Axially Symmetric Stagnation-Point Flow of a Rarefied Gas

By

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This paper treats theoretically the axially symmetric stagnation-point flow of a rarefied gas. Representative Mach number M_S defined in the continuum flow region is assumed to be small and thus the analysis is based on the linearized version of the B-G-K equation. A method of solution similar to that used previously in the analysis of the two dimensional stagnation-point flow is applied in which the continuum flow and the Knudsen layer flow are considered successively. Actual analysis has been put forward to the second approximation correct to the order of $\sqrt{M_S}$. The results of the distributions of flow velocity, density and temperature in the Knudsen layer as well as the shear stress on the wall are discussed in detail.

1. Introduction

In recent years, there have been published many studies on rarefied gas flows which are based on the so-called B-G-K model[1] of the Boltzmann equation in the kinetic theory. In particular, when the flow speed is much smaller than the mean thermal speed of the gas molecule, the B-G-K equation can be linearized and it becomes possible to treat some simple flows with mathematical rigor. Previous studies in this direction deal with the shear flow along an infinite plane wall[2], the temperature Knudsen layer adjacent to a plane wall[3], Couette flow[4], Poiseuille flow[5], Rayleigh problem[6,7], plane shock wave[8] and so on. These investigations indicate that the B-G-K model can simulate the Boltzmann equation very well. One common point in these studies is that they are confined exclusively to the case of one dimensional flow (parallel to the wall) for mathematical simplicity.

Quite recently, however, Prof. Tamada succeeded in an analytical treatment of the stagnation-point flow in two dimensions[9]. Later, the present author noticed that Prof. Tamada's analysis can be extended to the axially symmetric case. In view of the fundamental importance of this type of flow, detailed study has been carried out and the result is given in the present paper. The analysis proceeds almost parallel to that in Ref. 9: The well-known solution of the Navier-Stokes

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equation for the continuum flow in question suggests a particular form of the distribution function of the molecular velocity, which enables us to decompose the kinetic equation into a few simultaneous integral equations in one independent variable. A method of successive approximation to the solution used in Ref. 9 is also effective in which the continuum equation and kinetic equation are considered alternately in the progress of approximation. The actual analysis has been put forward to the second approximation. It turns out that the profiles of the velocity, density and temperature in the Knudsen layer in the present case are quite similar to those in the two-dimensional case. An expression of the shear stress acting on the wall is also derived. The result of the present study may serve to clarify the relationship between the quasi-equilibrium (continuum) region and non-equilibrium region (Knudsen layer) of the flow of a gas.

2. Solution for Continuum Flow

We take cylindrical coordinates (R, θ, Z) with origin O at the stagnation point and Z-axis normal to the wall. There is no characteristic length in the present flow except for the mean free path* l of the gas molecule. We therefore take a reference point S on the Z-axis at a definite distance $L=Al$ from the wall, A being a given numerical value. Let Q_S be the flow speed and c the sound speed at the reference point. Then, we may characterize the stagnation-point flow by the Mach number $M_S=Q_S/c$ at the reference point. In the present study, we take, for convenience, the reference point in the continuum flow region away from the wall, so that $A\gg 1$. We also confine the study to the case in which $M_S\ll 1$ and so the region of continuum flow may be described approximately by the Navier-Stokes equations for incompressible fluid:

$$\left.\begin{aligned}
&V_R\frac{\partial V_R}{\partial R}+V_Z\frac{\partial V_R}{\partial Z}=-\frac{1}{\rho}\frac{\partial p}{\partial R}+\nu\left(\frac{\partial^2 V_R}{\partial R^2}+\frac{1}{R}\frac{\partial V_R}{\partial R^2}-\frac{V_R}{R^2}+\frac{\partial^2 V_R}{\partial Z^2}\right)\\
&V_R\frac{\partial V_Z}{\partial R}+V_Z\frac{\partial V_Z}{\partial Z}=-\frac{1}{\rho}\frac{\partial p}{\partial Z}+\nu\left(\frac{\partial^2 V_Z}{\partial R^2}+\frac{1}{R}\frac{\partial V_Z}{\partial R^2}+\frac{\partial^2 V_Z}{\partial Z^2}\right)\\
&\frac{\partial V_R}{\partial R}+\frac{V_R}{R}+\frac{\partial V_Z}{\partial Z}=0
\end{aligned}\right\}\qquad (1)$$

where $V=(V_R,\ O,\ V_Z)$ is the flow velocity, p the pressure, ρ the density and ν the kinematic viscosity. There is a well-known solution of (1) representing the axially symmetric stagnation-point flow with no slip at the wall as follows[10]:

* In general, the mean free path is a function of position and l means its representative value defined appropriately (cf. eq. (8)).

