Control System Synthesis by Sensitivity Considerations

By

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The purpose of this paper is to provide a new open-loop control system synthesis method by active use of the system sensitivity.

The concept of the combined system is, at first, introduced, which consists of both the originally given model of a physical system and its sensitivity model. Secondly, the controllability of it is analysed, where some interesting properties of the combined system controllability are made clear. Thirdly, based on these controllability analyses, the method of the sensitivity synthesis is developed in the minimum energy problem with terminal constraints from the view point of making the terminal constraints more rigid against parameter perturbations. Several simple examples are given to illustrate the effectiveness of the present method. Finally, it is suggested that the undesirable effects of the existing slight nonlinearity neglected at the stage of modeling can be diminished in the same manner.

1. Introduction

Despite the rapid development in the theory of optimum control system design, the applications of it to practical systems seem to have been stagnant at some points. This is mainly because of such a basic assumption usually made in the study of optimum control systems that the system to be controlled is known exactly. This is, of course, often a gross idealization. Even a small difference, due to inadequacy of identification methods and change of parameter etc., between a given physical system and its mathematical model quite often results in degradation of the performance from its optimum value, or violations of given constraints on states or control variables. A good engineering design must, thus, take such effects into account.

From this point of view, P. Dorato[1] pointed out for the first time the importance of analysing parameter perturbation effects on the performance index in nominal optimum systems and this was soon followed and extended by B. Pagurek[2] in the case of a linear plant with quadratic performance criteria. The effect of erroneous estimate of the initial conditions on state variables to the performance

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index was discussed by J. Clark[3]. The sensitivity problem of specified terminal conditions to system parameter variations in optimum systems is also examined by J. Holtzman and S. Horing[4]. The above mentioned works are widely called *sensitivity analysis in optimum systems.*

On the other hand, another use of the sensitivity theory to synthesize the least sensitive control was introduced by M. Gavrilovic and P. Petrovic[5], which we may call *sensitivity synthesis.* The work by W. Tuel, Jr. et al[6] belongs to the same category.

In this paper, we expand in detail the concept of sensitivity in the synthesis of optimum control in the case of the minimum energy problem of a linear plant with terminal constraints. At first, we examine the controllability of the combined system which consists of the model of a given physical system and its sensitivity model. This is fundamental in the later synthesis. The synthesis problem of minimum energy control with zero sensitive terminal constraints is, then, discussed and simple examples of it are shown to demonstrate the superiority of the present method to the conventional one. At last, a suggestion is made that in an analogous way we can take the effects of slight nonlinearity into consideration, which are often englected in modeling process.

2. Basic Concepts

As the model of an original or a physical system to be controlled, consider the system of the differential equation;

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{q}, \boldsymbol{u}(t)), \quad \boldsymbol{x}(t_0)=\boldsymbol{x}^0, \tag{2–1}$$

where $\boldsymbol{x}=\text{col}\,(x_1, x_2, \cdots, x_n)$ is the state vector of the system, $\boldsymbol{u}=\text{col}\,(u_1, u_2, \cdots, u_m)$ is the control vector and $\boldsymbol{q}=\text{col}\,(q_1, q_2, \cdots, q_r)$ is the constant system parameter vector.

Based on Eq. (2–1), the optimum control $\boldsymbol{u}^*(t)$, which satisfies the given constraints, has been investigated up to now. Unfortunately, in actual practice, the value of the system parameter $\boldsymbol{q}$ in Eq. (2–1) seldom corresponds to the actual value of $\boldsymbol{q}$. This is due to such things as the inadequacy of identification or modeling methods, component inaccuracies, environmental effects, aging, etc.

Suppose, then, that the actual system may have a different parameter value $\boldsymbol{q}+\Delta\boldsymbol{q}$ from $\boldsymbol{q}$ used in the model equation (2–1), that is,

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{q}+\Delta\boldsymbol{q}, \boldsymbol{u}(t)), \quad \boldsymbol{x}(t_0)=\boldsymbol{x}^0. \tag{2–2}$$

The difference between the model state $\boldsymbol{x}(t, \boldsymbol{q})$ in Eq. (2–1) and the actual state $\boldsymbol{x}(t, \boldsymbol{q}+\Delta\boldsymbol{q})$ in Eq. (2–2) is formally expressed by

$$\begin{aligned}\Delta\boldsymbol{x}(t, \Delta\boldsymbol{q}) &= \boldsymbol{x}(t, \boldsymbol{q}+\Delta\boldsymbol{q})-\boldsymbol{x}(t, \boldsymbol{q}) \\ &= \sum_{i=1}^{r} {}_i\boldsymbol{y}(t)\Delta q_i + o[(\Delta\boldsymbol{q})^2]\,, \end{aligned} \tag{2–3}$$

by use of a Taylor series expansion, where the first order sensitivity vector with respect to q_i is defined to be

$${}_i\boldsymbol{y}(t)=\left.\frac{\partial\boldsymbol{x}(t, \boldsymbol{q})}{\partial q_i}\right|_{\Delta\boldsymbol{q}=\boldsymbol{0}} \qquad (i=1, 2, \cdots, r)\,. \tag{2–4}$$

Under suitable conditions[7] on the vector function $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{q}, \boldsymbol{u})$, roughly speaking, the continuity of $\boldsymbol{f}$, $\dfrac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}$ and $\dfrac{\partial\boldsymbol{f}}{\partial\boldsymbol{q}}$, ${}_i\boldsymbol{y}(t)$ becomes the solution of the differential equation

$$\frac{d\,{}_i\boldsymbol{y}}{dt}=\frac{\partial\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{q}, \boldsymbol{u})}{\partial\boldsymbol{x}}\,{}_i\boldsymbol{y}+\frac{\partial\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{q}, \boldsymbol{u})}{\partial q_i} \qquad (i=1, 2, \cdots, r)\,, \tag{2–5}$$

with initial condition

$${}_i\boldsymbol{y}(t_0)=\boldsymbol{0} \qquad (i=1, 2, \cdots, r)\,. \tag{2–6}$$

This is widely called the parameter sensitivity equation.

