A Study on Nonreciprocal Parametric Amplifiers

By

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Two or three methods are described to obtain a nonreciprocal parametric amplifier by use of two parametric elements in the frequency inverting case without using non-reciprocal elements such as a uniguide or a circulator. This kind of amplifier can be matched at the input terminal and/or at the output terminal. In a certain case the backward insertion loss can be made infinite. It was proved, however, in regard to the stability criterion that the two properties (matching and infinite backward insertion loss) can not be realized simultaneously. The other characteristics are almost the same as the usual negative resistance parametric amplifier. Some experimental results are presented with respect to one of the amplifiers. Though it has an electrical difficulty of construction, if once built, it may be convenient for low noise pre-amplifier.

1. Introduction

Parametric amplifiers have come to be used because of their low noise property. Most of the parametric amplifiers are of negative resistance type using a circulator. In the course of the study it was found that a nonreciprocal parametric amplifier can be built without using a nonreciprocal element such as a circulator or a uniguide, if two parametric elements are properly combined.

How one can build a nonreciprocal amplifier is suggested in the admittance matrix expression of a time-varing capacitance[1]

$$Y=\begin{pmatrix} G_0 & G_1 & G_2 \\ G_1^* & G_0 & G_1 \\ G_2^* & G_1^* & G_0 \end{pmatrix} + j\begin{pmatrix} \omega_u & 0 & 0 \\ 0 & \omega_s & 0 \\ 0 & 0 & -\omega_i \end{pmatrix}\begin{pmatrix} C_0 & C_1 & C_2 \\ C_1^* & C_0 & C_1 \\ C_2^* & C_1^* & C_0 \end{pmatrix}. \tag{1}$$

The off-diagonal parts are not symmetric but complex conjugates to each other. Keeping an eye on the conductance part of the parametric element, it is found that this nonreciprocity is associated with the phase but not with the amplitude, since the absolute value of elements are, with respect to the diagonal, equal to each other. On the other hand, the reactance part has the nonreciprocity in regard to both

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phase and amplitude. The amplitude nonreciprocity of the reactance part is in proportion to the associated frequencies enabling us to build a frequency up-converter with gain. If two parametric reactance elements are coupled in a cascade such that the input signal frequency is converted into some other frequency by one parametric element and that converted frequency is turned back to the original frequency by the other parametric element, the nonreciprocity of the amplitude may disappear and that of the phase may markedly appear, since the output frequency becomes equal to that of the input. The parallel combination of a phase nonreciprocal device with a reciprocal (passive) element may produce a nonreciprocal (in amplitude) device, when their phase shifts are determined such that the two signal waves are added in phase in the forward direction and in opposite phase in the backward direction. If the frequency inverting case is employed here, amplification gain may be obtained.†

Some reports regarding the nonreciprocal parametric amplifiers have appeared so far, among which the papers by R. Maurer and K.-H. Locherer(2) are most detaild. They used a lumped admittance as a reciprocal element but they did not perform the stability criterion. In this paper two or three types of such amplifiers are described, their characteristics being expressed by the scattering matrix which is the most convenient form for this type of amplifier. The realizability of the amplifier is discussed in terms of the stability criterion and some experimental results are given.

2. Analysis of a Nonreciprocal Parametric Amplifier by use of a Quarter-wave Transmission Line

This section tries to make clear some behavior of the nonreciprocal parametric amplifier shown in Fig. 1. The upper part $e^{-j\theta s}$ represents a transmission line

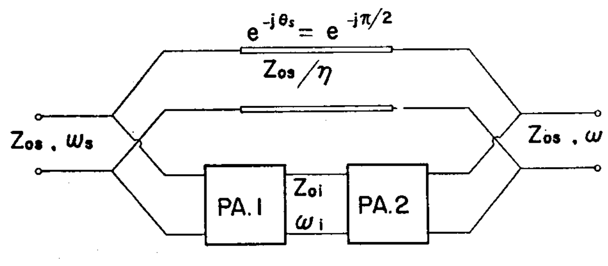

Fig. 1. Nonreciprocal parametric amplifier by use of a quarter-wave line.

† A nonreciprocal device is also possible by use of the time-varying conductance, but in this case gain can not be obtained unless the conductance has an average negative value.

which provides the phase shift of $\pi/2$ and the lower part the cascade-connected two-stage parametric amplifier, in which the both idler resonant circuits are coupled directly.

(1) Scattering Matrix

The admittance matrix of time-varying capacitance excited at the pump frequency $\omega_p/2\pi$,

$$c(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_p t}, \qquad C_{-n} = C_n^* \tag{2}$$

is written[1]

$$Y_c = \begin{pmatrix} j\omega_s C_0 & j\omega_s C_1 \\ -j\omega_i C_1^* & -j\omega_i C_0 \end{pmatrix} \tag{3}$$

between the two components at the signal and the idler frequencies.

This admittance matrix is transformed into a fundamental matrix

$$F_1 = \frac{1}{j\alpha_i}\begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix}, \tag{4}$$

where the elements of the matrix are normalized to the characteristic impedances of the transmission lines Z_{0s} and Z_{0i}, as

$$\alpha_s = \omega_s C_1 \sqrt{Z_{0s} Z_{0i}}, \qquad \alpha_i = \omega_i C_1^* \sqrt{Z_{0s} Z_{0i}}, \qquad \alpha^2 = \alpha_i \alpha_s .$$

The fundamental matrix of the second stage PA.2 is given by the inverse of Eq. (4);

$$F_2 = \frac{1}{-j\beta_s}\begin{pmatrix} 0 & 1 \\ -\beta^2 & 0 \end{pmatrix}, \tag{5}$$

where the pump factor α is replaced by β in order to distinguish the two pump factors for the first stage or for the second;

$$\beta_s = \omega_s C_2 \sqrt{Z_{0s} Z_{0i}}, \qquad \beta_i = \omega_i C_2^* \sqrt{Z_{0s} Z_{0i}}, \qquad \beta^2 = \beta_s \beta_i .$$

