$ is norm reducing for real $\boldsymbol{x}$ relative to the norm $||\boldsymbol{x}||=(\boldsymbol{x}\boldsymbol{S}^{-1}\boldsymbol{x}')^{1/2}$.

Proof. By definition, the condition of norm-reducing for the matrix $\boldsymbol{Y}$ is expressed,

$$||\boldsymbol{Y}|| = \sup_{x \neq 0} \frac{||\boldsymbol{Qx}||}{||\boldsymbol{x}||} < 1 , \tag{5.5}$$

then,

$$||(\boldsymbol{Q}-\boldsymbol{S})(\boldsymbol{P}+\boldsymbol{S})^{-1}\boldsymbol{x}|| > ||\boldsymbol{x}|| . \tag{5.6}$$

This is equivalent to the next equation for every nonzoro vector $\boldsymbol{y}=(\boldsymbol{P}+\boldsymbol{S})^{-1}\boldsymbol{x}$,

$$||(\boldsymbol{Q}-\boldsymbol{S})\boldsymbol{y}|| < ||(\boldsymbol{P}+\boldsymbol{S})\boldsymbol{y}|| . \tag{5.7}$$

Inturn, this is equivalent to the special Euclidian norm $||\boldsymbol{x}||=(\boldsymbol{x}\boldsymbol{S}^{-1}\boldsymbol{x}')^{1/2}$ to the next equation for all nonzero $\boldsymbol{y}$.

$$[(\boldsymbol{Q}-\boldsymbol{S})\boldsymbol{y}]'\boldsymbol{S}^{-1}[(\boldsymbol{Q}-\boldsymbol{S})\boldsymbol{y}] < [(\boldsymbol{P}+\boldsymbol{S})\boldsymbol{y}]'\boldsymbol{S}^{-1}[(\boldsymbol{P}+\boldsymbol{S})\boldsymbol{y}] . \tag{5.8}$$

Expanding the bilinear terms and canceling, this is equivalent to the following condition,

$$\boldsymbol{y}'(\boldsymbol{P}'\boldsymbol{S}^{-1}\boldsymbol{P}-\boldsymbol{Q}'\boldsymbol{S}^{-1}\boldsymbol{Q})\boldsymbol{y}+\boldsymbol{y}'(\boldsymbol{P}+\boldsymbol{P}'+\boldsymbol{Q}+\boldsymbol{Q}')\boldsymbol{y} > 0 . \tag{5.9}$$

The first term of this equation is reduced to next form,

$$\boldsymbol{y}'(\boldsymbol{P}'\boldsymbol{S}^{-1}\boldsymbol{P}-\boldsymbol{Q}'\boldsymbol{S}^{-1}\boldsymbol{Q})\boldsymbol{y}= \{(||\boldsymbol{Py}||/||\boldsymbol{y}||)^2-(||\boldsymbol{Qy}||/||\boldsymbol{y}||)^2\} ||\boldsymbol{y}||^2 . \tag{5.10}$$

From this equation, it is obvious that the first term of the equation (5.9) vanishes if and only if the norms of the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are equal. This condition is satisfied by the hypothesis.

Then, the condition (5.9) reduces that $\boldsymbol{y}'(\boldsymbol{P}+\boldsymbol{P}'+\boldsymbol{Q}+\boldsymbol{Q}')\boldsymbol{y}>0$ for all nonzero $\boldsymbol{y}$. But this is the hypothesis that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are positive definite.

Theorem 5.1. Any stationary interlacing scanning ADI process (4.1) and (4.2) is convergent, provided $\boldsymbol{H}_1$ and $\boldsymbol{V}_1$ are symmetric and positive definite, and $\boldsymbol{L}$ and $\boldsymbol{W}$ are positive definite and the norms of $\boldsymbol{L}$ and $\boldsymbol{W}$ are equal.