Regarding the initial condition $\boldsymbol{x}^0$ as parameter, we can derive analogous equation as follows;

$$\begin{aligned}\Delta\boldsymbol{x}(t, \Delta\boldsymbol{x}^0) &= \boldsymbol{x}(t, \boldsymbol{x}^0+\Delta\boldsymbol{x}^0)-\boldsymbol{x}(t, \boldsymbol{x}^0) \\ &= \sum_{i=1}^{n} {}_i\boldsymbol{z}(t)\Delta x_i^0 + o[(\Delta\boldsymbol{x}^0)^2]\,, \end{aligned} \tag{2–7}$$

where

$${}_i\boldsymbol{z}(t)=\left.\frac{\partial\boldsymbol{x}(t, \boldsymbol{x}^0)}{\partial x_i^0}\right|_{\Delta\boldsymbol{x}^0=\boldsymbol{0}} \qquad (i=1, 2, \cdots, n)\,. \tag{2–8}$$

${}_i\boldsymbol{z}(t)$ becomes the solution of a set of differential equations

$$\frac{d\,{}_i\boldsymbol{z}}{dt}=\frac{\partial\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{q}, \boldsymbol{u})}{\partial\boldsymbol{x}}\,{}_i\boldsymbol{z} \qquad (i=1, 2, \cdots, n)\,, \tag{2–9}$$

with initial condition

$${}_iz_j(t_0)=\delta_{ij} \qquad (i, j=1, 2, \cdots, n)\,. \tag{2–10}$$

From Eqs. (2–3) and (2–5) or Eqs. (2–7) and (2–9) effects of both parameter and initial condition perturbations on actual system state are able to be approximately analysed, provided that the nominal optimum control is given based on the system model of Eq. (2–1). This lays the foundation of *sensitivity analysis in optimum system.*

Going a step further, on the other hand, consider the model equation (2–1) and its parameter sensitivity equation (2–5) simultaneously

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t,\ \boldsymbol{x},\ \boldsymbol{q},\ \boldsymbol{u}(t)) \\ \dfrac{d_i\boldsymbol{y}}{dt} = \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \cdot {}_i\boldsymbol{y} + \dfrac{\partial \boldsymbol{f}}{\partial q_i} \qquad (i, = 1,\ 2,\ \cdots,\ r)\ , \end{cases} \tag{2–11}$$

or the model equation (2–1) and the sensitivity equation (2–9) to its initial condition perturbations

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t,\ \boldsymbol{x},\ \boldsymbol{q},\ \boldsymbol{u}(t)) \\ \dfrac{d_i\boldsymbol{z}}{dt} = \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \cdot {}_i\boldsymbol{z} \qquad (i=1,\ 2,\ \cdots,\ n)\ , \end{cases} \tag{2–12}$$

then it may be recognized that there is the possibility to control not only the original model but also its sensitivity by properly chosen control $\boldsymbol{u}(t)$. That is, we can take perturbation effects into consideration at the initial stage of synthesis. This is the basis of *sensitivity synthesis* to small parameter or initial condition perturbations.

As a matter of fact, can we really control both the given model and its sensitivity model at the same time by properly choosing control $\boldsymbol{u}(t)$? To answer this question, we must examine the controllability of the combined system expressed by Eq. (2–11) or (2–12).

3. Controllability of Combined System

Before entering into our controllability problems in hand, let us slightly refer to the meaning of controllability and its condition[8]. Generally speaking, if it is possible to drive a system from its initial state to the origin in some finite time, we say the system is completely controllable, or simply, controllable.

For a linear time-invariant system;

$$\frac{d\boldsymbol{x}}{dt} = \mathcal{A}\boldsymbol{x}(t) + \mathcal{B}\boldsymbol{u}(t)\ , \tag{3–1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are $n \times n$ and $n \times m$ constant matrices respectively, the controllability condition is stated as follows; the system (3–1) is completely controllable if and only if the rank of G is n, where the $n \times nm$ matrix G is defined by

$$G=[\mathscr{B},\ \mathscr{A}\mathscr{B},\ \mathscr{A}^2\mathscr{B},\ \cdots,\ \mathscr{A}^k\mathscr{B},\ \cdots,\ \mathscr{A}^{n-1}\mathscr{B}]\ . \tag{3-2}$$

We should notice here that if the rank of G-matrix is less than n, say $k(<n)$, the system is not controllable as stated above in the whole space $\boldsymbol{x}$, but is able to be controllable in the properly chosen (less than k-dimensional) subspace.

As the original system to be controlled, we assume a linear time-invariant system described by

$$\frac{d\boldsymbol{x}}{dt}=A\boldsymbol{x}(t)+B\boldsymbol{u}(t)\ , \tag{3-3}$$

where $A=[a_{ij}]$ and $B=[b_{ij}]$ are $n\times n$ and $n\times m$ constant matrices respectively.

At first, we examine the case where one element a_{ij} in the coefficient matrix A is changeable. (Hereafter we shall focuss our attention on only one parameter variation for simplicity.) The sensitivity equation due to deviation of a_{ij} is expressed, from Eq. (2–5), by

$$\frac{d\,{}_{ij}\boldsymbol{y}}{dt}=A\cdot{}_{ij}\boldsymbol{y}+E_{ij}\cdot\boldsymbol{x}\ , \tag{3-4}$$

where ${}_{ij}\boldsymbol{y}\equiv\partial\boldsymbol{x}/\partial a_{ij}$ is the sensitivity vector with respect to a_{ij} and the matrix E_{ij} denotes

$$E_{ij}=\begin{array}{c} \\ i \\ \\ \end{array}\!\!\begin{array}{c} j \\ \left(\begin{array}{ccc} & | & \\ \text{———} & 1 & \text{———} \\ & | & \end{array}\right) \end{array}, \tag{3-5}$$

namely E_{ij} has the unity value in i, j-component and otherwise zeros. The combined system is, thus, written as

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{x}\\ {}_{ij}\boldsymbol{y}\end{pmatrix}=\begin{pmatrix}A, & 0\\ E_{ij}, & A\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ {}_{ij}\boldsymbol{y}\end{pmatrix}+\begin{pmatrix}B\\ 0\end{pmatrix}\boldsymbol{u}\ . \tag{3-6}$$

In this case, G-matrix is reducible to the following $2n\times 2nm$ matrix after some manipulations;

$$G=\begin{pmatrix} B,\ AB\ , & \cdots, & A^kB & , \cdots, & A^{2n-1}B \\ 0,\ E_{ij}B, & \cdots, & \sum_{l=1}^{k} A^{l-1}E_{ij}A^{k-l}B, & \cdots, & \sum_{l=1}^{2n-1} A^{l-1}E_{ij}A^{2n-1-l}B \end{pmatrix}. \tag{3-7}$$

We can conclude, therefore, that if and only if the rank of G-matrix described by Eq. (3–7) is $2n$, the combined system (3–6) is controllable in the whole space ($\boldsymbol{x}$, ${}_{ij}\boldsymbol{y}$).