There are some relations between α and β;

$$\alpha_s \beta_i = \alpha\beta e^{j\varphi}, \qquad \alpha_i \beta_s = \alpha\beta e^{-j\varphi},$$

where φ is the phase difference bewteen the two pump excitation; $\varphi = \varphi_1 - \varphi_2$. The total fundamental matrix of the lower part is given by the multiplication of the fundamental matrix of each element;

$$F_t = \begin{pmatrix} 1 & 0 \\ y_{s1} & 1 \end{pmatrix} \frac{1}{j\alpha_i} \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_{i1} & 1 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ y_{i2} & 1 \end{pmatrix}^* \frac{1}{-j\beta_s} \begin{pmatrix} 0 & 1 \\ -\beta^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_{s2} & 1 \end{pmatrix}$$

$$= \frac{e^{j\varphi}}{\alpha\beta} \begin{bmatrix} y_{s2}(y_{i1}^* + y_{i2}^*) - \beta^2 & y_{i1}^* + y_{i2}^* \\ y_{s2}(y_{s1} y_{i1}^* - \alpha^2) + y_{s1}(y_{s2} y_{i2}^* - \beta^2) & y_{s1}(y_{i1}^* + y_{i2}^*) - \alpha^2 \end{bmatrix}, \tag{6}$$

where y_s and y_i are the resonant admittances of the signal and idler resonant circuits respectively which are normalized to the characteristic impedances of the transmission lines. This fundamental matrix is again transformed into the admittance matrix

$$Y_a = \frac{1}{y_i} \begin{bmatrix} y_{s1} y_i - \alpha^2 & -\alpha\beta e^{j\varphi} \\ -\alpha\beta e^{-j\varphi} & y_{s2} y_i - \beta^2 \end{bmatrix}, \tag{7}$$

where

$$y_{i1}^* + y_{i2}^* = y_i \,.$$

The admittance matrix of the transmission line of the upper part is

$$Y_l = j\eta \begin{pmatrix} -\cot\theta_s & \mathrm{cse}\,\theta_s \\ \mathrm{cse}\,\theta_s & -\cot\theta_s \end{pmatrix}, \tag{8}$$

where θ_s is the phase shift of the line and η is the characteristic admittance normalized to the input transmission line Z_{0s}. The overall admittance matrix of the nonreciprocal amplifier is then given by the summation of the two admittance matrices,

$$Y = Y_a + Y_l \,. \tag{9}$$

In order to obtain useful results, we set

$$\theta_s = \pi/2 + 2n\pi \,, \qquad \varphi = \pi/2 + 2m\pi \,,$$

having

$$Y = \begin{bmatrix} y_{s1} - \dfrac{\alpha^2}{y_i} & j\left(\eta - \dfrac{\alpha\beta}{y_i}\right) \\ j\left(\eta + \dfrac{\alpha\beta}{y_i}\right) & y_{s2} - \dfrac{\beta^2}{y_i} \end{bmatrix}. \tag{10}$$

To clarify the characteristics of this amplifier, we transform this matrix into the scattering matrix,

$$S = (1+Y)^{-1}(1-Y) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{11}$$

$$S_{11}\Delta = \{(1 - y_{s1}) y_i + \alpha^2\}\{(1 + y_{s2}) y_i - \beta^2\} - \eta^2 y_i^2 + \alpha^2\beta^2$$

$$S_{12}\Delta = -j2 y_i(\eta y_i - \alpha\beta)$$

$$S_{21}\Delta = -j2 y_i(\eta y_i + \alpha\beta)$$

$$S_{22}\Delta = \{(1+y_{s1})y_i - \alpha^2\}\{(1-y_{s2})y_i + \beta^2\} - \eta^2 y_i^2 + \alpha^2\beta^2 .$$
$$\Delta = \{(1+y_{s1})y_i - \alpha^2\}\{(1+y_{s2})y_i - \beta^2\} + \eta^2 y_i^2 - \alpha^2\beta^2 .$$

In this expression S_{11} represents the reflection coefficient at the input terminal, $|S_{12}|$ the backward transducer gain, and $|S_{21}|^2$ the forward transducer gain and $|S_{22}|$ represents the reflection coefficient at the output terminal. The fact, $S_{21} \neq S_{12}$, implies the nonreciprocity of this amplifier, which is already clear in Eq. (10) from $Y_{12} \neq Y_{21}$.

The physical significance of obtaining the nonreciprocity of gain may be understood as follows: In Fig. 1 the signal entered from the left is amplified by the negative resistance due to the first parametric element and comes out to the output terminal, one through the upper transmission line and the other through the cascade-connected parametric device. These two waves may correspond to the first term and the second of S_{21} in Eq. (11) respectively. These two waves are both retarded by $\theta = \varphi = \pi/2$, and are added in phase and dissipated at the load. Conversely the wave entered from the right is amplified and appears at the input terminal out of phase, one retarded by $\theta = \pi/2$ and the other advanced by $\varphi = \pi/2$. The backward gain, thus, are much reduced and, if the two amplitudes are equal, the backward gain completely vanishes.

(2) Condition for Stable Oparation

The characteristic equation of this amplifier is given by

$$f(j2\delta) \equiv \Delta = \{(1+g_{s1}+j2\delta Q_{s1})(1+j2\delta Q_i) - \alpha^2\}\{(1+g_{s2}+j2\delta Q_{s2})(1+j2\delta Q_i) - \beta^2\} + \eta^2(1+j2\delta Q_i)^2 - \alpha^2\beta^2 = 0 . \tag{12}$$

This equation has a common factor $1+j2\delta Q_i$, and is reduced to, by dividing Eq. (12) by that factor,

$$g(j2\delta) \equiv (1+j2\delta Q_i)\{(1+g_{s1}+j2\delta Q_{s1})(1+g_{s2}+j2\delta Q_{s2}) + \eta^2\} - \alpha^2(1+g_{s2}+j2\delta Q_{s2}) - \beta^2(1+g_{s1}+j2\delta Q_{s1}) = 0 \tag{13}$$

Since the equation $1+\lambda Q_i = 0$ has a definite stable root $\lambda = -1/Q_i$, this simplification is allowed. Solving Eq. (13) we have

$$2\delta = 0 \tag{14}$$

with

$$\alpha^2(1+g_{s2}) + \beta^2(1+g_{s1}) = (1+g_{s1})(1+g_{s2}) + \eta^2 , \tag{14'}$$

and

$$4\delta^2 = \frac{(1+g_{s1}-\alpha^2)(1+g_{s2}-\beta^2) - \alpha^2\beta^2 + \eta^2}{Q_i\{Q_{s1}(1+g_{s2}) + Q_{s2}(1+g_{s1})\} + Q_{s1}Q_{s2}} \tag{15}$$

with

$$Q_{s1}Q_{s2}(\alpha^2Q_{s2}+\beta^2Q_{s1}) = \{Q_{s1}(1+g_{s2})+Q_{s2}(1+g_{s1})\}$$
$$\times[Q_i^2\{(1+g_{s1})(1+g_{s2})+\eta^2\}+Q_i\{Q_{s1}(1+g_{s2})+Q_{s2}(1+g_{s1})\}+Q_{s1}Q_{s2}]\,.$$