$$\left.\begin{aligned} \bar{V}_R &= Q\frac{R}{L}f'(\zeta), \qquad \bar{V}_Z = -2\frac{Q}{\sqrt{R_e}}f(\zeta) \\ p-p_0 &= -\frac{\rho}{2}Q^2\left\{\frac{1}{R_e}P(\zeta)+\frac{R^2}{L^2}\right\}, \qquad \zeta = \sqrt{R_e}\frac{Z}{L}, \qquad R_e = \frac{LQ}{\nu} \end{aligned}\right\} \tag{2}$$

where Q is a velocity parameter, p_0 the pressure at the stagnation point and the functions f and P satisfy following equations

$$\left.\begin{aligned} f'^2-2ff'' &= 1+f''' \\ 2ff' &= \tfrac{1}{4}P'-f'' \qquad (f' = df/d\zeta \text{ etc.}) \end{aligned}\right\} \tag{3}$$

and the boundary conditions for $\zeta \gg 1$ (inviscid region)

$$f(\zeta)\sim\zeta \quad \left(\bar{V}_R\sim Q\frac{R}{L}, \quad \bar{V}_Z = -2Q\frac{Z}{L}\right) \tag{3a}$$

and at the wall $(\zeta=0)$

$$f=f'=0 \qquad (\bar{V}_R = \bar{V}_Z = 0) \tag{3b}$$

The solution may be expressed near the stagnation point $R=Z=0$ as follows[10]:

$$\left.\begin{aligned} \bar{V}_R &= Q(R/L)(2a\zeta+\cdots\cdots) \qquad a\simeq 0.6560 \\ \bar{V}_Z &= -2(Q/\sqrt{R_e})(a\zeta^2+\cdots\cdots) \\ p-p_0 &= -(\rho/2)Q^2\{(2/R_e)(4a\zeta+\cdots\cdots)+R^2/L^2\} \end{aligned}\right\} \tag{4}$$

This solution, however, becomes invalid in the Knudsen layer where $Z=O(l)$. The kinetic theory approach is needed to understand the behaviour of the gas there. The solution from the kinetic theory is expected to tend asymptotically to (4) for $Z\gg l$. Thus, eqs. (4) give boundary conditions at infinity in the kinetic theory treatment. A boundary condition as for the temperature T is also needed and we shall consider in this study the case of constant temperature in the continuum region, viz.

$$T\sim T_0 \qquad \text{for} \qquad Z\gg l \tag{4a}$$

3. Fundamental Equations in Kinetic Theory Analysis

We assume that the gas obeys the B-G-K equation as mentioned earlier. Let $\boldsymbol{V}=(V_R, V_\theta, V_Z)$ be the molecular velocity, $F(\boldsymbol{V}; R, Z)$ the velocity distribution function and n the number density of molecules. Then, the B-G-K equation for the present problem may be written as

$$V_R\frac{\partial F}{\partial R}+V_Z\frac{\partial F}{\partial Z}+\frac{V_\theta^2}{R}\frac{\partial F}{\partial V_R}-\frac{V_R V_\theta}{R}\frac{\partial F}{\partial V_\theta} = An(F_e-F) \tag{5}$$

where A is a constant, An the collision frequency. The function F_e is the local equilibrium distribution given by

$$F_e = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m}{2kT}(\boldsymbol{V}-\bar{\boldsymbol{V}})^2\right\} \tag{5a}$$

where m is the mass of the molecule and k the Boltzmann constant. The number density n, the flow velocity $\bar{\boldsymbol{V}}$, and the temperature T are related to F by the equations

$$\left.\begin{aligned} &n = \int F d\boldsymbol{V}, \qquad \bar{\boldsymbol{V}} = \frac{1}{n}\int \boldsymbol{V} F d\boldsymbol{V} \\ &\frac{3k}{m} T = \frac{1}{n}\int (\bar{\boldsymbol{V}} - \boldsymbol{V})^2 F d\boldsymbol{V}, \qquad d\boldsymbol{V} = dV_R dV_\theta dV_Z \end{aligned}\right\} \tag{5b}$$

integrations being carried out over the whole molecular velocity space. Let n_0 be the number density at $R=Z=0$ of the continuum flow when estended to that point with neglect of small but steep variation in the Knudsen layer. Then, $p_0 = kn_0 T_0$ is still the stagnation pressure of the extended continuum flow.*

Now, the flow velocity $\bar{\boldsymbol{V}}$ in the Knudsen layer is very small compared with the mean thermal speed of the molecule since we are considering the case of small Mach number in the continuum region. Therefore, the distribution of the molecular velocity should be close to an equilibrium distribution at rest with number density n_0 and temperature T_0. Thus, if we write

$$\left.\begin{aligned} &F = n_0\left(\frac{h}{\pi}\right)^{3/2} e^{-hV^2}(1+\Phi), \qquad h = \frac{m}{2kT_0} \\ &n = n_0(1+\Sigma), \qquad T = T_0(1+\Theta) \end{aligned}\right\} \tag{6}$$

the quantities Φ, Σ, Θ are all small compared with unity. Inserting (6) into (5), (5a), (5b) and retaining only the first order terms, we may have a linearized system of the fundamental equations. Comparison of the system with the boundary condition (4) suggests introduction of new variables

