On the Torsional Distorsions of Long-spanned, Four-walled Trussed Structures

By

Naruhito SHIRAISHI* and Ei-ichi WATANABE*

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In this paper the torsional deformation of four walled stiffening girders is investigated from the theoretical point of view and the results are compared with experimental results performed on the two types of models, namely the so-called plate-girder type model and the truss type model. The Torsion-box theory is applied for analysis of torsional behaviour of four trussed-walled box girder beams. However the truss type girder can resist torsion less compared with the ideal thin-walled girder and can be characterized by the modified form of the Two-spar theory rather than the Torsion-box theory.

1. Introduction

In connection with torsional deformation of stiffening girders of long-spanned suspension bridges, F. Bleich[1], Nan sze Sih[2], etc., applied the so called shear flow theory for estimation of torsional rigidity of girders by replacing trussed walls by equivalent thin walls. Recently T. Okumura & H. Watanabe[3] investigated the torsional rigidity of the truss bridge by taking into account the warping displacements proportional to the twisting angle. In spite of a number of investigations on this problem there appears to be no satisfactory theory on the torsional behaviours of trussed girders because of the fact that the truss system of long span can possibly resist less continuously to compared with the ideal thin-walled systems.

This paper present an investigation of the torsional deformation of trussed structures (i.e., girders) of four walled cross section mainly from a theoretical point of views. Experimental analysis was also performed in order to confirm the theoretical characteristics of torsional resistance of trussed girders by use of the model tests.

2. Theoretical Formulation of Torsional Deformations of Trussed Girder of Four Walled Cross Section

2.1 Theoretical analysis for an idealized continuous system

Torsional behaviour of truss girder of rectangular cross section is analyzed in

* Department of Civil Engineering

aeronautical engineering for design of wings and in civil engineering for design of stiffening girders of long-spanned suspension bridge. For analysis of torsion of four walled truss system, the system is generally idealized as either a torsion box or two spars with cross beams. Historically the latter idealization, namely the so called Two Spar Theory, anteceded the former, the Torsion Box Theory, and the latter theory is also considered mathematically to include the former theory, while the differences between two theories are often small[4]. In this investigation, we first study the torsion problem of four walled girder in terms of the Torsion Box Theory for the sake of briefness under the following assumptions:

1) The four flanges are capable of carrying the longitudinal normal stresses, while the walls carry only shear stresses.
2) The wall thickness and the cross sectional areas of flanges are constant between the bulkheads.
3) The cross sections are doubly symmetrical rectangular.
4) The width and the depth of the cross section remains constant along the span length.
5) The bulkheads are rigid against deformations within their planes with no resistance to deformations normal to their planes.

Assuming that the cross section of the system consists of four flanges with four thin walled webs as shown in Fig. 1, the equilibrium of forces in the longitudinal direction of the flanges is written as

$$\frac{dP}{dx} = \tau_v t_v - \tau_h t_h \tag{1}$$

and the stress strain relation for flanges is written as

$$P = -EA_0 \frac{du}{dx} \tag{2}$$

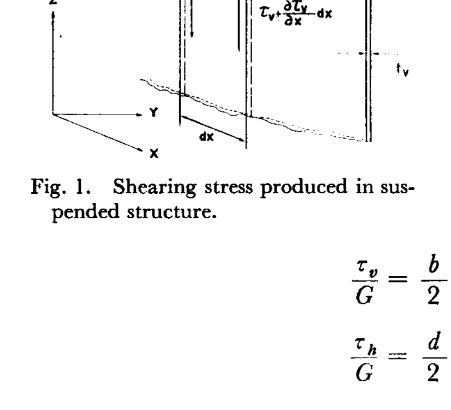

Fig. 1. Shearing stress produced in suspended structure.

The relationships for shear stresses and the distortions are of the form

$$\left.\begin{aligned} \frac{\tau_v}{G} &= \frac{b}{2}\frac{d\phi}{dx} - \frac{2u}{d} \\ \frac{\tau_h}{G} &= \frac{d}{2}\frac{d\phi}{dx} + \frac{2u}{b} \end{aligned}\right\} \tag{3}$$

and the torque moment is expressed as

$$M_t = (\tau_v t_v + \tau_h t_h) bd \tag{4}$$

By use of eq's (1)$\sim$(4), the elimination of ϕ' yields to

$$\frac{d^2u}{dx^2} - \frac{8G}{\alpha EA_0}u + \frac{\beta}{\alpha EA_0 bd}M_t = 0 \tag{5}$$

and one also has alternatively

$$\left.\begin{aligned} \frac{d\phi}{dx} &= \frac{\alpha}{2Gb^2d^2}M_t - \frac{\alpha EA_0}{2Gbd}u'' \\ \text{or}\qquad \frac{d^3\phi}{dx^3} - \frac{8G}{\alpha EA_0}\frac{d\phi}{dx} + \frac{4}{EA_0b^2d^2}M_t &= \frac{\alpha-\beta}{8Gb^2d^2}M_t'' \end{aligned}\right\} \tag{6}$$

where

$$\alpha = \frac{b}{t_h} + \frac{d}{t_v}, \quad \beta = \frac{b}{t_h} - \frac{d}{t_v} \tag{7}$$

Eq's (5) and (6) are thus considered as a set of fundamental equations for torsion problems of thin walled box girder and it is indicated from them that one can approximately estimate the torsional rigidity of a girder by taking into account only the lower order terms of deformations; in other words, the warping of the chord member u and the rate of twisting angle $d\phi/dx$ can be approximated as proportional to the applied twisting moment M_t. In derivation of the fundamental equations (5) and (6) it is assumed that the bulkheads are rigid enough to annihilate the shear deformations. If the shear deformations of the bulkhead are considered, one of the fundamental equations becomes the 4th order differential equation instead of the 2nd order differential equation, as shown in the paragraph 4.

2.2 Theoretical analysis of the four walled truss system (Fig. 2)

In order to express the forces in every member in terms of distorsions, a unit truss system bounded by the bulkheads is considered as shown in Fig. 3. Let the

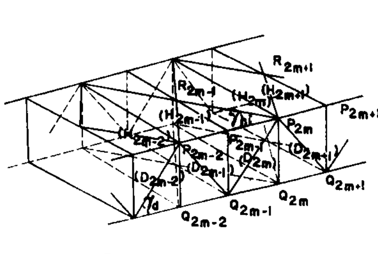

Fig. 2. Panel points of framework.

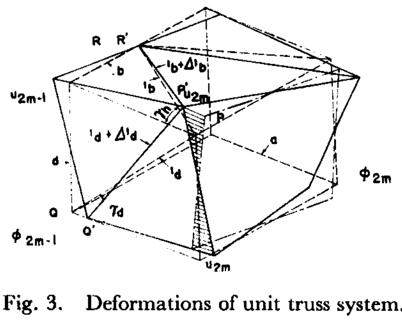

Fig. 3. Deformations of unit truss system.

longitudinal displacement u_j and the angle of distorsion ϕ_j be small, then the strains of the diagonal members in vertical plane, $(\varepsilon_h)_{2m}$, and those in horizontal plane, $(\varepsilon_d)_{2m}$, are written as

$$(\varepsilon_d)_{2m} = -\frac{\sin\gamma_d\cos\gamma_d}{2d}\left[2(u_{2m}+u_{2m-1})-bd\frac{\phi_{2m}-\phi_{2m-1}}{a}\right] \tag{8}$$

$$(\varepsilon_h)_{2m} = -\frac{\sin\gamma_h\cos\gamma_h}{2b}\left[4u_{2m}+bd\frac{\phi_{2m}-\phi_{2m-1}}{a}\right] \tag{9}$$

Now we can obtain the fundamental equations, considering the equilibrium of longitudinal forces at every panel points in terms of the respective strain components such as expressed by eq's (8) and (9).