The characteristic equation (12) or (13) has no unstable root when the pump power α^2 and β^2 are small. In reality, the characteristic equation

$$(1+\lambda Q_i)\{(1+g_{s1}+\lambda Q_{s1})(1+g_{s2}+\lambda Q_{s2})+\eta^2\} = 0$$

obtained by setting $\alpha^2=\beta^2=0$ in Eq. (13) has three roots

$$\lambda = -1/Q_i\,,\quad \frac{-\{Q_{s1}(1+g_{s2})+Q_{s2}(1+g_{s1})\}\pm\sqrt{\{Q_{s1}(1+g_{s2})-Q_{s2}(1+g_{s1})\}^2-4\eta^2}}{2Q_{s1}Q_{s2}}$$

which are all stable. The amplifier is then stable under the condition

$$\alpha^2(1+g_{s2})+\beta^2(1+g_{s1})<(1+g_{s1})(1+g_{s2})+\eta^2\,. \tag{16}$$

The oscillation frequency is given by $\delta=0$, the center frequency.

Under Ineq. (16), Eq. (15) is solved as

$$2\delta = \pm\sqrt{\frac{(1+g_{s1}-\alpha^2)(1+g_{s2}-\beta^2)-\alpha^2\beta^2+\eta^2}{Q_i\{Q_{s1}(1+g_{s2})+Q_{s2}(1+g_{s1})\}+Q_{s1}Q_{s2}}}\,. \tag{17}$$

This may give the order of bandwidth of the amplifier. When Eq. (14) holds[3]

$$d=\left[\frac{\partial g_r}{\partial 2\delta}\cdot\frac{\partial g_i}{\partial\alpha^2}-\frac{\partial g_i}{\partial 2\delta}\cdot\frac{\partial g_r}{\partial\alpha^2}\right]_{\delta=0}, \tag{14'}$$
$$= (1+g_{s2})[Q_i\{(1+g_{s1})(1+g_{s2})+\eta^2\}+Q_{s2}(1+g_{s1}-\alpha^2\}+Q_{s1}(1+g_{s2}-\beta^2)]>0\,.$$

The stable operation region is then shown in Fig. 2 according to Ineq. (16). It

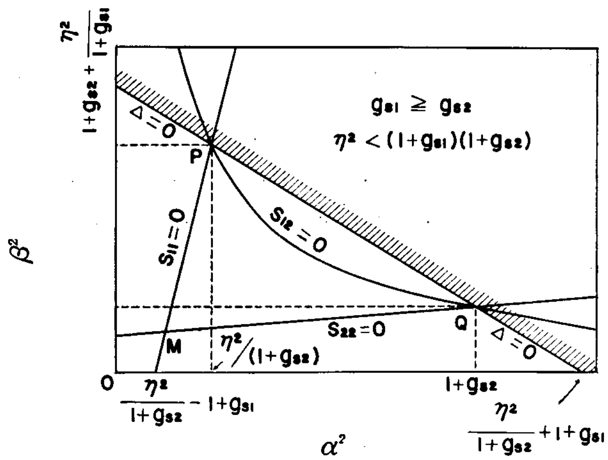

Fig. 2. Stable operation domain, the loci on which the amplifier is matched at the input or the output terminal and the locus of infinite backward insertion loss on $\alpha^2-\beta^2$ plane.

is assumed that $g_{s1} \geqq g_{s2}$, since the generality is not lost by doing so.

(3) Gain Characteristic

The ideal of this kind of amplifier may be that the forward gain $|S_{21}|^2$ is high and the backward gain $|S_{12}|^2$ is as low as possible. From Eq. (11) it is formally possible to make $S_{12}=0$ by choosing proper value of α^2 and β^2. The reflection gains at the input and output terminals must also be low. The boundary of stable operation ($\Delta=0$), the locus of infinite insertion loss in the backward direction ($S_{12}=0$), and the loci on which the amplifier is matched at the input or the output terminal ($S_{11}=S_{22}=0$) are shown in Fig. 2. If we express $|S|^2$ along z-axis with β^2 along x-axis and α^2 along y-axis, we could obtain a more vivid concept of the performance of the amplifier. The amplifier is stable inside the triangle enclosed by α^2-axis, β^2-axis and the line $\Delta=0$. The straight line $S_{11}=0$ intersects with the hyperbola $S_{12}=0$ at the point P just on the boundary of stable operation, and the straight line $S_{22}=0$ intersects with the hyperbola at the point Q. The point M at which the amplifier is matched at both terminals exists always inside the stable operating region. The match of this amplifier is thus always possible. The hyperbola $S_{12}=0$ enters into the stable operating region, if and only if

$$\eta^2 > (1+g_{s1})(1+g_{s2})\,. \tag{18}$$

In the limit of $\eta^2=(1+g_{s1})(1+g_{s2})$, the three points P, Q and M coincide on the boundary $\Delta=0$. That is, the ideal state where S_{11}, S_{22} and S_{12} simultaneously vanish can not be realized within the stable operation region. The forward gain $|S_{21}|^2$ is high with α^2 and β^2 increased. By selecting η^2 near $(1+g_{s1})(1+g_{s2})$ and increasing the pump power α^2 and β^2 near the stable operation boundary, better performance of the amplifier may be obtained, three points P, Q and M nearly coinciding with one another.

(4) Frequency Characteristic

It is difficult to discuss the frequency characteristic in general due to the complexity of Eq. (11). The frequency characteristic of the forward gain is supposed to be single peaked from the discussion in the previous paragraph (2). The forward gain at the center frequency is, from Eq. (11),

$$|S_{21}|^2 = \frac{4(\eta+\alpha\beta)^2}{(1+g_{s1}-\alpha^2)(1+g_{s2}-\beta^2)-\alpha^2\beta^2+\eta^2} \tag{19}$$

The gain-bandwidth product is approximately given by the multiplication of Eq. (19) by Eq. (17),

$$2\delta|S_{21}| = \frac{2(\eta+\alpha\beta)}{\sqrt{Q_i\{Q_{s1}(1+g_{s2})+Q_{s2}(1+g_{s1})\}+Q_{s1}Q_{s2}}} \tag{20}$$

An example of the calculated frequency characteristic of gains is shown in Fig. 3. The constants were determined as

$$g_{s1}=g_{s2}=0.5\,,\quad \eta=1.4\,,$$
$$\alpha=\beta^2=1.1\,,\quad Q_{s1}=Q_{s2}=Q_s\,,$$
$$Q_i/Q_s=0.5\,.$$

The forward transducer gain $|S_{21}|^2$ has a single peaked frequency characteristic, and the backward gain $|S_{12}|$ is less than unity having little of frequency characteristic. The reflection gains $|S_{11}|^2$ and $|S_{22}|^2$ are zeros at the center frequency and have small values elsewhere.