Now we consider a similar matrix $\tilde{T}_\rho$ with T_ρ.

$$\tilde{T}_\rho = (\rho W - H_1)(\rho L + H_1)^{-1}(\rho_2 L - V_1)(\rho W + V_1)^{-1}. \tag{5.11}$$

Then we obtain nest inequality by Theorem 5.1.

$$\begin{aligned} \Lambda(T_\rho) &= \Lambda(\tilde{T}_\rho) \leqq \|\tilde{T}_\rho\| \\ &\leq \|(\rho W - H_1)(\rho L + H_1)^{-1}\| \, \|(\rho L - V_1)(\rho W + V_1)^{-1}\| < 1 . \end{aligned} \tag{5.12}$$

For the case of acceleration sequence parameters ρ_n, we obtain next inequality.

$$\prod_{i=1}^{n} [\Lambda(T_{\rho_i})] = \prod_{i=1}^{n} [\Lambda(\tilde{T}_{\rho_i})] \leqq \prod_{i=1}^{n} \|\tilde{T}_{\rho_i}\| < 1 . \tag{5.13}$$

Then the scheme (4.5) and (4.6) is convergent.

6. Approximate Average Rate of Convergence

For any $n\times n$ complex matrix, the average rate of convergence for m iterations of the matrix A is defined

$$R(A^m) \equiv -\bar{l}_n[\|A^m\|^{1/m}] = -\bar{l}_n\|A^m\|/m . \tag{6.1}$$

Here, we survey the relation between the norms of ADI error reduction matrix and interlacing scanning ADI error reduction matrix.

On the interlacing scanning ADI method, the average rate of convergence is determined by $\|T_\rho\|$. Generally it is hard to obtain the eigenvalue of $\|T_\rho\|$. Therefore we obtained the eigenvalue of T_ρ where $\rho=2$. Thereafter we compared this value with the eigenvalue of error reduction matrix of ADI method.

Considering first row directional case, we put vector

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{6.2}$$

where u_1 and u_2 are the vectors associate with odd row and even row, respectively. Then row directional equation (4.1) is expressed in the following form

$$\begin{aligned} &\left\{ \begin{pmatrix} H_{1,1} & 0 \\ 0 & H_{2,2} \end{pmatrix} + \begin{pmatrix} \rho I & 0 \\ -\rho V_{2,1}/2 & \rho I \end{pmatrix} \right\} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^{n+1/2} \\ &\qquad = \left\{ \begin{pmatrix} \rho I & 0 \\ -\rho V_{2,1}/2 & \rho I \end{pmatrix} - \begin{pmatrix} V_{1,1} & -V_{1,2} \\ -V_{2,1} & V_{2,2} \end{pmatrix} \right\} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^{n} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \end{aligned} \tag{6.3}$$

where $H_{1,1}$ and $H_{2,2}$, $V_{1,1}$, $V_{2,1}$ and $V_{1,2}$ and $V_{2,2}$, K_1 and K_2 are the matrices and vectors associated with odd row and even row, respectively. This equation is reduced to the next equation.

$$\boldsymbol{u}^{n+1/2} = -(\boldsymbol{D}-\rho\boldsymbol{E}/2)^{-1}\{(2-\rho)\boldsymbol{I}-(1-\rho/2)\boldsymbol{E}-\boldsymbol{E}^*\}\boldsymbol{u}^n+(\boldsymbol{D}-\rho\boldsymbol{E}/2)^{-1}\boldsymbol{K} \quad (6.4)$$

where

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{H}_{1,1}+\rho\boldsymbol{I} & 0 \\ 0 & \boldsymbol{H}_{2,2}+\rho\boldsymbol{I} \end{pmatrix}, \quad \boldsymbol{E} = \begin{pmatrix} 0 & 0 \\ \boldsymbol{V}_{2,1} & 0 \end{pmatrix}, \quad \boldsymbol{E}^* = \begin{pmatrix} \boldsymbol{V}_{1,2} & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.5)$$
$$\begin{pmatrix} \boldsymbol{V}_{1,1}-\rho\boldsymbol{I} & 0 \\ 0 & \boldsymbol{V}_{2,2}-\rho\boldsymbol{I} \end{pmatrix} = (2-\rho)\boldsymbol{I}.$$