1) At Point P_{2m}

The equilibrium of longitudinal forces requires

$$P_{2m+1}-P_{2m}-D_{2m}\cos\gamma_d-H_{2m}\cos\gamma_h+D_{2m+1}\cos\gamma_d+H_{2m+1}\cos\gamma_h = 0 \tag{10}$$

where we have

$$\left.\begin{aligned} P_j &= -EA_0\frac{u_j-u_{j-1}}{a} \\ D_j &= -EA_v\frac{\sin\gamma_d\cos\gamma_d}{2d}\left\{2(u_j+u_{j-1})-bd\frac{\phi_j-\phi_{j-1}}{a}\right\} \\ H_j &= -EA_h\frac{\sin\gamma_h\cos\gamma_h}{2b}\left\{4u_j+bd\frac{\phi_j-\phi_{j-1}}{a}\right\} \end{aligned}\right\} \quad j=2m,\ 2m+1$$

On the other hand the torque moment at the j th panel connection (Fig. 2) is written as

$$\begin{aligned}(M_t)_j \equiv M_j &= D_jb\sin\gamma_d-2H_jd\sin\gamma_h \\ &= \frac{1}{2}Gbdt_vt_h\alpha\frac{\phi_j-\phi_{j-1}}{a}-2Gt_vt_h\beta u_j+Gbt_v(u_j-u_{j-1})\end{aligned} \tag{11}$$

for $j=2m,\ 2m+1$

where $$t_v = \frac{EA_v}{Gd}\sin^2\gamma_d\cos\gamma_d,\quad t_h = \frac{2EA_h}{Gb}\sin^2\gamma_h\cos\gamma_h \tag{12}$$

$$\left.\begin{aligned}\alpha &= \frac{b}{t_h}+\frac{d}{t_v} \\ \beta &= \frac{b}{t_h}-\frac{d}{t_v}\end{aligned}\right\} \tag{13}$$

Hence, eliminating ϕ_j and ϕ_{j-1} from eq's (10) and (11), the fundamental equation with respect to the panel point P_{2m} is written as

$$\frac{1}{(2a)^2}\{u_{2m+2}-2u_{2m}+u_{2m-2}\}-\frac{8G}{(\alpha EA_0-2Ga^2)}u_{2m}+\frac{1}{2}\left(\frac{\beta}{\alpha}\right)\frac{M_{2m}+M_{2m+1}}{bd\left(EA_0-2\dfrac{Ga^2}{\alpha}\right)}=0 \quad (14)$$

2) At Point Q_{2m}

Similarly as previously, the longitudinal equilibrium of forces requires

$$P_{2m+1}-P_{2m}+(H_{2m+1}-H_{2m})\cos\gamma_h=0 \quad (15)$$

Thus we have, in exactly the same manner,

$$\frac{1}{a^2}\{u_{2m+1}-2u_{2m}+u_{2m-1}\}-\frac{8G}{(\alpha EA_0-Ga^2)}u_{2m}+\frac{1}{2}\left(1-\frac{\beta}{\alpha}\right)\frac{M_{2m}+M_{2m-1}}{bd\left(EA_0-\dfrac{Ga^2}{\alpha}\right)}=0 \quad (16)$$

3) At Point Q_{2m-1}

The longitudinal equilibrium of forces requires

$$P_{2m}-P_{2m-1}+(D_{2m}-D_{2m-1})\cos\gamma_d=0 \quad (17)$$

which, therefore, is reduced to the fundamental equation with respect to the panel point Q_{2m-1} as follows

$$\frac{1}{a^2}\{u_{2m}-2u_{2m-1}+u_{2m-2}\}-\frac{8G}{\left\{\alpha EA_0-\dfrac{3}{4}Ga^2\left(1-\dfrac{\beta^2}{\alpha^2}\right)\right\}}u_{2m-1}$$
$$+\frac{1}{2}\left(1+\frac{\beta}{\alpha}\right)\frac{\alpha(M_{2m}+M_{2m-1})}{bd\left\{\alpha EA_0-\dfrac{3}{4}Ga^2\left(1-\dfrac{\beta^2}{\alpha^2}\right)\right\}}=0 \quad (18)$$

It should be noted that eq's (14), (16) and (18) yield to the following forms of expression when a, the length of unit panel, tends to zero; namely

$$u''-K^2u+\frac{1}{2}\left(\frac{\beta}{\alpha}\right)\frac{2M_t}{EA_0bd}=0 \quad (14')$$

$$u''-K^2u+\frac{1}{2}\left(1-\frac{\beta}{\alpha}\right)\frac{2M_t}{EA_0bd}=0 \quad (16')$$

$$u''-K^2u+\frac{1}{2}\left(1+\frac{\beta}{\alpha}\right)\frac{2M_t}{EA_0bd}=0 \quad (18')$$

in which eq (14′) coincide exactly with the form of eq. (5).

3. Torsional Analysis for the Model Beams in Terms of Torsion-box Theory

The deformations of stressed beam for this case are given by eq. (14′) with the boundary conditions (Fig. 4) as

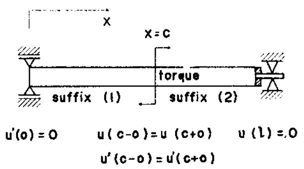

Fig. 4. Boundary conditions.

$$\begin{aligned} &\text{i) } u(l) = 0 && \text{ii) } u(c-o) = u(c+o) \\ &\text{iii) } u'(o) = 0 && \text{iv) } u'(c-o) = u'(c+o) \end{aligned} \tag{19}$$

The warping u is, thus, found to be

$$\left.\begin{aligned} u_1(x) &= \frac{1}{4}R\frac{bd}{GJ_T}M_t\{1-\tanh Kl \sinh Kc\,(\coth Kl \coth Kc-1)\cosh Kx\} \\ u_2(x) &= \frac{1}{4}R\frac{bd}{GJ_T}M_t \sinh Kc(\tanh Kl \cosh Kx-\sinh Kx) \end{aligned}\right\} \tag{20}$$

and the angle of twist is given as

$$\left.\begin{aligned} \phi_1(x) &= \frac{M_t}{GJ_T}\left\{x-\frac{1}{K}R^2(\cosh Kc-\tanh Kl \sinh Kc)\sinh Kx\right\} \\ \phi_2(x) &= \frac{M_t}{GJ_T}\left\{c-\frac{1}{K}R^2(\tanh Kl \sinh Kx-\cosh Kx)\sinh Kc\right\} \end{aligned}\right\} \tag{21}$$

Vertical shear forces are

$$\left.\begin{aligned} S_{v1}(x) &= \frac{M_t}{2b}\{1+R\tanh Kl \sinh Kc(\coth Kl \coth Kc-1)\cosh Kx\} \\ S_{v2}(x) &= -\frac{M_t}{2b}R\sinh Kc(\tanh Kl \cosh Kx-\sinh Kx) \end{aligned}\right\} \tag{22}$$

and the horizontal shear forces are

$$\left.\begin{aligned} S_{h1} &= \frac{M_t}{2d}\{1-R\tanh Kl \sinh Kc(\coth Kl \coth Kc-1)\cosh Kx\} \\ S_{h2} &= \frac{M_t}{2d}R\sinh Kc\,(\tanh Kl \cosh Kx-\sinh Kx) \end{aligned}\right\} \tag{23}$$

Strains in chord member are

$$\left.\begin{aligned} \varepsilon_1(x) &= R\frac{M_t}{KEbdA_0}\tanh Kl \sinh Kc(\coth Kl \coth Kc-1)\sinh Kx \\ \varepsilon_2(x) &= R\frac{M_t}{KEbdA_0}\sinh Kc \tanh Kl(\coth Kl \coth Kx-1)\sinh Kx \end{aligned}\right\} \tag{24}$$

where R is the ratio defined by

$$R = \beta/\alpha$$

Similar expressions are obtained for another set of such boundary conditions (Fig. 5) as

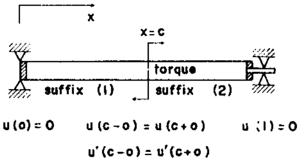

Fig. 5. Boundary conditions.

$$\begin{array}{ll} \text{i)}\ u(l) = 0 & \text{ii)}\ u(c-o) = u(c+o) \\ \text{iii)}\ u(o) = 0 & \text{iv)}\ u'(c-o) = u'(c+o) \end{array} \tag{25}$$

The warping u which is the solution of eq. (14′) under the conditions (25) is found to be

$$u_1(x) = \frac{1}{4}R\frac{M_t bd}{GJ_T}\left[1-\cosh Kx + \{\sinh Kc - \coth Kl(\cosh Kc-1)\}\sinh Kx\right]$$

$$u_2(x) = \frac{1}{4}R\frac{M_t bd}{GJ_T}\left[(\cosh Kc-1)\cosh Kx - \coth Kl(\cosh Kc-1)\sinh Kx\right]$$

The angle of twist is

$$\phi_1(x) = \frac{M_t}{GJ_T}\left[x - \frac{R^2}{K}\{\sinh Kx - (\sinh Kc - \coth Kl\cosh Kc + \coth Kl)(\cosh Kx - 1)\}\right]$$

$$\phi_2(x) = \frac{M_t}{GJ_T}\left[c - \frac{R^2}{K}\{\sinh Kc - (\sinh Kx - \coth Kl\cosh Kx + \coth Kl)(\cosh Kc - 1)\}\right]$$