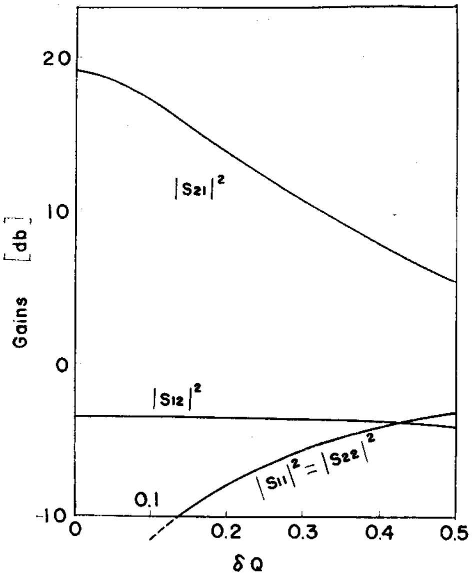

Fig. 3. Frequency characteristic of gains.

(5) Noise Figure

The noise figure of this amplifier is calculated as

$$F=1+g_{s1}+\frac{\omega_s}{\omega_i}\cdot\left\{\frac{\alpha\eta+\beta(1+g_{s1})}{\eta+\alpha\beta}\right\}^2+g_{s2}\left\{\frac{1+g_{s1}-\alpha^2}{\eta+\alpha\beta^2}\right\}. \tag{21}$$

The first and the second terms are the noise contributions from the input signal circuit. The third term is due to the idler loss conductance, which has the factor ω_s/ω_i similar to an ordinary parametric amplifier and can be made small by selecting the idler frequency as high as possible. The last term is due to the loss conductance in the output resonant circuit, which is negligibly small compared with the second term.

3. Analysis by use of a Signal Flow Graph

The amplifier mentioned thus far is not easy to realize as it is in Fig. 1. The more compact form to realize this in practice may be one as shown in Fig. 4.† Two parametric elements were imbedded in the transmission line at a proper

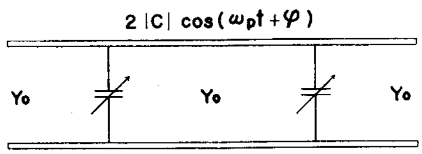

Fig. 4. Two parametric elements loaded in a transmission line.

† After the author's study J. Hamasaki published a paper similar to this device. Although his analysis is detailed, he did not take the circuit loss into account.

distance. The distance should be determined so as to provide $\pi/2$ phase shift for the signal wave and zero or π phase shift to the idler wave. The phase difference between the two pump supplies must be $\pi/2$.

The analisis by a signal flow graph was first suggested by S.J. Mason in 1953 associated with a feed-back system[4]. Later M.R. Leibowitz showed the usefluness of the analysis of microwave circuit.

We try, first, to find the scattering matrix of an admittance Y in Fig. 5, in which the currents I and voltages V are column matrices composed of the signal and idler components,

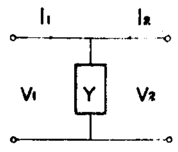

Fig. 5. Admittance representation of a parametric element.

$$I_1=\begin{pmatrix} I_{s1} \\ I_{i1}^* \end{pmatrix}, \qquad I_2=\begin{pmatrix} I_{s2} \\ I_{i2}^* \end{pmatrix}, \qquad V_1=V_2=\begin{pmatrix} V_s \\ V_i^* \end{pmatrix}. \tag{22}$$

There are relations between them

$$I_1+I_2=YV_1=YV_2, \qquad \text{or} \qquad V_1=V_2=Y^{-1}(I_1+I_2)$$

by Kirchhoff's theorem. We can write

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}=\begin{pmatrix} Y^{-1}(I_1+I_2) \\ Y^{-1}(I_1+I_2) \end{pmatrix}=Y^{-1}\begin{pmatrix} I_1+I_2 \\ I_1+I_2 \end{pmatrix}=Y^{-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}. \tag{23}$$

The inverse of the admittance matrix Y is, by Eq. (3),

$$Y^{-1}=\begin{pmatrix} 0 & -1/j\omega_i C^* \\ 1/j\omega_s C & 0 \end{pmatrix},$$

where the residual susceptances $j\omega C_0$ are assumed to be tuned out. The four-rank impedance matrix of Eq. (23) is written as

$$Z=\begin{pmatrix} 0 & -1/j\alpha_i & 0 & -1/j\alpha_i \\ 1/j\alpha_s & 0 & 1/j\alpha_s & 0 \\ 0 & -1/j\alpha_i & 0 & -1/j\alpha_i \\ 1/j\alpha_s & 0 & 1/j\alpha_s & 0 \end{pmatrix}, \tag{24}$$

which is normalized to the characteristic admittance Y_{0s} and Y_{0i} where

$$\alpha_s=\omega_s C/\sqrt{Y_{0s}Y_{0i}}, \qquad \alpha_i=\omega_i C^*/\sqrt{Y_{0s}Y_{0i}}.$$

The scattering matrix is then

$$S=(Z+1)^{-1}(Z-1)$$

or

$$\begin{pmatrix} b_{s1} \\ b_{i1}^* \\ b_{s2} \\ b_{i2}^* \end{pmatrix} = \frac{1}{4-\alpha^2} \begin{pmatrix} \alpha^2 & -j2\alpha_s & 4 & -j2\alpha_s \\ j2\alpha_i & \alpha^2 & j2\alpha_i & 4 \\ 4 & -j2\alpha_s & \alpha^2 & -j2\alpha_s \\ j2\alpha_i & 4 & j2\alpha_i & \alpha^2 \end{pmatrix} \begin{pmatrix} a_{s1} \\ a_{i1}^* \\ a_{s2} \\ a_{i2}^* \end{pmatrix}. \tag{25}$$