$$\left.\begin{aligned} &\lambda\sqrt{h}(R, Z) = (r, z), \qquad \sqrt{h}\boldsymbol{V} = \boldsymbol{v} = (v_r, v_\theta, v_z) \\ &\sqrt{h}\bar{\boldsymbol{V}}/\kappa = \bar{\boldsymbol{v}} = (\bar{v}_r, O, \bar{v}_z), \\ &(\Phi, \Sigma, \Theta)/\kappa = (\phi, \sigma, \tau) \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} &\lambda = An_0, \qquad \kappa = \frac{a\sqrt{R_e h Q}}{(\lambda\sqrt{h}L)^2} = \frac{a}{2}\left(\frac{\pi\gamma}{2}\right)^{3/4}\left(\frac{M}{A}\right)^{3/2} \\ &M = Q/c, \qquad c = \sqrt{\gamma k T_0/m} = \sqrt{\gamma/(2h)} \end{aligned}\right\} \tag{7a}$$

* The temperature in the continuum flow region is assumed to be constant T_0 (eq. (4a)).

γ being the adiabatic exponent. We have used in rewriting κ the following relations valid for the B-G-K gas:

$$l = 2/(\lambda\sqrt{\pi h}), \qquad \nu = (2h\lambda)^{-1}, \qquad R_e = 4\sqrt{\gamma/(2\pi)}AM \tag{8}$$

The parameters Q and M in (7a) are related respectively to the flow speed Q_S and the Mach number M_S at the reference point ($\zeta=\sqrt{R_e}$) by the equations (from (2))

$$(Q_S, M_S) = \{2f(\sqrt{R_e})/\sqrt{R_e}\}(Q, M) \tag{9}$$

These equations together with (8) enable us to calculate Q and M if Q_S, c (and hence M_S) and A (position of the reference point) be given. Now, in terms of the variables in (7), the linearized fundamental equations may be written as follows:

$$v_r\frac{\partial\phi}{\partial r}+v_z\frac{\partial\phi}{\partial z}+\frac{v_\theta^2}{r}\frac{\partial\phi}{\partial v_r}-\frac{v_rv_\theta}{r}\frac{\partial\phi}{\partial v_\theta} = -\phi+2v_r\bar{v}_r+2v_z\bar{v}_z+\sigma-\frac{3}{2}\tau+\tau\boldsymbol{v}^2 \tag{10}$$

$$\left.\begin{aligned} &\sigma = \int E\phi d\boldsymbol{v}, \qquad \bar{\boldsymbol{v}} = \int \boldsymbol{v}E\phi d\boldsymbol{v} \\ &\frac{3}{2}(\tau+\sigma) = \int \boldsymbol{v}^2E\phi d\boldsymbol{v}, \qquad E = \pi^{-3/2}e^{-v^2} \end{aligned}\right\} \tag{10a}$$

It can be shown as in Ref. 9 that the mean flow from the solution of the linearized B-G-K equation (10) tends asymptotically with distance from the wall to the solution of the Stokes equation in hydrodynamics. The forms (4) for $\zeta \ll 1$ in the stagnation-point flow under consideration satisfy the same equation and hence we may take the boundary conditions of eq. (10) for $z\to\infty$ as follows:

$$\left.\begin{aligned} &\bar{v}_r \sim 2rz, \qquad \bar{v}_z \sim -2z^2 \\ &\sigma \sim -4z, \qquad \tau \sim 0 \end{aligned}\right\} \tag{11}$$

We assume that molecules are reflected from the wall diffusely. Denoting by σ_W and τ_W the perturbation number density and temperature of the reflected molecules, we have the linearized forms of the boundary conditions at the wall $z=0$:

$$\left.\begin{aligned} &\phi_{v_z>0} = \sigma_W-\tfrac{3}{2}\tau_W+\tau_W\boldsymbol{v}^2 \\ &\bar{v}_z = 0 \end{aligned}\right\} \tag{12}$$

4. Distribution Function

Taking account of the axial symmetry of the flow together with the structure of eq. (10), the asymptotic form of ϕ and the boundary conditions (12), we assume the proper form of the distribution function to be

$$\phi = \varphi(v_r^2+v_\theta^2, v_z; z)+\psi(v_z; z)rv_r \tag{13}$$

Introducing this in (10a), we obtain

$$\bar{v}_r = \chi(z)r \qquad \chi = \int E\psi v_r^2 d\boldsymbol{v} \tag{14}$$

$$\bar{v}_z = \bar{v}_z(z) \qquad \bar{v}_z = \int E\varphi v_z d\boldsymbol{v} \tag{15}$$

$$\sigma = \sigma(z) \qquad \sigma = \int E\varphi d\boldsymbol{v} \tag{16}$$

$$\tau = \tau(z) \qquad \frac{3}{2}(\sigma+\tau) = \int E\varphi \boldsymbol{v}^2 d\boldsymbol{v} \tag{17}$$