Secondly, we turn our attention to another case where the parameter b_{ij} in the coefficient matrix B is changeable. In this case, as the sensitivity equation, we have

$$\frac{d_{ij}\boldsymbol{y}}{dt}=A\cdot{}_{ij}\boldsymbol{y}+E_{ij}\cdot\boldsymbol{u}\,, \tag{3–8}$$

where ${}_{ij}\boldsymbol{y}\equiv\partial\boldsymbol{x}/\partial b_{ij}$. The combined system is, thus, written by

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{x}\\ {}_{ij}\boldsymbol{y}\end{pmatrix}=\begin{pmatrix}A, & 0\\ 0, & A\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ {}_{ij}\boldsymbol{y}\end{pmatrix}+\begin{pmatrix}B\\ E_{ij}\end{pmatrix}\boldsymbol{u}\,, \tag{3–9}$$

then G-matrix of this combined system (3–9) is given by

$$G=\begin{pmatrix}B, & AB & , \cdots, & A^kB & , \cdots, & A^{2n-1}B\\ E_{ij}, & AE_{ij}, & \cdots, & A^kE_{ij}, & \cdots, & A^{2n-1}E_{ij}\end{pmatrix}. \tag{3–10}$$

The controllability of the combined system (3–9) depends on the rank of this G-matrix.

Thirdly, let us examine the controllability of the combined system which consists of the original system described by Eq. (3–3) and its sensitivity equations with respect to the initial condition $\boldsymbol{x}^0$. Corresponding to Eq. (2–9), the sensitivity equation with respect to one initial condition x_i^0 is given by

$$\frac{d_i\boldsymbol{z}}{dt}=A\cdot{}_i\boldsymbol{z}(t)\,. \tag{3–11}$$

It is immediately concluded that the combined system is not controllable in any larger dimensional space than the original one ($\boldsymbol{x}$) by reason of the complete separation of Eq. (3–11) from Eq. (3–3). But this is not generally the case in a nonlinear original system.

In order to derive more concrete and meaningful results, let us pay special attention to a particular form of model equation (3–3), that is,

$$\frac{d}{dt}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ \vdots\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & \cdots & \cdots & 0\\ \vdots & 0 & 1 & & & \vdots\\ \vdots & & & \ddots & & \vdots\\ \vdots & & & & \ddots & 0\\ 0 & \cdots & \cdots & \cdots & 0 & 1\\ a_1, & a_2, & \cdots, & a_\nu, & \cdots, & a_n\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ \vdots\\ \vdots\\ x_n\end{pmatrix}+\begin{pmatrix}0\\ 0\\ \vdots\\ \vdots\\ 0\\ b\end{pmatrix}u\,, \quad (b\neq 0)\,. \tag{3–12}$$

This is of the simplest form in the controllable systems.

Let G_ν denote G-matrix associated with the combined system where the sensitivity equation with respect to the parameter a_ν ($\nu=1, 2, \cdots, n$) in Eq. (3–12) is taken into consideration. After some calculations, we can find G_ν to be

$$G_\nu=\left(\begin{array}{ccc|ccc} & & 1 & {}_1t_n & \cdots\cdots & {}_nt_n \\ 0 & \cdot^{\cdot^{\cdot}} & \vdots & & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & {}_1t_n & \cdots\cdots & {}_nt_n & \cdots\cdots & {}_{2n-1}t_n \\ \hline & & & 1, & s_1, \cdots & s_{\nu-1} \\ & 0 & & 1 & \cdot^{\cdot^{\cdot}} & \vdots \\ & & 1 & \cdot^{\cdot^{\cdot}} & & \vdots \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \vdots \\ 1, & s_1 & \cdots & s_{\nu-1} & \cdots\cdots & s_{\nu+n-1} \end{array}\right)\times b^{2n} \qquad (\nu=1, 2, \cdots, n)\,, \tag{3–13}$$

where for $i=1, 2, \cdots, n-1$

$$ {}_0t_i=0, \quad {}_kt_i={}_{k-1}t_{i+1} \qquad (k\geq 1)\,, \tag{3–14}$$

$$ {}_0t_n=1, \quad {}_kt_n=\sum_{l=1}^{n} a_l\cdot{}_{k-l}t_l \qquad (k\geq 1)\,, \tag{3–15}$$

and

$$ s_0=1, \quad s_i=\sum_{k=0}^{i} {}_kt_n\cdot{}_{i-k}t_n \qquad (i\geq 1)\,. \tag{3–16}$$

It is immediately concluded from the form of G_ν that the combined system with respect to the parameter a_ν is controllable in the subspace–$(\boldsymbol{x}, {}_\nu y_1, {}_\nu y_2, \cdots, {}_\nu y_{n-\nu+1})$. (${}_\nu y_i$ denotes i-th component of the sensitivity vector ${}_\nu\boldsymbol{y}$ with respect to a_ν). On the other hand, the controllability in the whole space-$(\boldsymbol{x}, {}_\nu\boldsymbol{y})$ is reduced to the examination of the following determinant;

$$|G_\nu|=(-1)^{\sum_{i=1}^{n-1} i}\prod_{i=0}^{\nu-2}({}_is_1-{}_1t_n)^{\nu-1-i}\cdot b^{2n}\cdot\left|\begin{array}{cccc|ccc} & & & & \overbrace{1 \quad s_1 \quad \cdots\cdots\cdots \quad s_{\nu-1}}^{\nu} \\ & & \cdot^{\cdot^{\cdot}} & & \vdots \\ 1 & s_1 & \cdots\cdots & & s_{n-1} \\ \hline & & & 1 & {}_{\nu-1}s_1 \cdots {}_{\nu-1}s_{n-\nu+1} \\ & & \cdot^{\cdot^{\cdot}} & & \vdots \\ \underbrace{1 \quad {}_{\nu-1}s_1 \quad \cdots\cdots}_{\nu-1} & & & & {}_{\nu-1}s_{n-1} \end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} n-\nu+1 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \nu-1 \end{array}, \tag{13–17}$$

where

$$ {}_0s_1 = s_1, \quad {}_\mu s_i = \frac{{}_{\mu+1}s_{i+1} - {}_{i+1}t_n}{{}_{\mu-1}s_1 - {}_1t_n} \qquad (\mu \geq 1) . \tag{3–18} $$