Shear forces are

$$S_{v1}(x) = \frac{M_t}{2b}\left[1 + R\{\cosh Kx - 1(\sinh Kc - \coth Kl\cosh Kc + \coth Kl)\sinh Kx\}\right]$$

$$S_{v2}(x) = -R\frac{M_t}{2b}(\cosh Kc-1)(\cosh Kx - \coth Kl\sinh Kx)$$

$$S_{h1}(x) = \frac{M_t}{2b}\left[1 - R\{\cosh Kx - (\sinh Kc - \coth Kl\cosh Kc + \coth Kl)\sinh Kx\}\right]$$

$$S_{h2}(x) = R\frac{M_t}{2d}(\cosh Kc-1)(\cosh Kx - \coth Kl\sinh Kx)$$

Strains in chord memeber are

$$\varepsilon_1(x) = R\frac{M_t}{KEbdA_0}\left[\sinh Kx - \{\sinh Kc - \coth Kl(\cosh Kc-1)\}\cosh Kx\right]$$

$$\varepsilon_2(x) = R\frac{M_t}{KEbdA_0}(\cosh Kc-1)(\coth Kl\cosh Kx - \sinh Kx)$$

where

$$J_T = \frac{2Gb^2d^2}{\alpha}$$

4. The Idealized Analysis Taking the Shear Deformations of Bulkheads into Considerations

The above mentioned consideration is so far restricted to the torsional deformations of a rectangular girder under the assumption that there is no shear deformations of bulkheads to be taken into account, while, taking the shear deformations of bulkheads into consideration, the fundamental equation is derived as follows (Fig. 6):

$$\left\{\begin{aligned} P &= -EA_0\frac{du}{dx} \\ \frac{dP}{dx} &= \tau_v t_v - \tau_h t_h \\ \frac{\tau_v}{G} &= \frac{b}{2}\left(\frac{d\phi}{dx} - \frac{d\psi}{dx}\right) - \frac{2u}{d} \\ \frac{\tau_h}{G} &= \frac{d}{2}\left(\frac{d\phi}{dx} + \frac{d\psi}{dx}\right) + \frac{2u}{b} \end{aligned}\right.$$

and

$$\left\{\begin{aligned} M_t &= (\tau_v t_v + \tau_h t_h)bd \\ \tau_f &= 2\tilde{G}\psi \\ \frac{d}{dx}(\tau_v t_v) &= -\tau_f - \frac{P_z}{d} \\ \frac{d}{dx}(\tau_h t_h) &= -\tau_f - \frac{P_y}{b} \end{aligned}\right.$$

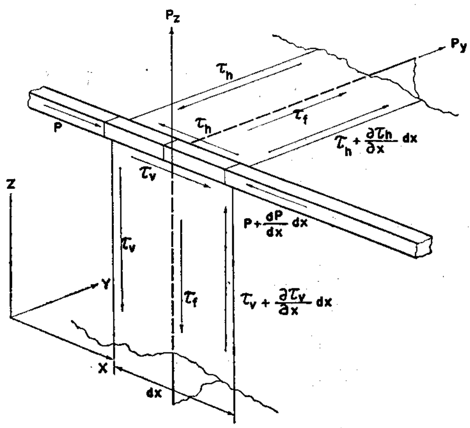

Fig. 6. Equilibrium state of stresses.

where $\tilde{G}$ and ψ are the shear modulus and the shear deformation of bulkhead. The above set of quations yields to

$$\frac{d^4u}{dx^4} - 2\frac{\tilde{G}\alpha}{Gbd}\frac{d^2u}{dx^2} + \frac{16}{bd}\frac{\tilde{G}}{EA_0}u = -\frac{2\beta\tilde{G}}{GEA_0b^2d^2}M_t + \frac{1}{EA_0}\left(\frac{1}{d}\frac{dP_z}{dx} - \frac{1}{b}\frac{dP_y}{dx}\right) \tag{26}$$

and

$$\frac{d\phi}{dx} = \frac{\alpha}{2Gb^2d^2}M_t - \frac{\beta EA_0}{2Gbd}\frac{d^2u}{dx^2} \tag{27}$$

Since eq. (26) is of the form of 4th order differential equation, the solution can be characterized according to the coefficients of the left handed sides of eq. (26); namely we have three types of solutions depending on the inequality expression of eq. (28),

$$\tilde{G}/G \gtreqless \frac{8bd}{(1+\nu)\alpha^2 A_0} \tag{28}$$

Considering eq. (28), the specific form of expressions for deformations and sectional forces of our torsion problem is obtained for two types of boundary conditions as shown in Fig's 7 and 8.

(i) When a concentrated torque is applied at $x=c$ (Fig. 7)

In this case the boundary conditions are given as specified in Fig. 7 and we have the following form of expressions.

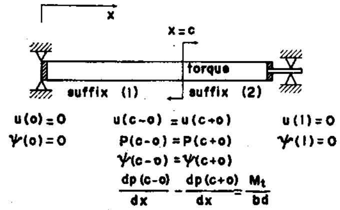

Fig. 7. Boundary conditions.

I) When $\tilde{G}/G < 8bd/(1+\nu)A_0\alpha^2$

Warping

$$\begin{cases} u_1(x) = \dfrac{\beta M_t}{8bdG}\left[1+\dfrac{8G}{\beta EA_0}(k_1e^{\lambda x}\sin\mu x+k_2e^{\lambda x}\cos\mu x+k_3e^{-\lambda x}\sin\mu x+k_4{}^{-\lambda x}e\cos\mu x\right] \\ u_2(x) = \dfrac{M_t}{EA_0bd}(k_5e^{\lambda x}\sin\mu x+k_6e^{\lambda x}\cos\mu x+k_7e^{-\lambda x}\sin x\mu+k_8e^{-\lambda x}\cos\mu x) \end{cases}$$

Angle of twist

$$\begin{cases} \phi_1(x) = \dfrac{M_t}{GJ_T}\Big\{x+\left(\dfrac{\beta}{\alpha}\right)[k_1(\lambda e^{\lambda x}\sin\mu x+\mu e^{\lambda x}\cos\mu x-\mu)+k_2(\lambda e^{\lambda x}\cos\mu x \\ \quad -\mu e^{\lambda x}\sin\mu x-\lambda)+k_3(-\lambda e^{-\lambda x}\sin\mu x+\mu e^{-\lambda x}\cos\mu x-\mu) \\ \quad +k_4(-\lambda e^{-\lambda x}\cos\mu x-\mu e^{-\lambda x}\sin\mu x+\lambda)]\Big\} \\ \phi_2(x) = \dfrac{M_t}{GJ_T}\Big\{c+\left(\dfrac{\beta}{\alpha}\right)[k_1(\lambda e^{\lambda c}\sin\mu c+\mu e^{\lambda c}\cos\mu c-\mu) \\ \quad +k_2(\lambda e^{\lambda c}\cos\mu c-\mu e^{\lambda c}\sin\mu c-\lambda)+k_3(-\lambda e^{-\lambda c}\sin\mu c+\mu e^{-\lambda c}\cos\mu c-\mu) \\ \quad +k_4(-\lambda e^{-\lambda c}\cos\mu c-\mu e^{-\lambda c}\sin\mu c+\lambda)-k_5(\lambda e^{\lambda c}\sin\mu c+\mu e^{\lambda c}\cos\mu c-\mu) \\ \quad -k_6(\lambda e^{\lambda c}\cos\mu c-\mu e^{\lambda c}\sin\mu c-\lambda)-k_7(\lambda e^{-\lambda c}\sin\mu c+\mu e^{-\lambda c}\cos\mu-\mu) \\ \quad -k_8(-\lambda e^{-\lambda c}\cos\mu c-\mu e^{-\lambda c}\sin\mu c+\lambda)+k_5(\lambda e^{\lambda x}\sin\mu x+\mu e^{\lambda x}\cos\mu x-\mu) \\ \quad +k_6(\lambda e^{\lambda x}\cos\mu x-\mu e^{\lambda x}\sin\mu x-\lambda)+k_7(-\lambda e^{-\lambda x}\sin\mu x+\mu e^{-\lambda x}\cos\mu x-\mu) \\ \quad +k_8(-\lambda e^{-\lambda x}\cos\mu x-\mu e^{-\lambda x}\sin\mu x+\lambda)]\Big\} \end{cases}$$