Then, substituting (13)~(17) into eq. (10) and comparing both sides, we get the next two equations:

$$v_z \frac{\partial \varphi}{\partial z} = -\varphi + 2v_z\bar{v}_z + \sigma - \frac{3}{2}\tau + \tau \boldsymbol{v}^2 - (v_r^2 + v_\theta^2)\psi \tag{18}$$

$$v_z \frac{\partial \psi}{\partial z} = -\psi + 2\chi \tag{19}$$

The boundary conditions (12) at $z=0$ become

$$\varphi_{v_z>0} = \sigma_W - \frac{3}{2}\tau_W + \tau_W \boldsymbol{v}^2 \tag{20}$$

$$\psi_{v_z>0} = 0 \tag{21}$$

$$\bar{v}_z = 0 \tag{22}$$

Also, the boundary conditions (11) for $z \to \infty$ are rewritten as

$$\begin{aligned} &\chi(z) \sim 2z, \qquad \bar{v}_z(z) \sim -2z^2 \\ &\sigma(z) \sim -4z, \qquad \tau(z) \sim 0 \end{aligned} \tag{23}$$

5. Integral Equations for Mean Fields and Their Solutions

To begin with, we shall obtain the solution for the flow velocity. Eq. (19) for ψ and the boundary conditions (21) and $\chi(z) \sim 2z$ (eq. (23)) are identical with those in the two dimensional case. Therefore, it may suffice to give a brief account of the solution. Solving eq. (19) for ψ under (21) and inserting the result into (14), we obtain an integral equation for $\chi(z)$ in the following form:

$$\sqrt{\pi}\,\chi(z) = \int_0^\infty J_{-1}(z-\eta)\chi(\eta)d\eta \tag{24}$$

where the function $J_n(x)$ is defined as

$$J_n(x) = (\text{sgn}\, x)^{n+1} \int_0^\infty \exp\left(-\xi^2 - \frac{|x|}{\xi}\right)\xi^n d\xi \tag{24a}$$

n being an integer. An approximate solution of eq. (24) has been given in Ref. 9. Namely,

$$\left.\begin{aligned} &\chi(z)=2z+\alpha+\chi^*(z)\,, \qquad \chi^*(z)=\sum_{i=0}^{3}a_iJ_i(z)\,,\\ &\alpha=2.032\,, \qquad a_0=-0.2378\,, \qquad a_1=-2.078\\ &\qquad\qquad\qquad a_2=3.530\,, \qquad a_3=-1.794 \end{aligned}\right\} \tag{25}$$

On the other hand, we multiply both sides of eq. (10) by E and integrate the result throughout the velocity space with (13)$\sim$(17). We then have a continuity equation

$$\frac{d\bar{v}_z}{dz}+2\chi=0 \tag{26}$$

Integrating this with (22) and inserting (25), we get the result

$$\left.\begin{aligned} &\bar{v}_z=-2z^2-2\alpha z+\beta+\bar{v}_z^*\\ &\bar{v}_z^*=\sum_{i=0}^{3}2a_iJ_{i+1}(z) \qquad \beta=0.9342 \end{aligned}\right\} \tag{27}$$

In the next place, solving eq. (18) for φ and applying the condition that φ should not be exponentially infinite for $z\to\infty$, $v_z<0$, and the condition (20) for $v_z>0$, we obtain the result

$$\left.\begin{aligned} \varphi_{v_z<0}=&-\int_z^\infty \exp\left(-\frac{z-\eta}{v_z}\right)\left\{2\bar{v}_z+\left(\sigma-\frac{3}{2}\tau\right)\frac{1}{v_z}+\tau\frac{\boldsymbol{v}^2}{v_z}\right.\\ &\left.+2\frac{v_r^2+v_\theta^2}{v_z}\int_\eta^\infty\exp\left(-\frac{\eta-\xi}{v_z}\right)\chi(\xi)d\xi\right\}d\eta\\ \varphi_{v_z>0}=&\left(\sigma_W-\frac{3}{2}\tau_W+\tau_W\boldsymbol{v}^2\right)e^{-z/v_z}+\int_0^z\exp\left(-\frac{z-\eta}{v_z}\right)\left\{2\bar{v}_z+\left(\sigma-\frac{3}{2}\tau\right)\frac{1}{v_z}\right.\\ &\left.+\tau\frac{\boldsymbol{v}^2}{v_z}-2\frac{v_r^2+v_\theta^2}{v_z}\int_0^\eta\exp\left(-\frac{\eta-\xi}{v_z}\right)\chi(\xi)d\xi\right\}d\eta \end{aligned}\right\} \tag{28}$$