Another combined system to be examined is the one with respect to the parameter b. In this case G_b-matrix is given by

$$ G_b = \begin{pmatrix} 0, & a_{1n}b, & a_{1n}^{(2)}b, & \cdots\cdots, & a_{1n}^{(2n-1)}b \\ 0, & a_{2n}b, & a_{2n}^{(2)}b, & \cdots\cdots, & a_{2n}^{(2n-1)}b \\ \vdots & \vdots & \vdots & & \vdots \\ 0, & a_{n-1n}b, & a_{n-1,n}^{(2)}b, & \cdots\cdots, & a_{n-1n}^{(2n-1)}b \\ b, & a_{nn}b, & a_{nn}^{(2)}b, & \cdots\cdots, & a_{nn}^{(2n-1)}b \\ 0, & a_{1n} & a_{1n}^{(2)}, & \cdots\cdots, & a_{1n}^{(2n-1)} \\ 0, & a_{2n} & a_{2n}^{(2)}, & \cdots\cdots, & a_{2n}^{(2n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0, & a_{n-1n}, & a_{n-1n}^{(2)}, & \cdots\cdots, & a_{n-1n}^{(2n-1)} \\ 1, & a_{nn}, & a_{nn}^{(2)}, & \cdots\cdots, & a_{nn}^{(2n-1)} \end{pmatrix}, \tag{3–19} $$

where $a_{ij}^{(k)}$ denotes i, j-component of the matrix A^k.

From Eq. (3–19), it is soon found that the combined system is not controllable in any space larger than $\boldsymbol{x}$-space.

Example-1 First-Order Original System

Consider a first-order original system given by

$$ \frac{dx}{dt} = ax(t) + bu(t) , \tag{3–20} $$

and its sensitivity equation with respect to a;

$$ \frac{d_a y}{dt} = a \cdot {}_a y(t) + x(t) , \tag{3–21} $$

then the determinant $|G_a|$ of the combined system is given by

$$ |G_a| = \begin{vmatrix} b, & ab \\ o, & b \end{vmatrix} = b^2 \neq 0 \quad . \tag{3–22} $$

The combined system with respect to a is controllable in the whole space-$(x, {}_a y)$.

Example-2 Second-Order Original System

Consider a second-order original system described by

$$\begin{cases} \dfrac{dx_1}{dt}=x_2 \\ \dfrac{dx_2}{dt}=a_1x_1+a_2x_2+bu\,, \end{cases} \tag{3–23}$$

and its sensitivity equations with respect to the parameter a_2;

$$\begin{cases} \dfrac{d_2y_1}{dt}={}_2y_2 \\ \dfrac{d_2y_2}{dt}=a_{12}y_1+a_{22}y_2+x_2\,, \end{cases} \tag{3–24}$$

where ${}_2y_1\equiv\partial x_1/\partial a_2$ and ${}_2y_2\equiv\partial x_2/\partial a_2$. The determinant $|G_2|$ of the combined system is given by

$$|G_2|=-(s_1-{}_1t_2)\cdot b^4\begin{vmatrix} 1, & s_1 \\ 1, & {}_1s_1 \end{vmatrix}=-a_2\cdot b^4\left(\frac{a_1+2a_2^2}{a_2}-2a_2\right)=-a_1b^4\,. \tag{3–25}$$

It can be found, therefore, that the combined system is controllable in the whole space-$(x_1, x_2, {}_2y_1, {}_2y_2)$ if and only if $a_1\neq 0$. It can easily be checked, however, that the combined system is controllable in the subspace-$(x_1, x_2, {}_2y_1)$ irrespective of the value of a_1.

It is also easily be checked that the combined system with respect to the parameter a_1 is, in any case, controllable in the whole space-$(x_1, x_2, {}_1y_1, {}_1y_2)$.

4. Minimum Energy Problem with Terminal Constraints

To demonstrate the effectiveness of the sensitivity synthesis method, we consider the minimum energy problem with terminal constraints as a typical example. An assumption is made that the combined system is controllable so that the solution should exist uniquely.

4.1. Conventional design and sensitivity of terminal constraints to parameter variations

Let us assume the original controlled system to be

$$\frac{d\boldsymbol{x}}{dt}=A\boldsymbol{x}(t)+B\boldsymbol{u}(t),\quad \boldsymbol{x}(0)=\boldsymbol{x}^0\,. \tag{4–1}$$

In a conventional sense, the minimum energy problem with terminal constraints is stated as follows: Determine the control $\boldsymbol{u}(t)$ that

(i) satisfies the terminal constraint;

$$\boldsymbol{x}(T)=\boldsymbol{x}^T\,, \tag{4-2}$$

(ii) minimizes the energy cost functional;

$$J(\boldsymbol{u})=\frac{1}{2}\int_0^T \boldsymbol{u}'R\boldsymbol{u}dt\,, \tag{4-3}$$

where T is a given final time, $\boldsymbol{x}^T$ is a given desired final state and R is a positive definite matrix.