The signal flow graph of the amplifier in Fig. 4 is drawn as Fig. 6, where

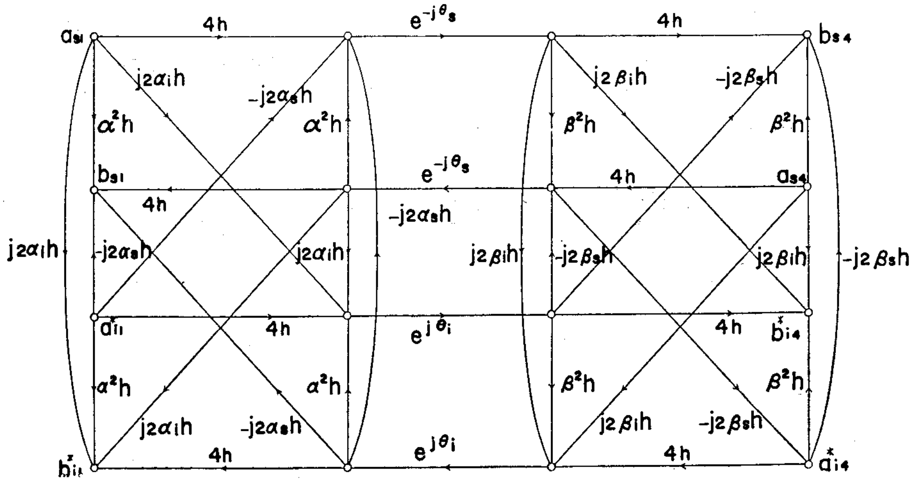

Fig. 6. Signal flow graph for Fig. 4.

$h=1/(4-\alpha^2)$. The left part represents the first parametric element and the right part the second parametric element. These two parts are coupled by a transmission line with phase shift angle θ_s for the signal wave and θ_i for the idler. Assuming the two pump powers are equal ($|\alpha|=|\beta|=r$), the determinant of Fig. 6 is written down

$$\Delta = h^2[r^4(1-e^{j2\theta_i})(1-e^{-j2\theta_s})-8r^2\{1+e^{j(\theta_i-\theta_s)}\cos\varphi\}+16]\,.$$

The element S_{11} is likewise obtained as

$$S_{11}\Delta = h^2r^2[-r^2(1-e^{j\theta_i})(1-e^{-j2\theta_s})+4\{2e^{j(\theta_i-\theta_s)}\cos\varphi+e^{-j2\theta_s}+1\}]\,.$$

$$S_{41}\Delta = 8h^2e^{j\theta_s}\{2+jr^2e^{j(\theta_i-\varphi)}\sin\theta_s\}\,.$$

The reflection coefficient at the input or output terminal is

$$S_{11} = S_{44} = \frac{-r^4(e^{-j2\theta_s}-1)(e^{j2\theta_i}-1)+4r^2\{1+e^{-j2\theta_s}+2e^{j(\theta_i-\theta_s)}\cos\varphi\}}{r^4(e^{-j2\theta_s}-1)(e^{j2\theta_i}-1)-8r^2\{1+e^{j(\theta_i-\theta_s)}\cos\varphi\}+1}\,, \tag{26}$$

and the forward and the backward gains are

$$S_{\substack{14\\41}} = \frac{8e^{-j\theta_s}\{2 \pm j\gamma^2 e^{j(\theta_i-\varphi)} \sin \theta_s\}}{\gamma^4(e^{-j2\theta_s}-1)(e^{j2\theta_i}-1)-8\gamma^2\{1+e^{j(\theta_i-\theta_s)}\cos\varphi\}+1}\,. \tag{27}$$

If we put here

$$\theta_s = \pi/2\,, \qquad \theta_i = 0\,, \qquad \text{and} \qquad \varphi = \pi/2\,,$$

we have

$$S_{11} = 0\,, \qquad S_{21} = -j\frac{2+\gamma^2}{2-\gamma^2} \qquad \text{and} \qquad S_{14} = -j\,.$$

This implies that the amplifier is matched at both terminals and the backward gain is unity. The forward gain characteristic is the same as that of an ordinary reflection type parametric amlifier. This result is consistent with that obtained in the previous section.

4. Nonreciprocal Parametric Amplifier by use of a Hybrid

In the foregoing sections the nonreciprocity of gain was obtained by coupling two parametric elements through the both waves. Next we shall consider the case where a hybrid is used in place of the transmission line shown in Fig. 7. The

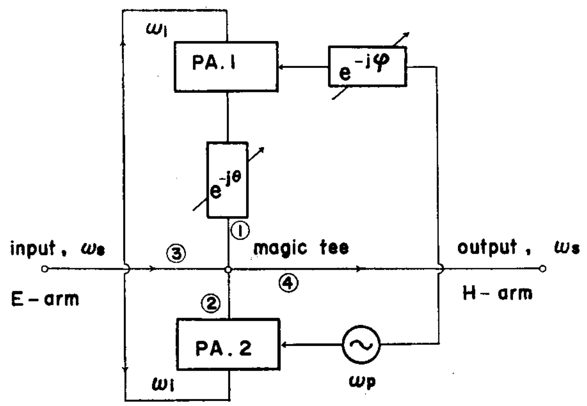

Fig. 7. Nonreciprocal parametric amplifier using a magic tee.

signal resonant circuits of two nondegenerate parametric amplifiers are coupled to the two main armes (1) and (2) of a magic tee and the idler resonant circuits of the two amplifiers are coupled by a transmission line, whose length is an integer multiple of a half-wave length. The two parametric elements are excited by a common pump source and the phase of difference φ between them can properly be adjusted by the use of a phase shifter. The direction of amplification is determined by the amount of this phase difference.