Inserting this in (15), (16), (17) and carring out some calculations, we get the simultaneous integral equations for $\bar{v}_z$, σ, τ:

$$\begin{aligned} \sqrt{\pi}\,\bar{v}_z(z)=&(\sigma_W-\tfrac{1}{2}\tau_W)J_1(z)+\tau_WJ_3(z)\\ &+\int_0^\infty\{2J_1\bar{v}_z+J_0\sigma+(J_2-\tfrac{1}{2}J_0)(\tau-4\chi)\}d\eta \end{aligned} \tag{29}$$

$$\begin{aligned} \sqrt{\pi}\,\sigma(z)=&(\sigma_W-\tfrac{1}{2}\tau_W)J_0(z)+\tau_WJ_2(z)\\ &+\int_0^\infty\{2J_0\bar{v}_z+J_{-1}\sigma+(J_1-\tfrac{1}{2}J_{-1})\tau-4J_1\chi\}d\eta \end{aligned} \tag{30}$$

$$\tfrac{3}{2}\sqrt{\pi}(\sigma+\tau) = (\sigma_W+\tfrac{1}{2}\tau_W)\{J_0(z)+J_2(z)\}+\tau_W J_4(z)$$
$$+\int_0^\infty \{2(J_0+J_2)\bar{v}_z+(J_{-1}+J_1)\sigma+\tfrac{1}{2}(J_{-1}+J_1+2J_3)\tau$$
$$-4(J_1+J_3)\chi\}d\eta \tag{31}$$

where the argument of J_n in the integrals is $(z-\eta)$.

If we make use of eqs. (22), (26) and the equation

$$\int_0^\infty J_0(z-\eta)\chi(\eta)d\eta = -\sqrt{\pi} \tag{32}$$

which can be obtained by integrating (24) with respect to z, we can eliminate χ, $\bar{v}_z$ from (29)$\sim$(31). Taking the asymptotic forms (23) into account, we write

$$\left.\begin{array}{lll} \sigma = -4z+\sigma^* & & \tau = \tau^* \\ \sigma^* \to 0\,, & \tau^* \to 0 & (z\to\infty) \end{array}\right\} \tag{33}$$

Then, elimination of χ, $\bar{v}_z$ results in the following equations:

$$\sqrt{\pi}\sigma^* = \sigma_W J_0(z)+\tau_W\{J_2(z)-\tfrac{1}{2}J_0(z)\}-4J_1(z)$$
$$+\int_0^\infty \{J_{-1}\sigma^*+(J_1-\tfrac{1}{2}J_{-1})\tau^*\}d\eta \tag{34}$$
$$\tfrac{3}{2}\sqrt{\pi}\tau^* = \sigma_W\{J_2(z)-\tfrac{1}{2}J_0(z)\}+\tau_W\{J_4(z)-J_2(z)+\tfrac{5}{4}J_0(z)\}-4J_3(z)+2J_1(z)$$
$$+\int_0^\infty \{(J_1-\tfrac{1}{2}J_{-1})\sigma^*+(J_3-J_1+\tfrac{5}{4}J_{-1})\tau^*\}d\eta \tag{35}$$

argument $(z-\eta)$ in the integrals being omitted as before, and an additional equation

$$\sigma_W = 2\sqrt{\pi}-\frac{\tau_W}{2}+\int_0^\infty [2J_0(\eta)\sigma^*+\{2J_2(\eta)-J_0(\eta)\}\tau^*]\,d\eta \tag{36}$$

These equations for σ^* and τ^* can be reduced to those in the two dimensional case, if we replace $(\sigma^*, \tau^*; \sigma_W, \tau_W)$ by *2* $(\sigma^*, \tau^*; \sigma_W, \tau_W)$. Thus, approximate solutions are at once obtained from Ref. 9 as follows:

$$\left.\begin{array}{llll} \sigma^* = \sum_{i=0}^{3} h_i J_i(z)\,, & \tau^* = \sum_{i=0}^{3} k_i J_i(z) & & \\ h_0 = 1.0816\,, & h_1 = -1.2654\,, & h_2 = 1.3722\,, & h_3 = -0.4306 \\ k_0 = 0.011052\,, & k_1 = -0.011142\,, & k_2 = 0.06812\,, & k_3 = 0.09492 \\ \sigma_W = 3.3706\,, & \tau_W = 0.89352 & & \end{array}\right\} \tag{37}$$

Introducing in (33) the Knudsen layer variations σ^* and τ^* thus found, we can understand the behaviours of density and temperature in the axially symmetric stagnation-point flow. It may be noted that there occurs a temperature jump between the gas and the wall.

It will be seen that the component of flow velocity parallel to the wall in the Knudsen layer of an axially symmetric stagnation-point flow is the same as in the two dimensional case, while the component of flow velocity perpendicular to the wall, density and temperature, including jump values at the wall, are just twice as much as those in the two dimensional case, if expressed in terms of the Knudsen layer variables.