To this problem, we can immediately obtain the optimum control as[9]

$$\boldsymbol{u}^*(t)=-R^{-1}B'\boldsymbol{p}(t)\,, \tag{4-4}$$

where $\boldsymbol{p}(t)$ is the adjoint vector for $\boldsymbol{x}(t)$ determined by the canonical equation

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{p}\end{pmatrix}=\begin{pmatrix}A, & M\\ 0, & -A'\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{p}\end{pmatrix},\quad M=-BR^{-1}B'\,, \tag{4-5}$$

with two boundary conditions;

$$\boldsymbol{x}(0)=\boldsymbol{x}^0,\quad \boldsymbol{x}(T)=\boldsymbol{x}^T\,. \tag{4-6}$$

Let $\Psi(t)$ be the fundamental matrix of the homogeneous part of Eq. (4–1) and $\Xi_1(t)$ be

$$\Xi_1(t)=\int_0^t \Psi^{-1}(\tau)M\Psi'^{-1}(\tau)d\tau\,, \tag{4-7}$$

then $\boldsymbol{u}^*(t)$ given by Eq. (4–4) is reduced to

$$\boldsymbol{u}^*(t)=-R^{-1}B\Psi'^{-1}(t)\boldsymbol{\rho}\,, \tag{4-8}$$

where

$$\boldsymbol{\rho}=\Xi^{-1}(T)\{\Psi^{-1}(T)\boldsymbol{x}^T-\boldsymbol{x}^0\}\,. \tag{4-9}$$

The optimum trajectory is

$$\boldsymbol{x}^*(t)=\Psi(t)(\boldsymbol{x}^0+\Xi_1(t))\boldsymbol{\rho}\,. \tag{4-10}$$

Let us call the control given by Eq. (4–8) *conventional* optimum control. As seen in the above development, the conventional optimum control is obtained based only on the original system.

In this type of problem, it is often very important to analyse how the condition (i) is violated by the parameter perturbations. Then, let us next estimate the variation of the state at the final time T from its desired value $\boldsymbol{x}^T$ induced by the parameter perturbation Δa_{ij} or Δb_{ij}. The sensitivity equation with respect to a_{ij} in A is given by Eq. (3–4);

$$\frac{d_{ij}\boldsymbol{y}}{dt}=A_{ij}\boldsymbol{y}(t)+E_{ij}\boldsymbol{x}^*(t), \quad {}_{ij}\boldsymbol{y}(0)=0\,, \tag{4–11}$$

Integrating Eq. (4–11) after the substitution of Eq. (4–10), we have

$${}_{ij}\boldsymbol{y}(t)=\Psi(t)\{\Xi_2(t)\boldsymbol{x}^0+\Xi_{21}(t)\boldsymbol{\rho}\}\,, \tag{4–12}$$

where

$$\Xi_2(t)=\int_0^t \Psi^{-1}(\tau)E_{ij}\Psi(\tau)d\tau\,, \tag{4–13}$$

$$\Xi_{21}(t)=\int_0^t \Psi^{-1}(\tau)E_{ij}\Psi(\tau)\Xi_1(\tau)d\tau\,. \tag{4–14}$$

From Eq. (2–3), as the variation of the state at the time T, we have approximately,

$$\Delta_A\boldsymbol{x}(T)\cong \sum_{i,j} {}_{ij}\boldsymbol{y}(T)\Delta a_{ij}\,, \tag{4–15}$$

under the assumption that Δa_{ij} is small enough.

In a similar manner, for the perturbations Δb_{ij} in B, we have

$$\Delta_B\boldsymbol{x}(T)\cong \sum_{i,j} {}_{ij}\boldsymbol{y}(T)\Delta b_{ij}\,, \tag{4–16}$$

where ${}_{ij}\boldsymbol{y}(t)$ is obtained from Eq. (3–8).

If these variations are estimated to be large, we possibly lose the chief design objective. Then, another new design concept is required.

Example-3

Let us consider, for example, the problem to minimize

$$J(u)=\int_0^T u^2(t)dt\,, \tag{4–17}$$

for the model system described by the scalar linear differential equation

$$\frac{dx}{dt}=ax(t)+bu(t), \quad x(0)=x^0\,, \tag{4–18}$$

with the terminal constraint $x(T)=x^T$. The conventional optimum control $u^*(t)$ and its response $x^*(t)$ are given by

$$u^*(t)=\frac{a}{b}\frac{x^T-x^0e^{aT}}{\sinh aT}e^{-at}\,, \tag{4–19}$$

$$x^*(t)=\frac{x^T\sinh at-x^0\sinh a(t-T)}{\sinh aT}\,. \tag{4–20}$$

On the other hand, the sensitivity function with respect to a is given by

$$ {}_a y(t) = \frac{1}{4a \sinh aT} [\{(2at-1)e^{at} + e^{-at}\} x^T - \{(2ate^{-aT} - e^{aT})e^{at} + e^{aT} e^{-at}\} x^0] \tag{4–21} $$

In Fig. 1, a nominal optimum trajectory by the conventional method and two perturbed trajectories by the variation Δa are shown. It can be seen that the estimated trajectroies by Eq. (4–15) coincide well with the actual ones in case of the small variation Δa.

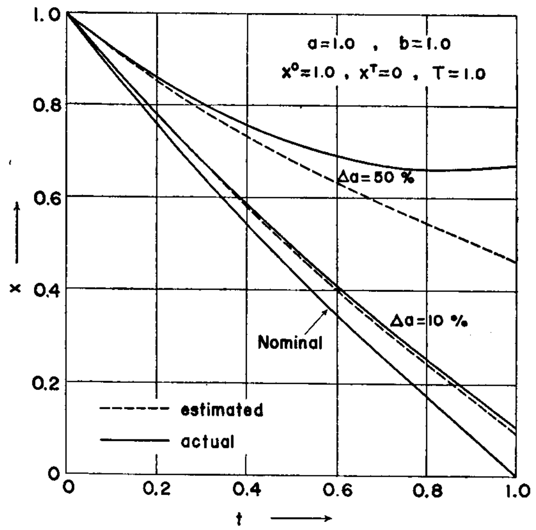

Fig. 1. The nominal optimum and perturbed trajectories by the conventional method.

4.2. New design: Sensitivity synthesis

It has been already pointed out in section 2 that we have a possibility to control both the model and its sensitivity with respect to the parameters, and in section 3 that the combined system becomes, in fact, controllable in some cases. Thus, the new synthesis concept is straightforward; the variation of the terminal constraints due to a certain variation Δa_{ij} is approximated to be $\Delta_{ij}\boldsymbol{x}(T) = {}_{ij}\boldsymbol{y}(T) \cdot \Delta a_{ij}$, then it is very desirable to choose such a control $\boldsymbol{u}(t)$ that it should satisfy ${}_{ij}\boldsymbol{y}(T) = \boldsymbol{0}$, which is always possible for the controllable combined system. From this point of view, the minimum energy problem with terminal constraints is, then, restated as follows:

For the controllable combined system;

$$ \frac{d}{dt}\begin{pmatrix} \boldsymbol{x} \\ {}_{ij}\boldsymbol{y} \end{pmatrix} = \begin{pmatrix} A, & 0 \\ E_{ij}, & A \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ {}_{ij}\boldsymbol{y} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix}\boldsymbol{u}, \quad \begin{pmatrix} \boldsymbol{x}(0) \\ {}_{ij}\boldsymbol{y}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^0 \\ \cdot \\ 0 \end{pmatrix}, \tag{4–22} $$

choose the optimum control $\boldsymbol{u}(t)$ that

(i′) satisfies the terminal constraints;

$$\boldsymbol{x}(T)=\boldsymbol{x}^T \quad \text{and} \quad {}_{ij}\boldsymbol{y}(T)=0\,, \tag{4–23}$$

(ii) minimizes the energy cost functional;

$$J=\frac{1}{2}\int_0^T \boldsymbol{u}'R\boldsymbol{u}dt\,. \tag{4–24}$$

The solution of this problem is obtained in a similar manner as in section 4.1. to be

$$\boldsymbol{u}^*(t)=-R^{-1}B'\Psi^{-1}(t)\{\boldsymbol{\rho}-\Xi_2(t)\boldsymbol{\tau}_{ij}\}\,, \tag{4–25}$$

by solving the following system of the canonical equations;

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{x} \\ {}_{ij}\boldsymbol{y} \\ \boldsymbol{p} \\ {}_{ij}\boldsymbol{q} \end{pmatrix}=\begin{pmatrix} A & & M & \\ E_{ij} & A & & \\ & & -A' & -E_{ij} \\ & & & -A' \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ {}_{ij}\boldsymbol{y} \\ \boldsymbol{p} \\ {}_{ij}\boldsymbol{q} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x}(0) \\ {}_{ij}\boldsymbol{y}(0) \end{pmatrix}=\begin{pmatrix} \boldsymbol{x}^0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x}(T) \\ {}_{ij}\boldsymbol{y}(T) \end{pmatrix}=\begin{pmatrix} \boldsymbol{x}^T \\ 0 \end{pmatrix}, \tag{4–26}$$

where the constant vectors $\boldsymbol{\rho}$ and $\boldsymbol{\tau}_{ij}$ are the solutions of the terminal condition equations

$$\begin{pmatrix} \Xi_1(T), & -\Xi_{12}(T) \\ \Xi_{21}(T), & -\Xi_{212}(T) \end{pmatrix}\begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{\tau}_{ij} \end{pmatrix}=\begin{pmatrix} \Psi^{-1}(T)\boldsymbol{x}^T-\boldsymbol{x}^0 \\ \Xi_2(T)\boldsymbol{x}^0 \end{pmatrix}, \tag{4–27}$$

and

$$\Xi_{12}(t)=\int_0^t \Psi^{-1}(\tau)M\Psi'^{-1}(\tau)\Xi_2(\tau)d\tau\,, \tag{4–28}$$

$$\Xi_{212}(t)=\int_0^t \Psi^{-1}(\tau)E_{ij}\Psi(\tau)\Xi_{12}(\tau)d\tau\,, \tag{4–29}$$

$$\Xi_2(t)=\int_0^t \Psi'(\tau)E_{ij}\Psi'^{-1}(\tau)d\tau\,. \tag{4–30}$$

The new nominal optimum responses are given by

$$\boldsymbol{x}^*(t)=\Psi(t)\{\boldsymbol{x}^0+\Xi_1(t)\boldsymbol{\rho}-\Xi_{12}(t)\boldsymbol{\tau}_{ij}\}\,, \tag{4–31}$$

$${}_{ij}\boldsymbol{y}^*(t)=\Psi(t)\{\Xi_2(t)\boldsymbol{x}^0+\Xi_{21}(t)\rho-\Xi_{212}(t)\boldsymbol{\tau}_{ij}\}\,. \tag{4–32}$$

Example-4 ·

Let us introduce the sensitivity aspect into the same problem discussed in the previous example-3. The combined system with respect to the parameter a is given by

$$\begin{cases} \dfrac{dx}{dt}=ax(t)+bu(t), \quad x(0)=x^0\,. \\ \dfrac{d_a y}{dt}=a\cdot{}_a y(t)+x(t), \quad {}_a y(0)=0\,, \end{cases} \tag{4-33}$$

which is already shown to be controllable in example-1.

By minimizing the energy cost (4–17) under the newly added constraint ${}_a y(T)=0$, we have

$$u^*(t)=-\frac{b}{2}\,(Pe^{-at}-Qte^{-at})\,, \tag{4-34}$$

$$x^*(t)=x^0e^{at}-\frac{b}{4a}(e^{at}-e^{-at})P+\frac{b^2}{8a^2}\{e^{at}-(2at+1)e^{-at}\}Q\,, \tag{4-35}$$

$$\begin{aligned} {}_a y^*(t)=x^0te^{at}-\frac{b^2}{8a^2}\{(2at-1)e^{at}+e^{-at}\}P+\frac{b^2}{8a^3} \\ \times\{(at-1)e^{at}+(at+1)e^{-at}\}Q\,, \end{aligned} \tag{4-36}$$

where the constants are

$$P=\frac{1}{\Delta}(p_1x^0+p_2x^T)\,, \quad Q=\frac{1}{\Delta}(q_1x^0+q_2x^T)\,, \tag{4-37}$$

$$\Delta=\frac{b^2}{8a^2}\{(e^{aT}-e^{-aT})^2-(2aT)^2\}\,, \tag{4-38}$$

$$p_1=\frac{1}{a}\{e^{2aT}-2aT(aT+1)-1\}, \quad p_2=\frac{1}{a}\{(aT-1)e^{aT}+(aT+1)e^{-aT}\}\,, \tag{4-39}$$

$$q_1=e^{2aT}-2aT-1, \quad q_2=(2aT-1)e^{aT}+e^{-aT}\,. \tag{4-40}$$

In Fig. 2, nominal optimum trajectories by the new method are shown, where the perturbed trajectories pass through the hatched areas for $|\Delta a|/|a|\leq 10\%$. From Fig. 2, we can see clearly the superiority of the new design method to the conventional one. (It should be noticed here that the differences between the nominal and the perturbed trajectories are multiplied by 10 for the visual perspicuity.)