Shearing forces

$$\begin{cases} S_{v1}(x) = \dfrac{M_t}{2b}[1-k_1(\lambda^2\sin\mu x+2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{\lambda x}-k_2(\lambda^2\cos\mu x \\ \qquad -2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{\lambda x}-k_3(\lambda^2\sin\mu x-2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{-\lambda x} \\ \qquad -k_4(\lambda^2\cos\mu x+2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{-\lambda x}] \\ S_{v2}(x) = -\dfrac{M_t}{2b}[k_5(\lambda^2\sin\mu x+2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{\lambda x}+k_6(\lambda^2\cos\mu x \\ \qquad +2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{\lambda x}+k_7(\lambda^2\sin\mu x-2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{-\lambda x} \\ \qquad +k_8(\lambda^2\cos\mu x+2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{-\lambda x}] \end{cases}$$

$$\begin{cases} S_{h1}(x) = \dfrac{M_t}{2d}[1+k_3(\lambda^2\sin\mu x+2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{\lambda x}+k_2(\lambda^2\cos\mu x \\ \qquad -2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{\lambda x}+k_3(\lambda^2\sin\mu x-2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{-\lambda x} \\ \qquad +k_4(\lambda^2\cos\mu x+2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{-\lambda x}] \\ S_{h2}(x) = \dfrac{M_t}{2d}[k_5(\lambda^2\sin\mu x+2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{\lambda x}+k_6(\lambda^2\cos\mu x \\ \qquad -2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{\lambda x}+k_7(\lambda^2\sin\mu x-2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{-\lambda x} \\ \qquad +k_8(\lambda^2\cos\mu x+2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{-\lambda x}] \end{cases}$$

Strain in chord member

$$\begin{cases} \varepsilon_3(x) = \dfrac{M_t}{bdEA_0}[k_1(\lambda\sin\mu x+\mu\cos\mu x)e^{\lambda x}+k_2e^{\lambda x}(\lambda\cos\mu x-\mu\sin\mu x) \\ \qquad +k_3(-\lambda\sin\mu x+\mu\cos\mu x)e^{-\lambda x}+k_4(-\lambda\cos\mu x-\mu\sin\mu x)e^{-\lambda x} \\ \varepsilon_2(x) = \dfrac{M_t}{bdEA_0}[k_5(\lambda\sin\mu x+\mu\cos\mu x)e^{\lambda x}+k_6(\lambda\cos\mu-\mu\sin\mu x)e^{\lambda x} \\ \qquad +k_7(-\lambda\sin\mu x+\mu\cos\mu x)e^{-\lambda x}+k_8(-\lambda\cos\mu x-\mu\sin\mu x)e^{-\lambda x}] \end{cases}$$

where $k_j(j=1, 2, \cdots, 7, 8)$ are determined by the equation

$$[a_{ij}][k_j] = [b_j],$$

$$[k_j] \equiv \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \end{pmatrix} \qquad [b_j] \equiv \begin{pmatrix} -\dfrac{\beta EA_0}{8G} \\ 0 \\ 0 \\ 0 \\ -\dfrac{\beta EA_0}{8G} \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

where

$$\lambda=\frac{1}{2\sqrt{S}}\sqrt{2K\sqrt{S}+1}\,,\quad \mu=\frac{1}{2\sqrt{S}}\sqrt{2K\sqrt{S}-1}$$

$$S=\frac{bd}{2\alpha}\frac{G}{G}\qquad K^2=\frac{8G}{\alpha EA_0}$$

$$[a_{ij}]=\left\{\begin{array}{ccc}
0, & 1, & 0,\\
3\lambda^2\mu-\mu^3, & \lambda^3-3\lambda\mu^2, & 3\lambda^2\mu-\mu^3,\\
0, & 0, & 0,\\
0, & 0, & 0,\\
e^{\lambda c}\sin\mu c, & e^{\lambda c}\cos\mu c, & e^{-\lambda c}\sin\mu c,\\
e^{\lambda c}(\lambda\sin\mu c+\mu\cos\mu c), & e^{\lambda c}(\lambda\cos\mu c-\mu\sin\mu c), & e^{-\lambda c}(-\lambda\sin\mu c+\mu\cos\mu c),\\
\begin{array}{c}e^{\lambda c}(\lambda^3\sin\mu c+3\lambda^2\mu\cos\mu c\\-3\lambda\mu^2\sin\mu c-\mu^3\cos\mu c),\end{array} & \begin{array}{c}e^{\lambda c}(\lambda^3\cos\mu c-3\lambda^2\mu\sin\mu c\\-3\lambda\mu^2\cos\mu c+\mu^3\sin\mu c),\end{array} & \begin{array}{c}e^{-\lambda c}(-\lambda^3\sin\mu c+3\lambda^2\mu\cos\mu c\\+3\lambda\mu^2\sin\mu c-\mu^3\cos\mu c),\end{array}\\
\begin{array}{c}e^{\lambda c}(\lambda^2\sin\mu\, c+2\lambda\mu\cos\mu c\\-\mu^2\sin\mu c),\end{array} & \begin{array}{c}e^{\lambda c}(\lambda^2\cos\mu c-2\lambda\mu\sin\mu c\\-\mu^2\cos\mu c),\end{array} & \begin{array}{c}e^{-\lambda c}(\lambda^2\sin\mu c-2\lambda\mu\cos\mu c\\-\mu^2\sin\mu c),\end{array}
\end{array}\right.$$

$$\begin{array}{ccc}
1, & 0, & 0,\\
-\lambda^3+3\lambda\mu^2, & 0, & 0,\\
0, & e^{\lambda l}\sin\mu l, & e^{\lambda l}\cos\mu l,\\
0, & \begin{array}{c}e^{\lambda l}(\lambda^3\sin\lambda l+3\lambda^2\mu\cos\mu l\\-3\lambda\mu^2\sin\mu l-\mu^3\cos\mu l),\end{array} & \begin{array}{c}e^{\lambda l}(\mu^3\cos\mu l-3\lambda^2\mu\sin\mu l\\-3\lambda\mu^2\cos\mu l+\mu^3\sin\mu l),\end{array}\\
e^{-\lambda c}\cos\mu c, & -e^{\lambda c}\sin\mu c, & -e^{\lambda c}\cos\mu c,\\
e^{-\lambda c}(-\lambda\cos\mu c-\mu\sin\mu c), & -e^{\lambda c}(\lambda\sin\mu c+\mu\cos\mu c), & -e^{-\lambda c}(\lambda\cos\mu c-\mu\sin\mu c),\\
\begin{array}{c}e^{-\lambda c}(-\lambda^3\sin\mu c-3\lambda^2\mu\sin\mu c\\+3\lambda\mu^2\cos\mu c+\mu^3\sin\mu c),\end{array} & \begin{array}{c}-e^{\lambda c}(\lambda^3\sin\mu c+3\lambda^2\mu\cos\mu c\\-3\lambda\mu^2\sin\mu c-\mu^3\cos\mu c),\end{array} & \begin{array}{c}-e^{\lambda c}(\lambda^3\cos\mu c-3\lambda^2\mu\cos\mu c\\-3\lambda\mu^2\cos\mu c+\mu^3\sin\mu c),\end{array}\\
\begin{array}{c}e^{-\lambda c}(\lambda^2\cos\mu c+2\lambda\mu\sin\mu c\\-\mu^2\cos\mu c),\end{array} & \begin{array}{c}-e^{\lambda c}(\lambda^2\sin\mu c+2\lambda\mu\cos\mu c\\-\mu^2\sin\mu c),\end{array} & \begin{array}{c}-e^{\lambda c}(\lambda^2\cos\mu c-2\lambda\mu\sin\mu c\\-\mu^2\cos\mu c),\end{array}
\end{array}$$

$$\left.\begin{array}{cc}
0, & 0,\\
0, & 0,\\
e^{-\lambda l}\sin\mu l, & e^{-\lambda l}\cos\mu l,\\
\begin{array}{c}e^{-\lambda l}(-\lambda^3\sin\mu l+3\lambda^2\mu\cos\mu l\\+3\lambda\mu^2\sin\mu l-\mu^3\cos\mu l),\end{array} & \begin{array}{c}e^{-\lambda l}(-\lambda^3\cos\mu l-3\lambda^2\mu\sin\mu l\\+3\lambda\mu^2\cos\mu l+\mu^3\sin\mu l),\end{array}\\
-e^{-\lambda c}\sin\mu c, & -e^{-\lambda c}\cos\mu c,\\
-e^{-\lambda c}(-\lambda\sin\mu c+\mu\cos\mu c), & -e^{-\lambda c}(-\lambda\cos\mu c-\mu\sin\mu c),\\
\begin{array}{c}-e^{-\lambda c}(-\lambda^3\cos\mu c+3\lambda^2\mu\cos\mu c\\+3\lambda\mu^2\sin\mu c-\mu^3\cos\mu c),\end{array} & \begin{array}{c}-e^{-\lambda c}(-\lambda^3\sin\mu c+3\lambda^2\mu\cos\mu c\\+3\lambda\mu^2\sin\mu c-\mu^3\cos\mu c),\end{array}\\
\begin{array}{c}-e^{-\lambda c}(\lambda^2\sin\mu c-2\lambda\mu\cos\mu c\\-\mu^2\sin\mu c),\end{array} & \begin{array}{c}-e^{-\lambda c}(\lambda^2\cos\mu c+2\lambda\mu\sin\mu c\\-\mu^2\cos\mu c),\end{array}
\end{array}\right\}$$