(1) Analysis

Denoting the incident wave by b' and the reflected wave by a' (note that the reflection between incident and reflected is reversed according to our standpoint, viz., seen from the parametric device or from the magic tee), the scattering matrix of the cascade connecte parametric amplitier is written as

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} b_1' \\ b_2' \end{pmatrix}, \tag{28}$$

where, using Eq. (6),

$$S_{11} = \frac{y_i(1-y_{s1})(1+y_{s2})+\alpha^2(1+y_{s2})-\beta^2(1-y_{s1})}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})},$$

$$S_{12} = \frac{2\alpha\beta e^{j\varphi}}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})},$$

$$S_{21} = \frac{2\alpha\beta e^{-j\varphi}}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})},$$

$$S_{22} = \frac{y_i(1+y_{s1})(1-y_{s2})-\alpha^2(1-y_{s2})+\beta^2(1+y_{s1})}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})}.$$

The scattering matrix when the phase shifter $e^{-j\theta}$ is inserted before the input terminal of PA. 1 is

$$\boldsymbol{S}_{12} = \begin{pmatrix} e^{-j\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} e^{-j\theta} & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

We can use the abbreviated form for the part inserted between the main arms of the magic tee (1) and (2) as

$$\boldsymbol{a}_{12} = \boldsymbol{S}_{12}\boldsymbol{b}_{12}, \tag{30}$$

where

$$\boldsymbol{a}_{12} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \boldsymbol{b}_{12} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{31}$$

Since the main arms (1) and (2) of an ideal magic tee is independent of the side arms (3) and (4), the scattering matrix of the magic tee

$$\boldsymbol{S} = \frac{j}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \tag{32}$$

can be resolved into two scattering matrices of rank two, i.e.,

$$\boldsymbol{b}_{12} = \frac{j}{\sqrt{2}}\boldsymbol{\sigma}\boldsymbol{a}_{34}, \qquad \boldsymbol{b}_{34} = \frac{j}{\sqrt{2}}\boldsymbol{\sigma}^t\boldsymbol{a}_{12} \tag{33}$$

with Eq. (31) and

$$\boldsymbol{b}_{34} = \begin{pmatrix} b_3 \\ b_4 \end{pmatrix}, \qquad \boldsymbol{a}_{34} = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix},$$

where the matrix of rank two is the submatrix of Eq. (32)

$$\boldsymbol{\sigma} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

and where the subscript t denotes the transposed matrix. Elimination of $\boldsymbol{a}_{12}$ and $\boldsymbol{b}_{12}$ from Eqs. (30) and (33) yields

$$\boldsymbol{b}_{34} = -\tfrac{1}{2}\,\boldsymbol{\sigma}^t\boldsymbol{S}_{12}\boldsymbol{\sigma}\boldsymbol{a}_{34}\,, \tag{34}$$

or

$$\begin{aligned}\boldsymbol{S}_{34} &\equiv \begin{pmatrix} S_{33} & S_{34} \\ S_{43} & S_{44} \end{pmatrix} = -\tfrac{1}{2}\,\boldsymbol{o}^t\boldsymbol{S}_{12}\boldsymbol{\sigma} \\ &= -\frac{1}{2}\begin{pmatrix} S_{11}e^{-j2\theta}-S_{12}e^{-j\theta}-S_{21}e^{-j\theta}+S_{22} & S_{11}e^{-j2\theta}+S_{12}e^{-j\theta}-S_{21}e^{-j\theta}-S_{22} \\ S_{11}e^{-j2\theta}-S_{12}e^{-j\theta}+S_{21}e^{-j\theta}-S_{22} & S_{11}e^{-j2\theta}+S_{12}e^{-j\theta}+S_{21}e^{-j\theta}+S_{22} \end{pmatrix}.\end{aligned} \tag{35}$$

Using further Eq. (28), and selecting the parameters θ and φ as

$$\theta = \pi/2+2n\pi\,, \qquad \varphi = \varphi_1-\varphi_2 = \pi/2+2m\pi\,,$$

we have

$$\begin{aligned}
S_{33} &= \frac{\alpha^2-\beta^2+y_i(y_{s2}-y_{s1})}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})} \\
S_{34} &= \frac{y_{s2}\alpha^2+{}_{s1}\beta^2-2\alpha\beta+y_i(1-y_{s1}y_{s2})}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})} \\
S_{43} &= \frac{y_{s2}\alpha^2+y_{s1}\beta^2+2\alpha\beta+y_i(1-y_{s1}y_{s2})}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})} \\
S_{44} &= \frac{\alpha^2-\beta^2+y_i(y_{s2}-y_{s1})}{y_i(1+y_{s1})(1+y_{s2})-\alpha^2(1+y_{s2})-\beta^2(1+y_{s1})}\,.
\end{aligned} \tag{36}$$

where

$$y_i = (y_{i1}+y_{i2})/2\,.$$

It is seen that the reflections $S_{33}=S_{44}$ are vanished when the two parametric amplifiers PA.1 and PA.2 are of the same characteristics. By setting

$$\alpha^2 = \beta^2 = \gamma^2\,, \qquad y_{s1} = y_{s2} = y_s$$

we have

$$S_{33} = S_{44} = 0\,,$$
$$S_{34} = \frac{1-y_s}{1+y_s}\,, \qquad S_{43} = \frac{(1-y_s)y_i+2r^2}{(1+y_s)y_i-2r^2}\,. \tag{37}$$

With the symmetrical construction, this amplifier is matched at both terminals and the backward gain $|S_{34}|^2$ is less than unity. The forward gain $|S_{43}|^2$ is in the same form as an ordinary reflection-type parametric amplifier.

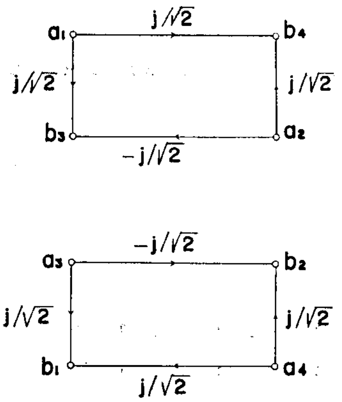

Fig. 8. Signal flow graph for a magic tee.

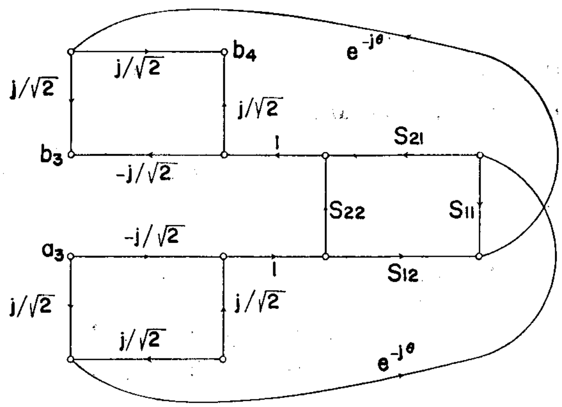

Fig. 9. Signal flow graph for the nonreciprocal parametric amplifier using a magic tee.