Furthermore, we put $\boldsymbol{D}^{-1}\boldsymbol{E}=L$, $\boldsymbol{D}^{-1}\boldsymbol{E}^*=U$. Then equation (6.5) becomes

$$\boldsymbol{u}^{n+1/2} = -(\boldsymbol{I}-\rho L/2)^{-1}\{(2-\rho)\boldsymbol{D}^{-1}-(1-\rho/2)L-U\}\boldsymbol{u}^n+(\boldsymbol{D}-\rho\boldsymbol{E}/2)^{-1}\boldsymbol{K}. \quad (6.6)$$

The error matrix of interlacing scanning ADI method $\boldsymbol{H}_I$ is that

$$\boldsymbol{H}_I = (\boldsymbol{I}-\rho L/2)^{-1}\{(\rho-2)\boldsymbol{D}^{-1}-(1-\rho/2)L-U\}. \quad (6.7)$$

The error matrix of ADI method $\boldsymbol{H}_A$ is that

$$\boldsymbol{H}_A = \{(\rho-2)\boldsymbol{D}^{-1}-L-U\}. \quad (6.8)$$

Now, we express the eigenvalues of interlacing scanning ADI method and ADI method to be λ_H and σ_H, respectively. The λ_H is determined, solving the equation det $(\lambda\boldsymbol{I}-\boldsymbol{H}_I)=0$. Since L is strictly lower trianguler matrix, then $(\boldsymbol{I}-\rho L/2)$ is nonsingular and det $(\boldsymbol{I}-\rho L/2)=1$. Then, λ_H is determined by next equation.

$$\begin{aligned} 0 = \phi(\lambda_H) &= \det(\boldsymbol{I}-\rho L/2)\det(\lambda_H\boldsymbol{I}-\boldsymbol{H}_I) \\ &= \lambda_H\boldsymbol{I}-(\rho-2)\boldsymbol{D}^{-1}+(1-\rho/2-\lambda_H\rho/2)L+U. \end{aligned} \quad (6.9)$$

For the ADI method, the eigenvalue σ_H is determined by solving that

$$\det\{\sigma_H\boldsymbol{I}-(\rho-2)\boldsymbol{D}+L+U\} = 0. \quad (6.10)$$

Putting $\rho=2$, we obtain the following relation.

$$\lambda_H = \sigma_H^2. \quad (6.11)$$

For vertical iteration, we will consider the same as above. The eigenvalues of vertical iteration matrix of the interlacing ADI and normal ADI method are expressed λ_V and σ_V, respectively.

Then,

$$\lambda_V = \sigma_V^2. \quad (6.12)$$

The interlacing ADI method by (4.1) and (4.2) with $\rho=2$ is equivalent to the block Gauss-sediel iteration. And Peaceman-Rachford ADI method is equivalent to the block Jacobi iteration. So we approximate the eigenvalue of interlacing

ADI method by $\lambda_V\lambda_H$. Therefore,

$$\lambda_V\lambda_H = (\sigma_V\sigma_H)^2 . \tag{6.13}$$

This relation is valid for spectral radius and norm. Then, the relation between the average rate of convergences of interlacing ADI method ($R_{m\lambda}$) and normal ADI method ($R_{m\sigma}$), will be estimated as follows.

$$R_{m\lambda} \doteqdot 2R_{m\sigma} . \tag{6.14}$$

For the scheme (4.5) and (4.6), we assume this relation will be valid.

7. Numerical Examples

In this section we describe one set of experiments which involved the solution of the Dirichlet problem with Laplace's equation for the regions shown in Fig. 1. These experiment were run using the Kyoto University Computer KDC-II.