II) When $\tilde{G}/G=8bd/(1+)A_0\alpha^2$

Warping is found to be

$$\begin{cases} u_1(x) = \dfrac{\beta M_t}{8bdG}\left[1+\dfrac{8G}{\beta EA_0}\left(\dfrac{k_1}{\xi}e^{\xi x}+k_2xe^{\xi x}+\dfrac{k_3}{\xi}e^{-\xi x}+k_4xe^{-\xi x}\right)\right] \\ u_2(x) = \dfrac{M_t}{bdEA_0}\left(\dfrac{k_5}{\xi}e^{\xi x}+k_6xe^{\xi x}+\dfrac{k_7}{\xi}e^{-\xi x}+k_8xe^{-\xi x}\right) \end{cases}$$

Angle of twist

$$\begin{cases} \phi_1(x) = \dfrac{M_t}{GJ_T}\left\{x+\left(\dfrac{\beta}{\alpha}\right)[k_1e^{\xi x}+k_2(\xi xe^{\xi x}+e^{\xi x}+1)-k_3e^{-\xi x}-k_4(x\xi e^{-\xi x}-e^{-\xi x}-1)]\right. \\ \phi_2(x) = \dfrac{M_t}{GJ_T}\left\{c+\left(\dfrac{\beta}{\alpha}\right)[k_5e^{\xi x}+k_6(\xi x+1)e^{\xi x}-k_7e^{-\xi x}-k_8(x\xi-1)e^{-\xi x}+(k_1-k_5)e^{\xi x}\right. \\ \qquad +(k_2-k_6)(\xi c+1)e^{\xi c}+k_2-(k_3-k_7)e^{-\xi c}-(k_4-k_8)(\xi c-1)e^{-\xi c}+k_4]\Big\} \end{cases}$$

Shearing force

$$\begin{cases} S_{v1}(x) = \dfrac{M_t}{2b}\{1-k_1e^{\xi x}-k_2(2+\xi x)e^{\xi x}-k_3e^{-\xi x}-k_4(-2+\xi x)e^{-\xi x}\} \\ S_{v2}(x) = -\dfrac{M_t}{2b}\{k_5e^{\xi x}+k_6(2+\xi x)e^{\xi x}+k_7e^{-\xi x}+k_8(-2+\xi x)e^{-\xi x}\} \end{cases}$$

$$\begin{cases} S_{h1}(x) = \dfrac{M_t}{2d}\{1+k_1e^{\xi x}+k_2(2+\xi x)e^{\xi x}+k_3e^{-\xi x}+k_4(-2+\xi x)e^{-\xi x}\} \\ S_{h2}(x) = \dfrac{M_t}{2d}\{k_5e^{\xi x}+k_6(2+\xi x)e^{\xi x}+k_7e^{-\xi x}+k_8(-2+\xi x)e^{-\xi x}\} \end{cases}$$

Strain in chord member

$$\begin{cases} \varepsilon_1(x) = \dfrac{M_t}{bdEA_0}[k_1e^{\xi x}+k_2(1+\xi x)e^{\xi x}-k_3e^{-\xi x}+k_4(1-\xi x)e^{-\xi x}] \\ \varepsilon_2(x) = \dfrac{M_t}{bdEA_0}[k_5e^{\xi x}+k_6(1+\xi x)e^{\xi x}-k_7e^{-\xi x}+k_8(1-\xi x)e^{-\xi x}] \end{cases}$$

where

$$\xi = \sqrt[4]{\frac{1}{2S}}, \quad S=\frac{bd}{2\alpha}\frac{\tilde{G}}{G}$$

where $k_j(j=1, 2, \cdots, 7, 8)$ are determined by the equations

$$\begin{cases} 1) \quad k_1+_3k = -\dfrac{\xi\beta EA_0}{8G} \\ 2) \quad k_4+3k_3-k_2+3k_4 = 0 \end{cases}$$

$$\left\{\begin{aligned}
&3)\quad \frac{e^{\xi l}}{\xi}k_5+le^{\xi l}k_6+\frac{e^{-\xi l}}{\xi}k_7+le^{-\xi l}k_8=0\\
&4)\quad e^{\xi l}k_5+(3+\xi l)e^{\xi l}k_6-e^{-\xi l}k_7+(3-\xi l)e^{-\xi l}k_8=0\\
&5)\quad \frac{e^{\xi c}}{\xi}k_1+ce^{\xi c}k_2+\frac{e^{-\xi c}}{\xi}k_3+ce^{-\xi c}k_4-\frac{e^{\xi c}}{\xi}k_5-ce^{\xi c}k_6-\frac{e^{-\xi c}}{\xi}k_7-ce^{-\xi c}k_8=-\frac{\beta EA_0}{8G}\\
&6)\quad e^{\xi c}k_1+(1+\xi c)e^{\xi c}k_2-e^{-\xi c}k_3+(1-\xi c)e^{-\xi c}k_4-e^{\xi c}k_5-(1+\xi c)e^{\xi c}k_6+e^{-\xi c}k_7\\
&\qquad -(1-c\xi)e^{-\xi c}k_8=0\\
&7)\quad e^{\xi c}{}_1k+(3+\xi c)e^{\xi c}k_2-e^{-\xi c}k_3+(3-\xi c)e^{-\xi c}k_4-e^{\xi c}k_5-(3+\xi c)e^{\xi c}k_6+e^{-\xi c}k_7\\
&\qquad -(3-\xi c)e^{-\xi c}k_8=0\\
&8)\quad \xi e^{\xi c}k_1+(2\xi+\xi^2c)e^{\xi c}k_2+\xi e^{-\xi c}k_3+(-2\xi+\xi^2c)e^{-\xi c}k_4-\xi e^{\xi c}k_5-(2\xi+\xi^2c)e^{\xi c}k_6\\
&\qquad -\xi e^{-\xi c}k_7-(-2\xi+\xi^2c)e^{-\xi c}k_8=-1
\end{aligned}\right.$$

III) When $\tilde{G}/G>8bd/(1+\nu)A_0\alpha^2$

Warping is expressed by

$$\left\{\begin{aligned}
u_1(x)&=\frac{\beta M_t}{8bdG}\left[1+\frac{8G}{EA_0\beta}(k_1e^{\xi_1x}+k_2e^{\xi_2x}+k_3e^{-\xi_1x}+k_4e^{-\xi_2x})\right]\\
u_2(x)&=\frac{M_t}{bdEA_0}(k_5e^{\xi_1x}+k_6e^{\xi_2x}+k_7e^{-\xi_1x}+k_8e^{-\xi_2x})
\end{aligned}\right.$$

Angle of twist

$$\left\{\begin{aligned}
\phi_1(x)&=\frac{M_t}{GJ_T}\Big\{x+\left(\frac{\beta}{\alpha}\right)[k_1\xi_1e^{\xi_1x}+k_2\xi_2e^{\xi_2x}-k_3\xi_1e^{-\xi_1x}-k_4\xi_2e^{-\xi_2x}\\
&\qquad -(k_1-{}_3k)\xi_1-(k_2-k_4)\xi_2]\Big\}\\
\phi_2(x)&=\frac{M_t}{GJ_T}\Big\{c+\left(\frac{\beta}{\alpha}\right)[k_5\xi_1e^{\xi_1x}+k_6\xi_2e^{\xi_2x}-k_7\xi_1e^{-\xi_1x}-k_8\xi_2e^{-\xi_2x}\\
&\qquad +(k_1-k_5)\xi_1e^{\xi_1c}+(k_2-k_6)\xi_2e^{\xi_2c}-(k_3-k_7)\xi_1e^{-\xi_1c}\\
&\qquad -(k_4-k_8)\xi_2e^{-\xi_2c}-(k_1-k_3)\xi_1-(k_2-k_4)\xi_2]\Big\}
\end{aligned}\right.$$

Shearing force

$$\left\{\begin{aligned}
S_{v1}(x)&=\frac{M_t}{2b}\{1-k_1\xi_1^2e^{\xi_1x}-k_2\xi_2^2e^{\xi_2x}-k_3\xi_1^2e^{-\xi_1x}-k_4\xi_2^2e^{\xi_2x}\}\\
S_{v2}(x)&=-\frac{M_t}{2b}(k_5\xi_1^2e^{\xi_1x}+k_6\xi_2^2e^{\xi_2x}+k_7\xi_1^2e^{-\xi_1x}+k_8\xi_2^2e^{-\xi_2x})
\end{aligned}\right.$$

$$\left\{\begin{aligned}
S_{h2}(x)&=\frac{M_t}{2d}(1+k_1\xi_1^2e^{\xi_1x}+k_2\xi_2^2e^{\xi_2x}+k_3\xi_1^2e^{-\xi_1x}+k_4\xi_2^2e^{-\xi_2x})\\
S_{h2}(x)&=\frac{M_t}{2d}(k_5\xi_1^2e^{\xi_1x}+k_6\xi_2^2e^{\xi_2x}+k_7\xi_1^2e^{-\xi_1x}+k_8\xi_2^2e^{-\xi_2x})
\end{aligned}\right.$$