(2) Physical Interpretation

The scattering matrix (32) of a magic tee is represented by the flow graph of Fig. 8, which is divided into independent subgraphs corresponding to Eqs. (33). Combination of this graph with that of the parametric device (28) yields Fig. 9. Noting that there is no feedback loop in the graph, the gain from port (3) to port (4) becomes

$$\frac{b_4}{a_3} = S_{43} = -\tfrac{1}{2}\,(S_{11}e^{-j2\theta}-S_{12}e^{-j\theta}+S_{21}e^{-j\theta}-S_{22})\,, \tag{38}$$

and the reflection gain at port (3) is

$$\frac{b_3}{a_3} = S_{33} = -\tfrac{1}{2}\,(S_{11}e^{-j2\theta}-S_{12}e^{-j\theta}-S_{21}e^{-j\theta}+S_{22})\,. \tag{39}$$

The above results are the same as Eq. (35). The other elements S_{34} and S_{44} are obtained in the same way. The signla flow graph method is closely related to the physical significance of performance.

In Fig. 7 the input signal wave entered into port (3) of the magic tee is split off into ports (1) and (2) in reverse phase. These two waves are partly amplified and reflected back and partly converted into the idler wave. The reflected wave at port (1) appears at the junction point of the magic tee retarded by $2\theta=\pi$ in phase, while the reflected wave at port (2) appears at the junction point suffering no phase shift. Since the reflected waves are in phase (and in magnitude), these waves flow out of port (4). This corresponds to $S_{11}e^{-j2\theta}-S_{22}=-S_{11}-S_{22}$ in Eq. (38). The converted idler waves are again converted back to signal frequency by the other parametric elements being subjected to phase shift. The amount of the phase shift from port (1) to port (2) is $-\theta+(\varphi_2-\varphi_1)=-\pi/2-\pi/2=-\pi$, while the phase shift in the reverse direction is $(\varphi_1-\varphi_2)-\theta \doteq 0$. Because the input signal wave is divided in opposite phase, these two converted signal waves are also added up in phase into port (4). These situation corresponds to

$$\begin{aligned}-S_{12}e^{-j\theta}+S_{21}e^{-j\theta} &= -|S_{12}|e^{j(\varphi_1-\varphi_2)-j\theta}+|S_{21}|e^{j(\varphi_2-\varphi_1)-j\theta} \\ &= -|S_{12}|-|S_{21}|\end{aligned}$$

in Eq. (38). But it is not yet known whether the reflected wave and the converted wave be added up in phase into port (4). A little minute inspection of the phase tells us that this is true, that is, the wave reflected by PA.1 is subjected to the phase shift of $-\theta-\theta=-\pi$ and the converted wave from PA. 1 to PA. 2 to the phase shift of $-\theta+(\varphi_2-\varphi_1)=-\pi$. This is illustrated by writing

$$\begin{aligned}S_{43} &= -(S_{11}e^{-j\pi}-|S_{11}|e^{j\pi/2}e^{-j\pi/2}+|S_{21}|e^{-j\pi/2}e^{-j\pi/2}-S_{22})/2 \\ &= (S_{11}+|S_{12}|+|S_{21}|+S_{22})/2\,. \end{aligned} \tag{40}$$

The case where the input signal is applied to port (4) is likewise interpreted obtaining

$$S_{34}=(S_{11}-|S_{12}|-|S_{21}|+S_{22}) \rightarrow 0\,.$$

(3) Stability Condition and Gain Characteristic

The characteristic equation of this amplifier is obtained from the denominator of Eq. (36),

$$\begin{aligned}f(j2\delta) \equiv \Delta = (1+j2\delta Q_i)(1+g_{s1}+j2\delta Q_{s1})(1+g_{s2}+j2\delta Q_{s2}) \\ -\alpha^2(1+g_{s2}+j2\delta Q_{s2})-\beta^2(1+g_{s1}+j2\delta Q_{s1})\,.\end{aligned} \tag{41}$$

This is identical to Eq. (13), if we set $\eta=0$ in it. The stability condition is then given by

$$\alpha^2(1+g_{s2})+\beta^2(1+g_{s1})<(1+g_{s1})(1+g_{s2})\,. \tag{42}$$

The stable operating domain is shown in Fig. 10.

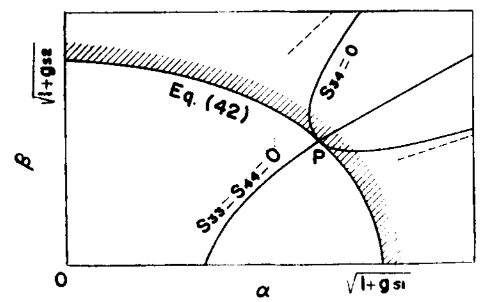

Fig. 10. Stable operation domain on $\alpha-\beta$ plane and the locus on which the amplifier is matched and the locus of infinite backward insertion loss.

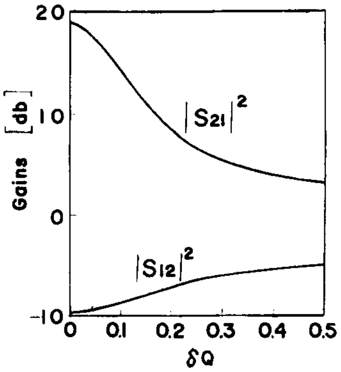

Fig. 11. Frequency characteristic of gains of the nonreciprocal parametric amplifier with a magic tee.

The locus on which the amplifier is matched ($S_{33}=S_{44}=0$) and that of the infinite backward loss ($S_{34}=0$) at the center frequency are also shown in Fig. 10 in case the parameters of the two element amplifiers are not necessarily the same ($g_{s1}\geqq g_{s2}$). The hyperbola $S_{33}=S_{44}=0$ crosses at point P with the hyperbola (42). Since the amplification gain is high near the stability boundary, it is desireble to operate the amplifier close to the point P.

An example of frequency characteristic of gains is shown in Fig. 11. The parameters are fixed as

$$\alpha=\beta=1.14\,,\qquad g_{s1}=g_{s1}=0.5$$
$$Q_{s1}=Q_{s2}=Q\,,\qquad k=0.5\,.$$

The reflection gains are always zero in an ideal case.

(4) Noise Figure

Noise figure can easily be calculated by use of the signal flow graph method. The result is

$$F=1+\frac{g_{s1}+g_{s2}}{2}+\frac{\omega_s}{\omega_i}\left\{\frac{2\alpha(1+g_{s2})+2\beta(1+g_{s1})}{\alpha^2 g_{s2}+\beta^2 g_{s1}+2\alpha\beta+2-2g_{s1}g_{s2}}\right\}^2.$$

This is the same order with that of an ordinary reflection type parametric amplifier. In the limit of $g_{s1}=g_{s2}\rightarrow 0$, $\alpha^2=\beta^2\rightarrow 1$, the noise figure F approaches to $1+\omega_s/\omega_i$, the minimum noise figure of a parametric amplifier.