For each of the regions shown in Fig. 1, the five point finite difference equation

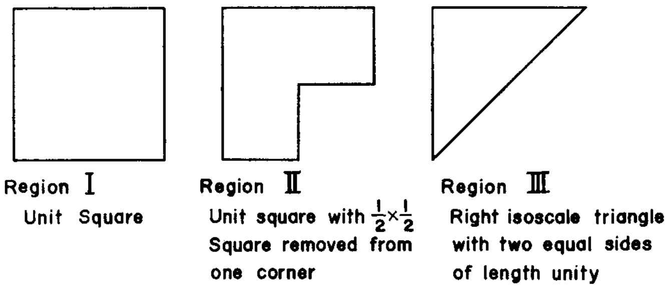

Fig. 1. Regions considered.

analog of the Dirichlet problem with Laplace's equation was solved for a number of mesh sizes using the interlacing scanning ADI method.

In every case, the boundary values were assumed to vanish, hence both the exact solution of the Dirichlet problem and the finite difference analog vanish identically. The advantage of this choice is that at each stage the approximate value at a given point is exactly equal to the error at that point. In each experiment, the starting values of unity were assumed at each interior mesh point, and the iterative process was terminated when the approximate values at all mesh points became less than 10^{-6}, in absolute value.

For the interlacing ADI method three choices of iteration parameters were used; the Peaceman-Rachford parameters; the Wachspress parameters and the optimum parameters which are given in the reference 1). The optimum para-

Table 1. The number of iterations by interlacing ADI (IADI) method and ADI method.

Region		I						II						III					
Method		IADI			ADI			IADI			ADI			IADI			ADI		
N	m	P	W	OP	P	W	OP	P	W	OP	P	W	OP	P	W	OP	P	W	OP
10	1	14		14	23		23	11		11	17		17	12		12	16		16
	2	10	17	12	16	18	12	10	24	12	12	20	13	10	20	10	11	19	11
	3	8	12	9	15	8	12	10	12	10	14	14	14	8	12	10	11	13	11
	4	9	11	10	15	9	11	9	11	11	13	13	13	9	11	10	11	13	12
	5		10			7			11			12			12			13	
20	1	28		28	46		46	20		20	37		37	23		23	33		33
	2	14	43	15	24	37	18	13	40	13	18	35	17	13	42	15	17	41	16
	3	11	17	12	21	14	14	16	17	15	17	18	18	11	18	14	15	17	14
	4	11	15	13	20	11	12	14	15	14	19	19	19	12	15	14	15	17	15
	5		15			11			17			19			14			17	
40	1	57		57	91		91	40		40	75		75	40		40	67		67
	2	27	80	25	36	73	25	19	66	23	28	73	26	20	84	27	45	80	22
	3	15	24	17	27	22	18	19	26	22	23	26	24	15	27	23	19	23	17
	4	15	21	15	27	15	18	21	19	18	23	22	24	15	20	17	19	19	19
	5		17			14			21			25			19			20	

P: Peaceman-Rachford Parameter, W: Wachspress Parameter, OP: Optimum parameter.