Strain in chord member

$$\begin{cases} \varepsilon_1(x) = \dfrac{M_t}{bdEA_0}(k_1\xi_1 e^{\xi_1 x} + k_2\xi_2 e^{\xi_2 x} - k_3\xi_1 e^{-\xi_1 x} - k_4\xi_2 e^{-\xi_2 x}) \\ \varepsilon_2(x) = \dfrac{M_t}{bdEA_0}(k_5\xi_1 e^{\xi_1 x} + k_6\xi_2 e^{\xi_2 x} - k_7\xi_1 e^{-\xi_1 x} - k_8\xi_2 e^{-\xi_2 x}) \end{cases}$$

where $k_j(j=1, 2, \cdots, 7,8)$ are determined by the equation

$$\begin{cases} 1) \quad k_1+k_2+k_3+k_4 = -\dfrac{\beta EA_0}{8G} \\ 2) \quad \xi_1^3 k_1+\xi_2^3 k_2-\xi_3^3 k_3-\xi_2^3 k_4 = 0 \\ 3) \quad e^{\xi_1 l}k_5+e^{\xi_2 l}k_6+e^{-\xi_1 l}k_7+e^{-\xi_2 l}k_8 = 0 \\ 4) \quad \xi_1^3 e^{\xi_1 l}k_5+\xi_2^3 e^{\xi_2 l}k_6-\xi_1^3 e^{-\xi_1 l}k_7-\xi_1^3 e^{-\xi_2 l}k_8 = 0 \\ 5) \quad e^{\xi_1 c}k_1+e^{\xi_2 c}k_2+e^{-\xi_1 x}k_3+e^{-\xi_2 c}k_4-e^{\xi_1 c}k_5-e^{\xi_2 c}k_6-e^{-\xi_1 c}k_7-e^{-\xi_2 c}k_8 = -\dfrac{\beta EA_0}{8G} \\ 6) \quad \xi_1 e^{\xi_1 c}k_1+\xi_2 e^{\xi_2 c}k_2-\xi_1 e^{-\xi_1 c}k_3-\xi_2 e^{\xi_2 c}k_4 \\ \qquad -\xi_1 e^{\xi_1 c}k_5-\xi_2 e^{\xi_2 c}k_6+\xi_1 e^{-\xi_1 c}k_7+\xi_2 e^{-\xi_2 c}k_8 = 0 \\ 7) \quad \xi_1^3 e^{\xi_1 c}k_1+\xi_2^3 e^{\xi_2 c}k_2-\xi_1^3 e^{-\xi_1 c}k_3-\xi_2^3 e^{-\xi_2 c}k_4 \\ \qquad -\xi_1^3 e^{\xi_1 c}k_5-\xi_2^3 e^{\xi_2 c}k_6+\xi_1^3 e^{-\xi_1 c}k_7+\xi_2^2 e^{-\xi_2 c}k_8 = 0 \\ 8) \quad \xi_1^2 e^{\xi_1 c}k_1+\xi_2^2 e^{\xi_2 c}k_2+\xi_1^2 e^{-\xi_1 c}k_3+\xi_2^2 e^{-\xi_2 c}k_4 \\ \qquad -\xi_1^2 e^{\xi_1 c}k_5-\xi_2^2 e^{\xi_2 c}k_6-\xi_1^2 e^{-\xi_1 c}k_7-\xi_2^2 e^{-\xi_2 c}k_8 = -1 \end{cases}$$

And

$$\xi_7 = \sqrt{\frac{1+\sqrt{1-4K^2S}}{2S}}$$

$$\xi_2 = \sqrt{\frac{1-\sqrt{1-4K^2S}}{2S}}$$

$$S = \frac{bd}{2\alpha}\frac{G}{\tilde{G}}, \quad K^2 = \frac{8G}{\alpha EA_0}$$

(ii) When a concentrated torque is applied at $x=l$ (Fig. 8)

Previously we derived the equations with regard to the torsional behaviour of the structure subjected to the concentrative torque load which acts between two

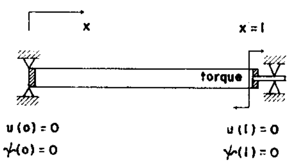

Fig. 8. Boundary conditions.

ends of the beam. These equations are reduced to the more simple form when the torque load is applied at the end of beam as shown in Fig. 8.

Boundary conditions:

$$\begin{aligned} &\text{i)}\ \ u(0)=0 \qquad &&\text{iii)}\ \ u(l)=0 \\ &\text{ii)}\ \ \psi(0)=0 \qquad &&\text{iv)}\ \ \psi(l)=0 \end{aligned} \tag{29}$$

I) When $G/G<8bd/(1+\nu)A_0\alpha^2$

Warping is expressed as

$$u(x)=\frac{\beta M_t}{8bdG}[1+C_1e^{\lambda x}\sin\mu x+C_2e^{\lambda x}\cos\mu x+C_3e^{-\lambda x}\sin\mu x+C_4e^{-\lambda x}\cos\mu x] \tag{30}$$

Angle of twist

$$\begin{aligned}\phi(x)=\frac{M_t}{GJ_T}\Big\{x+\left(\frac{\beta}{\alpha}\right)^2\frac{1}{K^2}&[C_1(\lambda e^{\lambda x}\sin\mu x+\mu e^{\lambda x}\cos\mu x-\mu)\\ &+C_2(\lambda e^{\lambda x}\cos\mu x-\mu e^{\lambda x}\sin\mu x-\lambda)+C_3(-\lambda e^{-\lambda x}\sin\mu x+\mu e^{-\lambda x}\cos\mu x-\mu)\\ &+C_4(-\lambda e^{-\lambda x}\cos\mu x-\mu e^{-\lambda x}\sin\mu x+\lambda)]\Big\}\end{aligned} \tag{31}$$

Shearing force

$$\begin{aligned}S_v(x)=\frac{M_t}{2b}\Big\{1-\left(\frac{\beta}{\alpha}\right)\left(\frac{1}{K_2}\right)&[C_1(\lambda^2\sin\mu x+2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{\lambda x}\\ &+C_2(\lambda^2\cos\mu x-2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{\lambda x}\\ &\qquad+C_3(\lambda^2\sin\mu x-2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{-\lambda x}\\ &+C_4(\lambda^2\cos\mu x+2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{-\lambda x}]\Big\}\end{aligned} \tag{32}$$

$$\begin{aligned}S_h(x)=\frac{M_t}{2d}\Big\{1+\left(\frac{\beta}{\alpha}\right)\left(\frac{1}{K^2}\right)&[C_1(\lambda^2\sin\mu x+2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{\lambda x}\\ &+C_2(\lambda^2\cos\mu x-2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{\lambda x}\\ &\qquad+C_3(\lambda^2\sin\mu x-2\lambda\mu\cos\mu x-\mu^2\sin\mu x)e^{-\lambda x}\\ &+C_4(\lambda^2\cos\mu x+2\lambda\mu\sin\mu x-\mu^2\cos\mu x)e^{-\lambda x}]\Big\}\end{aligned}$$

Strain in chord member

$$\begin{aligned}\varepsilon(x)=-\left(\frac{\beta}{\alpha}\right)\frac{1}{K^2}\frac{M_t}{bdEA_0}&[C_1(\lambda\sin\mu x+\mu\cos\mu x)e^{\lambda x}+C_2(\lambda\cos\mu x-\mu\sin\mu x)e^{\lambda x}\\ &+C_3(-\lambda\sin x\mu x+\mu\cos\mu x)e^{-\lambda x}+C_4(-\lambda\cos\mu x-\mu\sin\mu x)e^{-\lambda x}\end{aligned} \tag{33}$$

Where C_j $(j=1,2,3,4)$ are determined by the equations

$$\begin{cases} 1) & C_2+C_4=-1\,, \\ 2) & (3\lambda^2\mu-\mu^3)\,C_1+(\lambda^3-3\lambda\mu^2)\,C_2+(3\lambda^2\mu-\mu^3)\,C_3-(\lambda^3-3\lambda\mu^2)\,C_4=0\,, \\ 3) & e^{\lambda l}\sin\mu l C_1+e^{\lambda l}\cos\mu l C_2+e^{-\lambda l}\sin\mu l C_3+e^{-\lambda l}\cos\mu l C_4=-1\,, \\ 4) & (\lambda^3\sin\mu l+3\lambda^2\mu\cos\mu l-3\lambda\mu^2\sin\mu l-\mu^3\cos\mu l)\,e^{\lambda l}C_1 \\ & \quad +(\lambda^3\cos\mu l-3\lambda^2\mu\sin\mu l-3\lambda\mu^2\cos\mu l+\mu^3\sin\mu l)\,e^{\lambda l}C_2 \\ & \quad +(-\lambda^3\sin\mu l+3\lambda^2\mu\cos\mu l+3\lambda\mu^2\sin l\,\mu l-\mu^3\cos\mu l)\,e^{-\lambda l}C_3 \\ & \quad +(-\lambda^3\cos\mu l-3\lambda^2\mu\sin\mu l+3\lambda\mu^2\cos\mu l+\mu^3\sin\mu l)\,e^{-\lambda l}C_4=0 \end{cases}$$