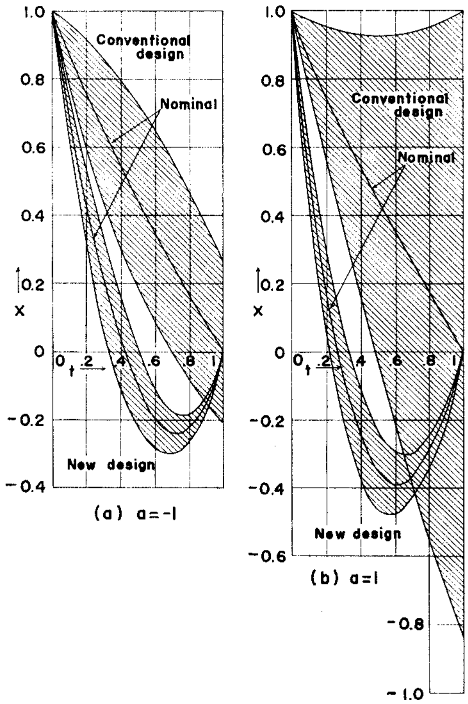

Fig. 2. The comparison of the conventional design method and the new one with ${}_a y(1)=0$.

Example-5

If we wish to make the terminal constraint $x(T)=x^T$ more rigid, we must take higher sensitivity equations into consideration.

The combined system with both the first- and the second-order sensitivity equations is given by

$$\frac{d}{dt}\begin{pmatrix} x \\ {}^{(1)}_{a}y \\ {}^{(2)}_{a}y \end{pmatrix}=\begin{pmatrix} a, & 0, & 0 \\ 1, & 0, & 0 \\ 0, & 2, & a \end{pmatrix}\begin{pmatrix} x \\ {}^{(1)}_{a}y \\ {}^{(2)}_{a}y \end{pmatrix}+\begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}u\;, \quad \begin{pmatrix} x(0) \\ {}^{(1)}_{a}y(0) \\ {}^{(2)}_{a}y(0) \end{pmatrix}=\begin{pmatrix} x^0 \\ 0 \\ 0 \end{pmatrix}, \tag{4–41}$$

where ${}^{(i)}_{a}y=\dfrac{\partial^i x}{\partial a^i}$ $(i=1,2)$. It is easily checked that the combined system (4–41) is controllable.

Minimization of the energy cost (4–17) under one more constraint ${}^{(2)}_{a}y(T)=0$ gives

$$u^*(t)=-\frac{b}{2}\,(Pe^{-at}-te^{-aT}Q+t^2e^{-at}R)\;, \tag{4–42}$$

$$x^*(t)=e^{at}X+\frac{b^2}{4a}e^{-at}P-\frac{b^2}{8a^2}(2at+1)e^{-at}Q$$
$$+\frac{b^2}{8a^3}\{2at(at+1)+1\}e^{-at}R, \tag{4–43}$$

$${}^{(1)}_{\ a}y^*(t)=e^{at(1)}Y+te^{at}X-\frac{b^2}{8a^2}e^{-at}P+\frac{b^2}{8a^3}(at+1)e^{-at}Q$$
$$-\frac{b^2}{16a^4}\{2(at+1)^2+1\}e^{-at}R, \tag{4–44}$$

$${}^{(2)}_{\ a}y(t)=e^{at}\cdot{}^{(2)}Y+2te^{at}\cdot{}^{(1)}Y+t^2e^{at}X+\frac{b^2}{8a^3}e^{-at}P$$
$$-\frac{b^2}{16a^4}(2at+3)e^{-at}Q+\frac{b^2}{8a^5}\{(at+1)(at+2)+1\}e^{-at}R, \tag{4–45}$$

where the constants are determined by the following 6-dimensional matrix equation

$$\begin{pmatrix} 1, & 0, & 0, & \frac{b^2}{4a}, & -\frac{b^2}{8a^2}, & \frac{b^2}{8a^3} \\ 0, & 1, & 0, & -\frac{b^2}{8a^2}, & \frac{b^2}{8a^3}, & -\frac{3b^2}{16a^4} \\ 0, & 0, & 1, & \frac{b^2}{8a^3}, & -\frac{3b^2}{16a^4}; & \frac{3b^2}{8a^5} \\ e^{aT}, & 0, & 0, & \frac{b^2}{4a}a^{-aT}, & -\frac{b^2}{8a^2}(2aT+1)e^{-aT}, & \frac{b^2}{8a^2}\{2aT(aT+1)+1\}e^{-aT} \\ Te^{aT}, & e^{aT}, & 0, & -\frac{b^2}{8a^2}e^{-aT}, & \frac{b^2}{8a^3}(aT+1)e^{-aT}, & -\frac{b^2}{16a^4}\{2(aT+1)^2+1\}^{-aT} \\ T^2e^{aT}, & 2Te^{aT}, & e^{aT}, & \frac{b^2}{8a^3}e^{-aT}, & -\frac{b^2}{16a^4}(2aT+3)e^{-aT}, & \frac{b^2}{8a^5}\{(aT+1)(aT+2)+1\}e^{-aT} \end{pmatrix} \begin{pmatrix} X \\ {}^{(1)}Y \\ {}^{(2)}Y \\ P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} x^0 \\ 0 \\ 0 \\ x^T \\ 0 \\ 0 \end{pmatrix} \tag{4–46}$$

Table 1. The numerical comparison of the variations of the terminal constraint $x^T=0$ and the energy consumption.

a	Method	$y(1)\,\Delta a$	$x(1)$	Energy
−5	Conv.	$-.303\times10^{-2}$	$-.233\times10^{-2}$	$.454\times10^{-3}$
	New 1	0	$-.422\times10^{-3}$	$.374\times10^{-1}$
	New 2	0	—	.517
−1	Conv.	$-.242\times10^{-1}$	$-.227\times10^{-1}$	.313
	New 1	0	$-.425\times10^{-3}$	$.227\times10$
	New 2	0	—	$.626\times10$
1	Conv.	.093	.100	$.231\times10$
	New 1	0	$.160\times10^{-2}$	$.627\times10$
	New 2	0	$.106\times10^{-4}$	$.123\times10^2$
5	Conv.	$.705\times10$	$.116\times10^2$	$.100\times10^2$
	New 1	0	.545	$.200\times10^2$
	New 2	0	$.231\times10^{-1}$	$.305\times10^2$

New 1; New design with ${}_ay(1)=0$, New 2; with ${}^{(1)}_{\ a}y(1)={}^{(2)}_{\ a}y(1)=0$.