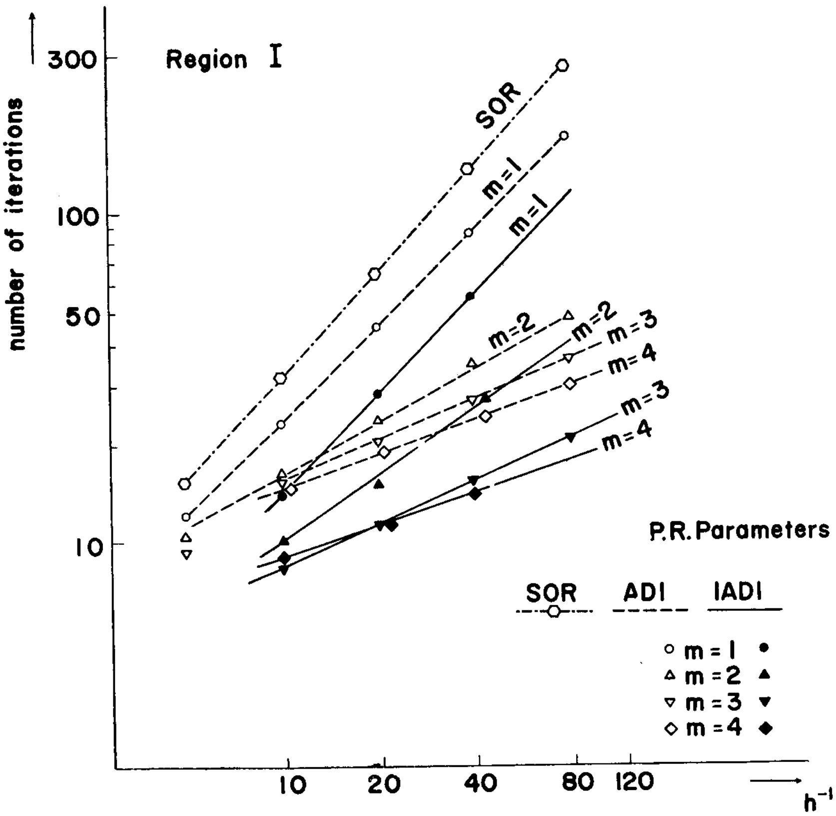

Fig. 2. Graphs of number of iterations versus h^{-1} for region I.

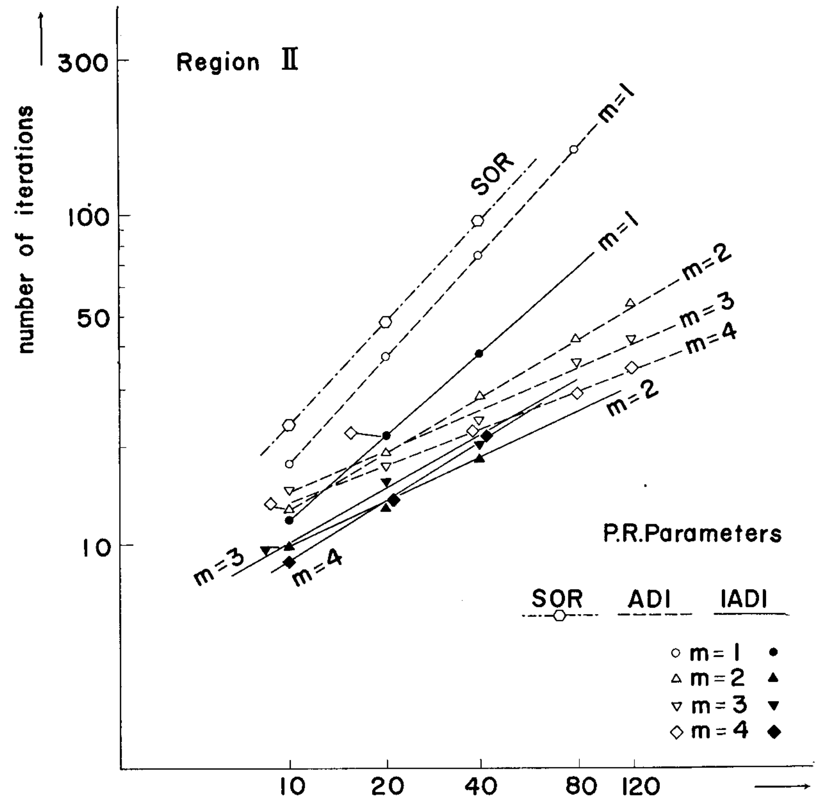

Fig. 3. Graphs of number of iterations versus h^{-1} for region II.

meters and the Peaceman-Rachford parameters were chosen for $m=1$, 2, 3 and 4. While Wachspress parameters were chosen for $m=2$, 3, 4 and 5. Mesh sizes of $h=1/10$, $1/20$ and $1/40$ were used.