6. Refinement of Continuum Flow

As we have seen in the foregoing analysis (eqs. (25), (27)), the velocity of the continuum flow when extended to the wall accompany slip and jump of the forms:

$$\left.\begin{aligned} &\bar{V}_R = Q(R/L)(2a^2\alpha\varepsilon) \\ &\bar{V}_Z = (Q/\sqrt{R_e})(4a^2\beta\varepsilon^2) = O(\varepsilon^2) \\ &a \simeq 0.6560\,, \qquad \varepsilon = \frac{1}{2a}\frac{\sqrt{R_e}}{\lambda\sqrt{hL}} = \frac{1}{2a}\left(\frac{\pi\gamma}{2}\right)^{1/4}\left(\frac{M}{A}\right)^{1/2} \end{aligned}\right\} \tag{38}$$

The solution for the continuum flow should therefore be corrected according to this result in the next step. Eq. (38) suggests that the correction to be made is proportional to ε. It can be shown that the field equation (1) for the continuum flow remains unchanged to this order. Thus, our task is to solve eqs. (3) with modified boundary conditions for f:

$$\left.\begin{aligned} &f = 0\,, \qquad f' = 2a^2\alpha\varepsilon \qquad \text{at} \quad \zeta = 0 \\ &f \sim \zeta\,, \qquad\qquad\qquad\quad \text{at} \quad \zeta \to \infty \end{aligned}\right\} \tag{39}$$

Denoting by f_0 the non-slip solution given in **2**, the solution of the first equation in (3) subject to (39) can be easily obtained by perturbing f_0. Thus,

$$f = f_0 + a\alpha\varepsilon f_0{}' \tag{40}$$

On the other hand, we have from (33)

$$\sigma + \tau \equiv (p - p_0)/(\kappa p_0) = -4z \tag{41}$$

at the outer edge of the Knudsen layer, so that $p - p_0 = 0$ at $\zeta = R = 0$ in (2), and so

$$P(0) = 0 \tag{41a}$$

Integrating the second equation in (3) under (41a), we have

$$\frac{1}{4}P = f_0' + f_0^2 + a\alpha\varepsilon(f_0'' - 2a + 2f_0 f_0') \tag{42}$$

The results (40), (42) and (2) constitute the first approximation to the continuum stagnation-point flow. This is the so-called slip flow. Expanding these solutions for small ζ and rewriting the results in terms of the Knudsen layer variables (7), we have

$$\left.\begin{aligned} \bar{v}_r &\sim r\{2z+\alpha+\varepsilon(-z^2-\alpha z)\} \\ \bar{v}_z &\sim -2z^2-2\alpha z+\beta+\varepsilon(\tfrac{2}{3}z^3+\alpha z^2) \\ \sigma &\sim -4z+\varepsilon\{2z^2+2\alpha z-r^2\} \\ \tau &\sim 0 \end{aligned}\right\} \tag{43}$$

These equations again afford boundary conditions at $z\to\infty$ for the second approximation (correct to $O(\varepsilon)$) to the flow in the Knudsen layer (cf. (11)). One thing to be added is the correction to the relation (9) between the parameters. From (2) and (40), we have

$$V_Z = -2(Q/\sqrt{R_e})\{f_0(\zeta)+a\alpha\varepsilon f_0'(\zeta)\}$$

At the reference point, $\zeta=\sqrt{R_e}$ and $|V_Z|=Q_S$, so that*

$$(Q_S,\ M_S) = (2/\sqrt{R_e})\{f_0(\sqrt{R_e})+a\alpha\varepsilon f_0'(\sqrt{R_e})\}(Q,\ M) \tag{44}$$

7. Refinement of Knudsen Layer Flow

We next proceed to the second approximation to the Knudsen layer flow. It can be shown that the linearized B-G-K equation (10) does not change to this order of approximation. We indicate hereafter by suffix 1 the quantities to the first approximation obtained in the preceding paragraphs and by suffix 2 the correction terms proportional to ε to be obtained below. Thus, bearing (43) in mind, we assume that the perturbation distribution function ϕ in the form:

$$\phi = \varphi_1(v_r^2+v_\theta^2,\ v_z;\ z)+\psi_1(v_z;\ z)rv_r+\varepsilon\{\varphi_2(v_r^2+v_\theta^2,\ v_z;\ z)+\psi_2(v_z;\ z)rv_r-r^2\} \tag{45}$$

Inserting this in (10a), we have

$$\bar{v}_r = r(\chi_1+\varepsilon\chi_2)\ , \qquad \chi_2 = \int E\psi_2 v_r^2 d\boldsymbol{v} \tag{46}$$

$$\bar{v}_z = \bar{v}_{z,1}+\varepsilon\bar{v}_{z,2}\ , \qquad \bar{v}_{z,2} = \int E\varphi_2 v_z d\boldsymbol{v} \tag{47}$$

$$\sigma = \sigma_1+\varepsilon(\sigma_2-r^2)\ , \qquad \sigma_2 = \int E\varphi_2 d\boldsymbol{v} \tag{48}$$

$$\tau = \tau_1+\varepsilon\tau_2\ , \qquad \tfrac{3}{2}(\sigma_2+\tau_2) = \int E\varphi_2 \boldsymbol{v}^2 d\boldsymbol{v} \tag{49}$$

* The speed of sound c is associated with the temperature T_0, which remains unchanged.