II) When $\tilde{G}/G=8bd/(1+\nu)\,A_0\alpha^2$

Warping is written as

$$u(x)=\frac{\beta M_t}{8bdG}(1+k_1e^{\xi x}+k_2\xi x e^{\xi x}+k_3e^{-\xi x}+k_4\xi x e^{-\xi x}) \tag{34}$$

Strain in chord member

$$\varepsilon(x)=\frac{\xi M_t}{bdEA_0K^2}\left(\frac{\beta}{\alpha}\right)[e^{\xi x}k_1+e^{\xi x}(1+\xi x)\,k_2-e^{-\xi x}k_3-e^{-\xi x}(1-\xi x)\,k_4] \tag{35}$$

Shearing force

$$\begin{cases} S_v(x)=\dfrac{M_t}{2b}\left\{1-\left(\dfrac{\beta}{\alpha}\right)\left(\dfrac{\xi}{K}\right)^2[k_1e^{\xi x}+k_2(2+\xi x)\,e^{\xi x}+k_3e^{-\xi x}+k_4(-2+\xi x)\,e^{-\xi x}]\right\} \\ S_h(x)=\dfrac{M_t}{2d}\left\{1+\left(\dfrac{\beta}{\alpha}\right)\left(\dfrac{\xi}{K}\right)^2[k_1e^{\xi x}+k_2(2+\xi x)\,e^{\xi x}+k_3e^{-\xi x}+k_3(-2+\xi x)\,e^{-\xi x}]\right\} \end{cases} \tag{36}$$

where k_j $(j=1, 2, 3, 4)$ are to be determined by

$$\begin{aligned} & k_1+k_3=-1 \\ & \xi k_1+3k_2-\xi k_3+3k_4=0 \\ & e^{l\xi}k_1+le^{\xi l}k_2+e^{-\xi l}k_3+le^{-\xi l}k_4=-1 \\ & \xi e^{\xi l}k_1+(3+\xi l)\,e^{\xi l}k_3-\xi e^{-\xi l}k_4+(3-\beta l)\,e^{-\xi l}k_4=0 \end{aligned}$$

5. Experimental Result and Numerical Illustrations

In order to perform the experimental work for torsional behaviour of four walled beams, two types of models are considered here. The first is a four-walled structure with two vertical trusses, whereas the second one is built of two stiffening girders with two horizontal trusses. The models are supported by four steel balls on the supporting frame rigid enough to prevent the beam from moving around the axis of the beam at one end, while at the other end a ballbearing is fitted so that the beam may rotate freely about the axis of the beam and the plane of the ball-bearing is permitted to move freely around the horizontal axis, perpendicular to the beam axis. The principal portion of the models were made from the Methacrylic Acids. (Photo 1 & 2) The model tests were carried out by measuring the displacements of the model as well as strains produced using the electric strain gage and the differential transformers. The results obtained are as follows.

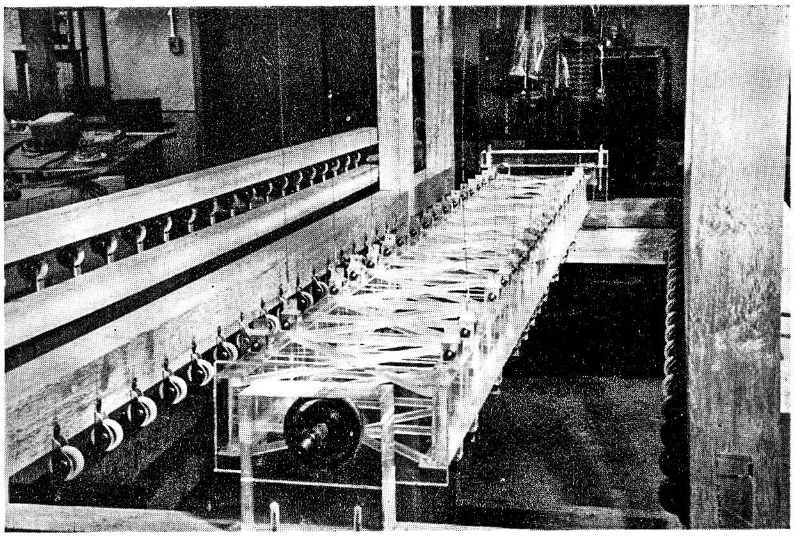

Photo. 1. Model.

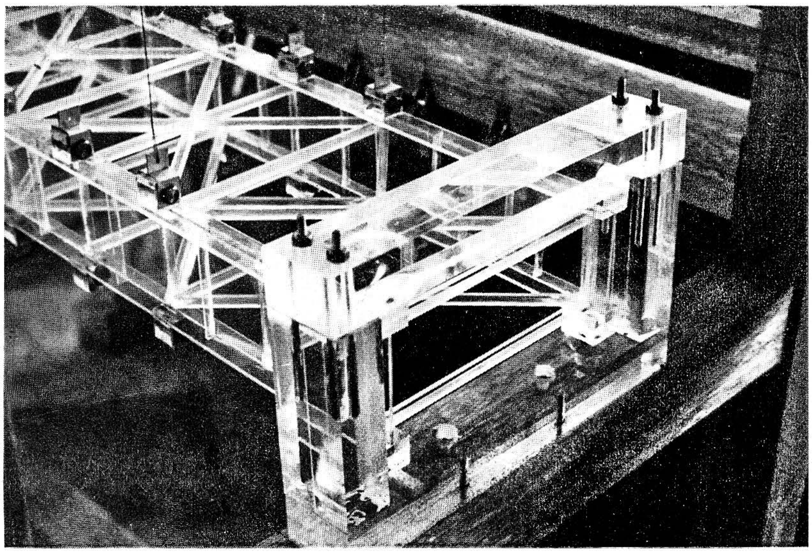

Photo. 2. Model.

5.1 Plate grider type model

a. Strains in the chord members (Fig. 9)

At or near the loading point, flanges underwent considerable stress concentrations due to the constraint of warping. The absolute maximum experimental value showed respectable agreement with the one obtained by the idealized theories. The small stress deviations appear from the middle of the span to the right-hand support of the beam, which becomes more obvious as the load moves toward the right-hand side of the beam.

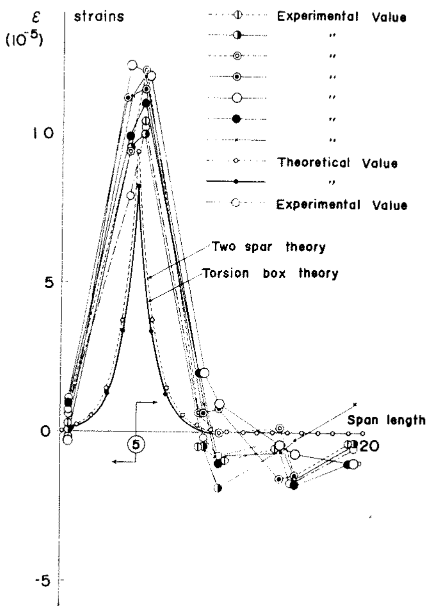

Fig. 9. Strains at chord members when a torque is applied at point 5.

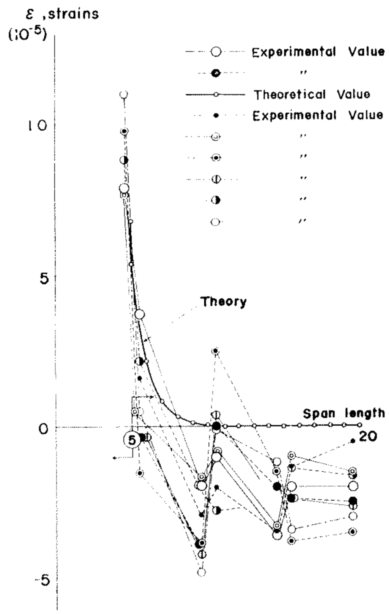

Fig. 10. Strains at lateral bracing of horizontal truss when a torque is applied at point 5.

b. Strains of the lateral bracing of the horizontal truss (Fig. 10)

It is noted that there are considerable deviations between the theoretical values and the experimental results. However it may be considered improper to criticize the idealized theory because of lack of reliable data with respect to the strains in the the inclined members, which seems therefore to call for the more elaborate works to reach to the final conclusion.

c. Angle of twist (Fig. 11)

The theoretical computation may be thought reasonable except for the one near the center of the span.