Table 1 gives observed numbers of iterations using above three types of parameters, for the interlacing ADI method and the Peaceman-Rachford method. The later result is given by reference 1).

Figs. 2, 3 and 4 show graphs, with logarithimic scales, of the observed number of iterations versus h^{-1} for the point successive overrelaxation method, and for the interlacing ADI and the Peaceman-Rachford method with Peaceman-Rachford parameters.

Table 1 and Figs. 2, 3 and 4 show that the interlacing ADI method with the Peaceman-Rachford parameters is superior to the Peaceman-Rachford ADI method, for number of iterations. Although, the approximated relation (6.12) is not valid unless $\rho=2$, nevertheless, it seems that the ratio of number of iterations for the Peaceman-Rachford method and the interlacing ADI method is two on a maximum

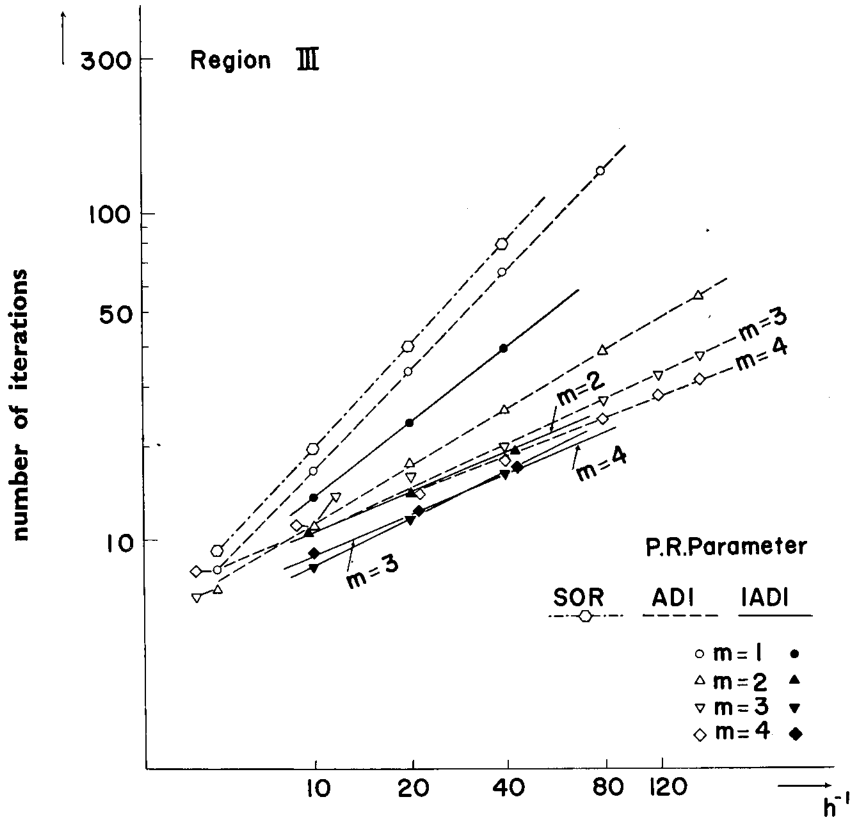

Fig. 4. Graphs of number of iterations versus h^{-1} for region III

value.

Table 1. shows that the minimum number of iterations is given by the interlacing ADI method with the Peaceman-Rachford parameters for any value of m.

8. Conclusion

We propose the interlacing ADI method which gives the minimum number of iterations for the solution of a model problem. The convergence of this method is proved theoretically and the experiment for a model problem shows good convergence property. But we could not give the rate of convergence of this method, rigorously.

The preceding experiments show that the number of iterations required for region II and III is usually almost equal to that required for the square. The Peaceman-Rachford parameters are recommended in preference to the Wachspress and optimum parameters, for the interlacing ADI method.

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