Substitution of (45)$\sim$(49) into eq. (10) and comparison of both sides yield the following equations for φ_2 and ψ_2:

$$v_z \frac{\partial \varphi_2}{\partial z} = -\varphi_2 + 2v_z \bar{v}_{z,2} + \sigma_2 - \tfrac{3}{2}\tau_2 + \tau_2 \boldsymbol{v}^2 - (v_r^2 + v_\theta^2)\psi_2 \tag{50}$$

$$v_z \frac{\partial \psi_2}{\partial z} = -\psi_2 + 2\chi_2 + 2 \tag{51}$$

The boundary conditions at the wall (cf. (12)) become

$$\left.\begin{aligned} &\varphi_{2,v_z>0} = \sigma_{W,2} - \tfrac{3}{2}\tau_{W,2} + \tau_{W,2}\boldsymbol{v}^2 \\ &\psi_{2,v_z>0} = 0, \qquad \bar{v}_{z,2} = 0 \end{aligned}\right\} \tag{52}$$

Also, the boundary conditions (43) for $z \to \infty$ give

$$\left.\begin{aligned} &\chi_2 \sim -z^2 - \alpha_1 z\,, \qquad \bar{v}_{z,2} \sim \tfrac{2}{3}z^3 + \alpha_1 z^2 \\ &\sigma_2 \sim 2z^2 + 2\alpha_1 z\,, \qquad \tau_2 \sim 0 \end{aligned}\right\} \tag{53}$$

We first solve eq. (51) for ψ_2 under (52) and introduce the result in (46). We then obtain the equation for χ_2:

$$\left.\begin{aligned} &\chi_2 = -z^2 - \alpha_1 z + \alpha_2 + \chi_2^* \\ &\sqrt{\pi}\,\chi_2^* = 2J_2(z) - \alpha_1 J_1(z) - (1+\alpha_2)J_0(z) + \int_0^\infty J_{-1}(z-\eta)\chi_2^* d\eta \end{aligned}\right\} \tag{54}$$

An approximate solution of this integral equation is again found in Ref. 9. Thus,

$$\left.\begin{aligned} &\chi_2^* = \sum_{i=0}^{3} a_i^{(2)} J_i(z)\,, \qquad \alpha_2 = -0.533 \\ &a_0^{(2)} = 1.479\,, \quad a_1^{(2)} = -9.34\,, \quad a_2^{(2)} = 13.23\,, \quad a_3^{(2)} = -5.74 \end{aligned}\right\} \tag{55}$$

Substituting χ_2 from (54) into the continuity equation (cf. (26)) and integrating under the condition $\bar{v}_{z,2}(0)=0$, we get

$$\left.\begin{aligned} &\bar{v}_{z,2} = \tfrac{2}{3}z^3 + \alpha_1 z^2 - 2\alpha_2 z + \beta_2 + \bar{v}_{z,2}^* \\ &\bar{v}_{z,2}^* = \sum_{i=0}^{3} 2a_i^{(2)} J_{i+1}(z) \qquad \beta_2 = 1.192 \end{aligned}\right\} \tag{56}$$

Now, we refer briefly to the shear stress acting on the wall. It is well known that the gradient law for the stress breaks down when the velocity distribution function changes appreciably in the interval of mean free path or time. The shear stress is written, by its definition, as

$$\begin{aligned} S_{RZ}/(2p_0) &= -\kappa \int E[\phi]_{z=0} v_r v_z d\boldsymbol{v} \\ &= -\kappa \int E\{\psi_1(v_z, 0) + \varepsilon\psi_2(v_z, 0)\} v_r^2 v_z r d\boldsymbol{v} \end{aligned} \tag{57}$$

Inserting $\psi_1(v_z, 0)$, $\psi_2(v_z, 0)$ and making some calculations, we are led to the result

$$S_{RZ}/(2p_0) = \kappa \left(1 - \frac{\alpha_1}{2} \varepsilon\right) r \tag{58}$$

The same result is also obtained by a continuum calculation basing on the slip flow as given by (43).

We finally proceed to obtain σ_2 and τ_2. We solve eq. (50) with known ψ_2 taking account of (52), and insert the resulting φ_2 in (47)~(49) as before. We then have three equations involving χ_2, $\bar{v}_{z,2}$, σ_2 and τ_2. From these equations, we can eliminate χ_2, $\bar{v}_{z,2}$ as in the previous case. The results are simultaneous integral equations for σ_2 and τ_2. Remembering (53), we write

$$\left.\begin{array}{lll} \sigma_2 = 2z^2+2\alpha_1 z+\sigma_2^*, & \tau_2 = \tau_2^* & \\ \sigma_2^* \to 0, & \tau_2^* \to 0, & (z\to\infty) \end{array}\right\} \tag{59}$$

Then, it is found that the equations for σ_2^* and τ_2^* just coincide with (34)~(36) if we take

$$(\sigma_2^*, \tau_2^*; \sigma_{W,2}, \tau_{W,2}) = -\frac{\alpha_1}{2}(\sigma_1^*, \tau_1^*; \sigma_{W,1}, \tau_{W,1}) \tag{60}$$