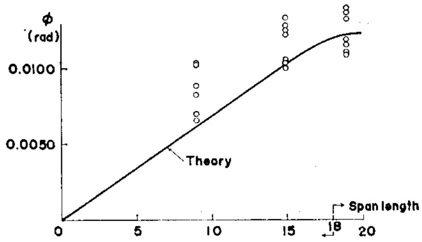

Fig. 11. Angle of twist of girder when a torque is applied at point 5.

5.2 Truss type model

a. Strains in chord members (Fig. 12)

The general configuration of the theoretical analysis in terms of Two-Spar theory ($\beta/\alpha=1$) may be thought to agree with the experimental results, while the Torsion-Box theory was proved to be less exact compared with the Two-Spar theory.

b. Stains in the diagonal members of the stiffening truss (Fig. 13)

The similar results are obtained as in the case of the plate girder model.

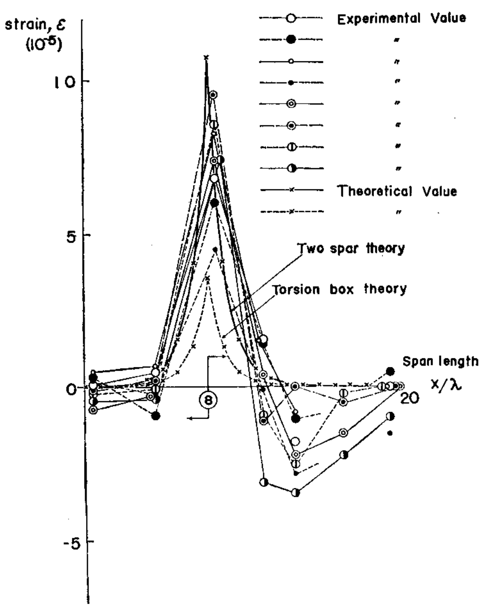

Fig. 12. Strains of the chord member when a torque load is applied at point 8.

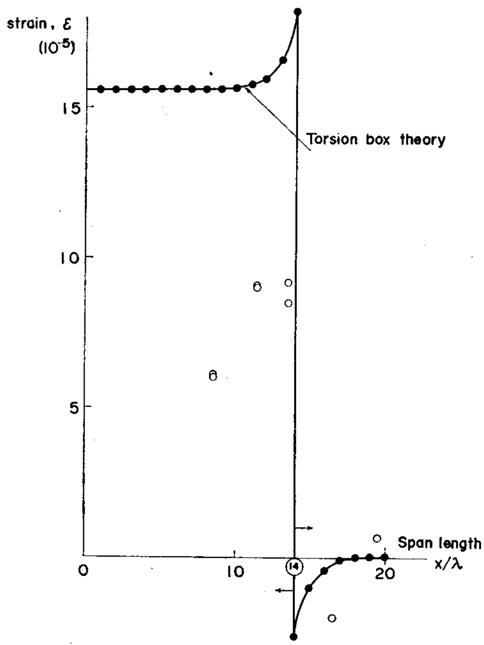

Fig. 13. Strains in the digonal members of the truss girder when a torque is applied at point 14.

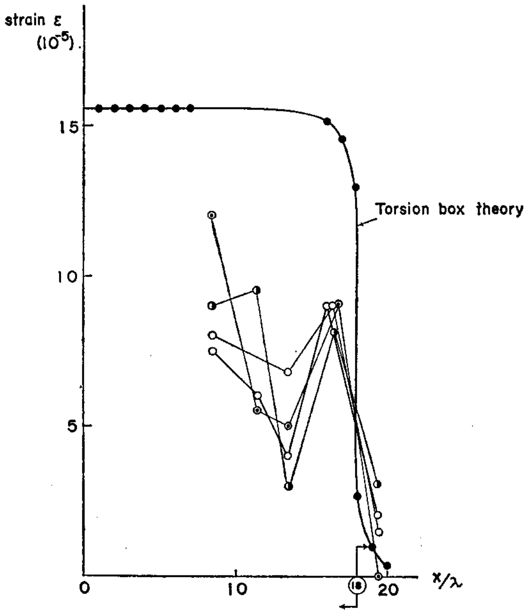

Fig. 14. Strains in the lateral system of the horizontal truss, torque acting at point 18.

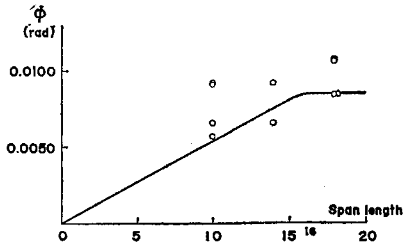

Fig. 15. Angle of twistwhen a torque is applied at point 16.

c. Strains in the lateral system of the horizontal truss (Fig. 14)

d. Angle of twist (Fig. 15)

The experimental values are considered to agree with the theoretical values to some extent.

5.3 Numerical illustration on the influence of bulkhead deformations.

By use of eq. (33) the effects of shear deformations of bulkheads are estimated under the boundary conditions, eq. (29), together with eq. (35). It should be noted that owing to the finite deformations of the bulkheads, beams having less stronger bulkhead rigidity undergo the undulatory deformations of flanges.

6. Discussion and Conclusion

As one of the most important factors in the torsion analysis of truss systems we should pay deliberate attention to the parameter $R=\beta/\alpha$ in eq's (14′), (16′), and (18′). which has direct influence upon the magnitude of flange force due to the constraint of warping. Since this parameter has close relation to the form in which the beam resists torsional deformation, an experimental analysis on the flange force together with the shearing force acting on the inclined members of models becomes necessary to judge the torsional mode of the four-walled truss systems. The ratios of the magnitude of the strains in the chord member calculated by eq's (14′), (16′), and (18′) are R, $1-R$ and $1+R$ respectively; however, the average value $\overline{R}$ of these values would be convenient to characterize the truss systems. For example, in the case of the plate girder type beam, $R=0.8899$, $1+R=1.8899$, $1-R=0.1101$ and the average ratio $\overline{R}=0.9633$, while in the case of the truss type beam $R=0.3333$, $1+R=1.333$, $1-R=0.6667$ and the average ratio $R=0.7778$. Although there is little difference between the values R and $\overline{R}$ in the former, the difference in the latter is too large to be neglected, which may give a reasonable explanation to the statement described in the preceding paragraph.

Thus we come to the conclusion from a number of our experiments on models besides theoretical analysis that the four-walled truss system amy be considered to resist the external concentrative torque load in the modified form of Two-spar structure ($R=1$) rather than Torsion-box, where the term two spars denotes originally a structure in which the strength and stiffness are derived mainly from two rods with connecting ribs.

Notation

a ..... Panel length

A_0 .... Cross-sectional area of a truss chord

A_h .... Cross-sectional area of a lateral bracing

A_v .... Cross-sectional area of a diagonal member of the stiffening truss

b ..... Width of bridge

c ...... Abscissa, horizontal distance of a cross section of the loaded point from one end of beam

d ..... Depth of stiffening truss or plate girder

E ..... Modulus of elasticity of the stiffening frame

G ..... Modulus of rigidity of the stiffening frame

GJ_T .. Torsional rigidity of suspended structure $=2Gb^2d^2/\alpha$

K ..... Characteristic value defined by $\sqrt{\frac{8G}{\alpha EA_0}}$

l ...... Span length

M_t .... Twisting torque acting on the cross frame of suspended structure

P...... Normal force, acting on the truss chord or the flange of plate girder

R ..... Ratio defined by β/α

S_h ..... Shearing force produced in the plane of horizontal truss

S_v ..... Shearing force produced in the plane of vertical stiffening truss or girder

t_h .... Thickness of the web plate of vertical stiffening truss or girder ($2\sin^2\gamma_h\cos\gamma_h\,EA_h/Gb$ for truss)

t_v ...... Thickness of the web plate of vertical stiffening truss or girder ($\sin^2 \gamma_d \cos \gamma_d EA_v/Gd$ for truss)

u ..... Warping of the chord member

x ..... Abscissa, horizontal distance of a cross section of the frame from the fixed end

α .... Quantity defined by $b/t_h + d/t_v$

β .... Quantity defined by $b/t_h - d/t_h$

γ_h .... Angle of slope of truss diagonal (horizontal)

γ_d .... Angle of slope of truss diagonal (vertical)

ε .... Strain in chord member

ν .... Poisson's ratio of the stiffening frame

ϕ .... Angle of twist of beam

ψ .... Shear deformation of the bulkheads

τ .... Shearing stress

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