5. Experimental Proof of the Nonreciprocal Parametric Amplifier

Since the nonreciprocal parametric amplifier described in Sec. 2 or 3 is difficult to construct, the experiment of the type described in Sec. 3 was carried out. The block diagram of the experiment is shown in Fig. 12. The two idler tuned circuits are coupled through a phase shifter PH_i and an attenuator ATTi. The proper idler coupling phase is determined by experiment. Pump power is supplied from a common klystron V-55 at 8.6 GC.

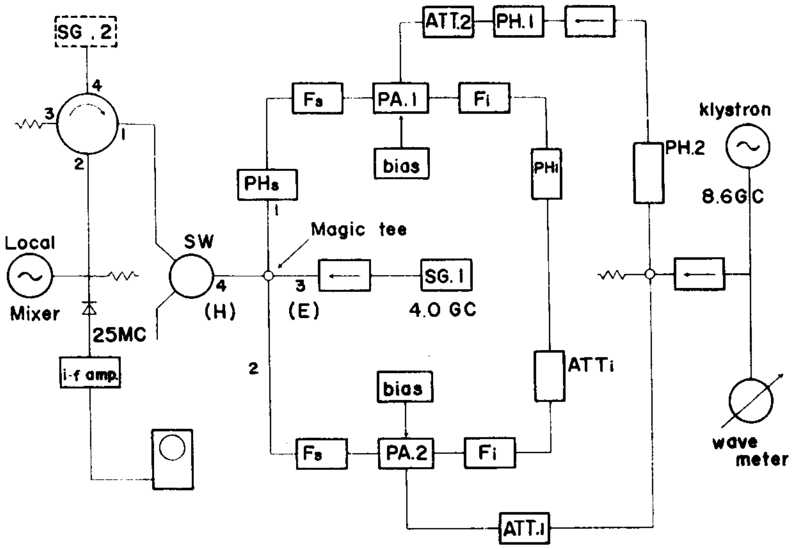

Fig. 12. Block diagram of experiment.

Microwave signal from the signal generator at 4.0 GC is impressed to the E-arm of the magic tee, which is divided into the two parametric amplifiers PA.1 and PA.2, and amplified signal appears at the H-arm. This signal is detected by a heterodyne receiver with a local oscillator and an i-f amplifier.

If the attenuator ATT_i is set to the maximum, the idler coupling is shut off and the amplifier will operate as a bilateral amplifier. In this situation the phase shifter PH_i before a varactor mount PA.1 is adjusted to the best condition, that is, the transducer gain from the E-arm to the H-arm should be high and the reflection gain at the E- or H-arm should be small. The pump phase difference (PH.1 and PH.2) and the idler coupling phase (PH_i) are determined by cut-and-try method. The relation of gains to the pump phase difference is seen from Eq. (35). Setting

$$\alpha = \beta = \gamma\,, \qquad y_{s1}=y_{s2} \qquad \text{and} \qquad \theta=\pi/2\,,$$

it becomes

$$|S_{33}|^2 = \frac{\gamma^4\cos^2\varphi}{(1-\gamma^2)^2}\,, \qquad |S_{34}|^2 = \left(\frac{1-\gamma^2\sin\varphi}{1-\gamma^2}\right)^2,$$

$$|S_{43}|^2 = \left(\frac{1+\gamma^2\sin\varphi}{1-\gamma^2}\right), \qquad |S_{44}|^2 = \frac{\gamma^4\cos^2\varphi}{(1-\gamma^2)^2}\,.$$

It is seen above that the forward gain $|S_{43}|^2$ interchanges with the backward gain $|S_{34}|^2$ as the pump phase φ is changed. The difference between these two gains is maximum when $\varphi=\pi/2+m\pi$ and simultaneously the reflection gains $|S_{33}|^2$ and $|S_{44}|^2$ are zero. When $\varphi=m\pi$ the backward gain is equal to the forward gain and the reflection gains are maximum. An example of experimental results is shown in Fig. 13.

Fig. 14 is the frequency characteristic of gains. The backward and the reflected gains are always less than unity.

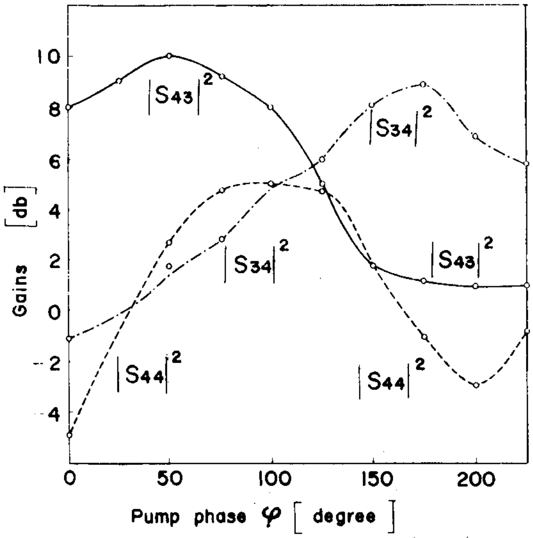

Fig. 13. Gains as a function of the pump phase difference φ.

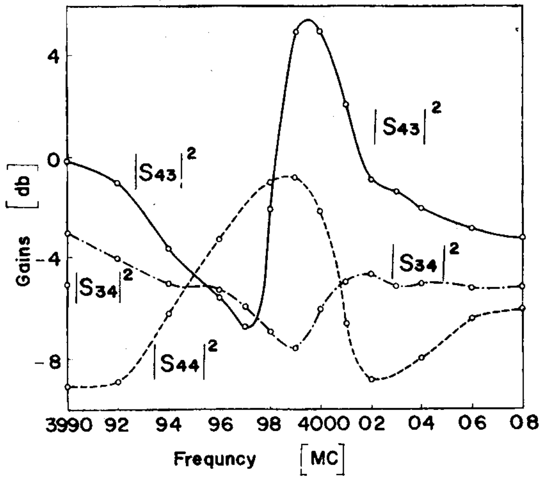

Fig. 14. Frequency characteristic of gains of the non-reciprocal parametric amplifier.

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