Thus, the final results for the Knudsen layer flow may be summarized as follows:

$$\left.\begin{array}{ll} \bar{v}_r & = r\{2z+\alpha_1+\chi_1^*+\varepsilon(-z^2-\alpha_1 z+\alpha_2+\chi_2^*)\} \\ \bar{v}_z & = -2z^2-2\alpha_1 z+\beta_1+\bar{v}_{z,1}^*+\varepsilon(\frac{2}{3}z^3+\alpha_1 z^2-2\alpha_2 z+\beta_2+v_{z,2}^*) \\ \sigma & = -4z+\sigma_1^*+\varepsilon\{(2z^2+2\alpha_1 z+\sigma_2^*)-r^2\} \\ \tau & = \tau_1^*+\varepsilon\tau_2^* \end{array}\right\} \tag{61}$$

where α_1, χ_1^* are given by (25), β_1, $v_{z,1}^*$ by (27), σ_1^*, τ_1^* by (37), α_2, χ_2^* by (55) β_2, $v_{z,2}^*$ by (56) and σ_2^*, τ_2^* by (60). Further, the wall temperature variation becomes

$$\tau_W = \tau_{W,1}+\varepsilon\tau_{W,2}$$

where the values of $\tau_{W,1}$ and $\tau_{W,2}$ are given in (37) and (60) respectively.

In Fig. 1 are shown plotted v_r/r and v_z for three values of ε. Plots of $[\sigma]_{r=0}$ and τ for the same values of ε are given in Fig. 2. Thin lines represent asymptotic behaviors. We can see in these figures the first order effect of ε or $\sqrt{M_S}$ to the Knudsen layer flow. It will be seen also that the flow velocity v_r does not vanish at the wall as noticed earlier. The asymptotic form of v_r takes also a finite value when extended to the wall. The temperature in the Knudsen layer increases with approach to the wall. The temperature of the wall is higher

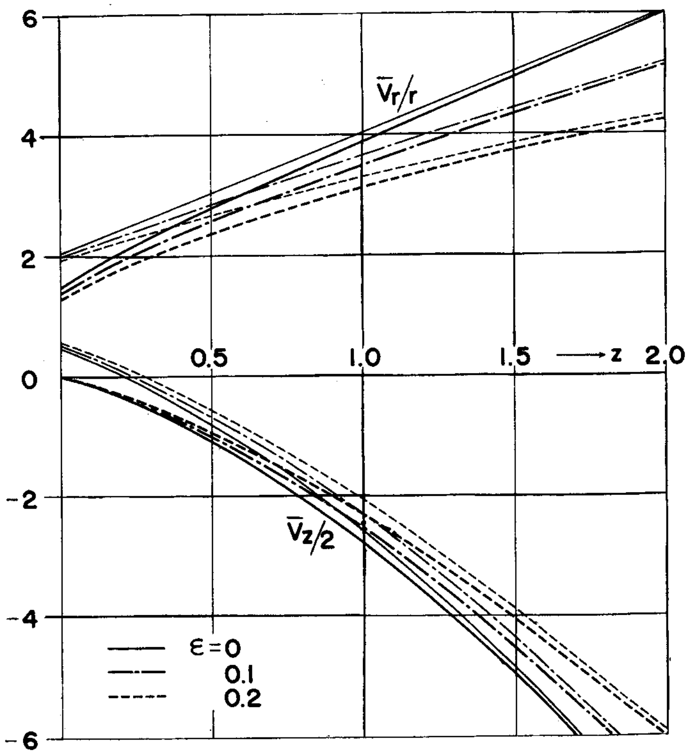

Fig. 1. Distributions of radial and vertical velocities. Thin lines represent asymptotic behaviours.

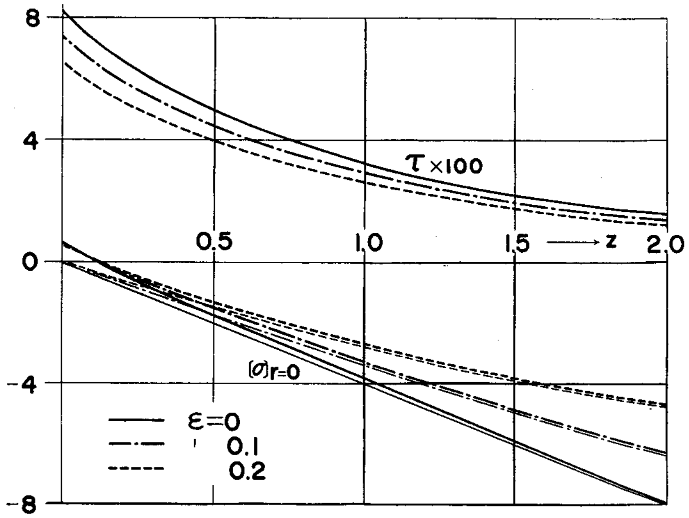

Fig. 2. Distributions of density and temperature on the z-aixs. Thin lines represent asymptotic behaviours.

than that of the gas neighbouring the wall by an amount $\tau_W - \tau(0)$.

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