Fig. 3 shows the nominal optimum and the perturbed trajectories by the new design method taking sensitivity equations up to second order into consideration. The detailed variations of x at the final time $T=1$ from its desired value $x^T=0$ are listed in Tab. 1.

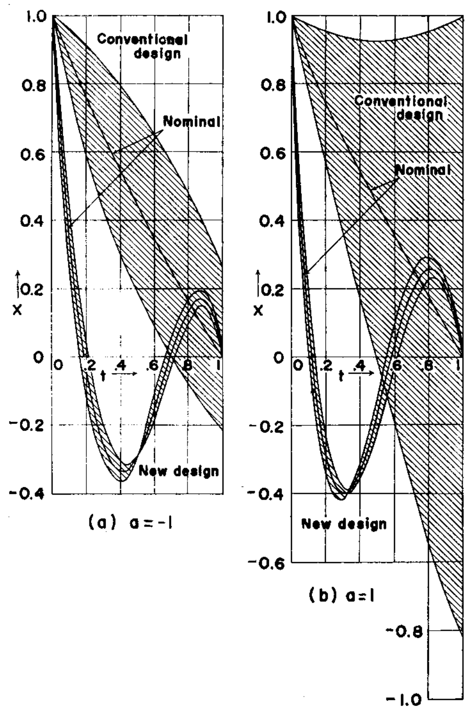

Fig. 3. The comparison of the conventional design method and the new one with ${}^{(1)}_{a}y(1)={}^{(2)}_{a}y(1)=0$.

Example-6

The addition of extra constraints ${}_{a}y(T)=0$ or both ${}^{(1)}_{a}y(T)=0$ and ${}^{(2)}_{a}y(T)=0$ results in a smaller terminal constraint error, but unfortunately in more energy consumption as seen in Tab. 1. To compromise, therefore, the increase of the energy consumption and the decrease of the terminal constraint error is another way of the sensitivity synthesis. This can clearly be accomplished by modifying the performance index (4–17) into

$$J'(u)=w_1\int_0^T u^2(t)dt+w_{2a}y^2(T)\,, \tag{4–47}$$

instead of adding extra constriaint ${}_{a}y(T)=0$ in the combined system (4–33). In Eq. (4–47), w_1 and w_2 are possitive weight constants.

In this case, we obtain

$$u^*(t)=\frac{b}{2w_1}(Qt-P)e^{-at}, \tag{4–48}$$

$$x^*(t)=x^0e^{at}-\frac{b^2}{4aw_1}(e^{at}-e^{-at})P+\frac{b^2}{8a^2w_1}\{e^{at}-(2at+1)e^{-at}\}Q, \tag{4–49}$$

where

$$P=\frac{1}{\varDelta}(p_1x^0+p_2x^T), \quad Q=\frac{1}{\varDelta}(q_1x^0+q_2x^T), \tag{4–50}$$

$$\varDelta=\frac{b^2}{8a^2w_1}\{(e^{aT}-e^{-aT})^2-(2aT)^2+2w(1-e^{-2aT})\}, \tag{4–51}$$

$$p_1=\frac{1}{a}\{e^{2aT}-2aT(aT+1)-1+w\},$$

$$p_2=\frac{1}{a}\{(aT-1)e^{aT}+(aT+1-w)e^{-aT}\}, \tag{4–52}$$

$$q_1=e^{2aT}-2aT-1, \quad q_2=(2aT-1)e^{aT}+e^{-aT}, \tag{4–53}$$

$$w=4a^3w_1/b^2w_2. \tag{4–54}$$

Fig. 4 illustrates the relation between the energy consumtpion and the terminal sensitivity value in the same system considered in **Ex.-4.**

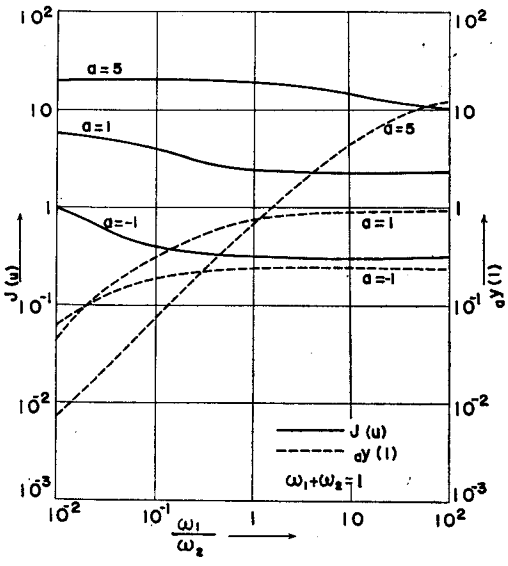

Fig. 4. The relation between the energy consumption and the final sensitivity.

4.3. Extended effectiveness of sensitivity considerations

In the preceding paragraphs, our attentions are focussed on the effect caused by the parameter variation. Here, let us point out that we can, in like manner, devise a counterplot against the effect caused by existing slight nonlinearity which is consciously or unconsciously neglected in the stage of modeling.

We often establish a simplified model

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{f}(t,\ \boldsymbol{x},\ \boldsymbol{q}),\quad \boldsymbol{x}(t_0)=\boldsymbol{x}^0\ , \tag{4–55}$$

by neglecting the existing small $\varepsilon(t,\ \boldsymbol{x}_p)$ in the real system

$$\frac{d\boldsymbol{x}_p}{dt}=\boldsymbol{f}(t,\ \boldsymbol{x}_p,\ \boldsymbol{q}+\varepsilon(t,\ \boldsymbol{x}_p)),\quad \boldsymbol{x}_p(t_0)=\boldsymbol{x}^0\ . \tag{4–56}$$

Our first interest is to evaluate the difference between $\boldsymbol{x}(t)$ and $\boldsymbol{x}_p(t)$. Making assumptions that

(i) $\boldsymbol{f}$ and its derivatives $\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$ and $\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}$ are continuous in a closed domain in $(t,\ \boldsymbol{x},\ \boldsymbol{q})$-space, and

(ii) there exists a positive constant vector $\boldsymbol{\partial}$ such that

$$||\varepsilon(t,\ \boldsymbol{x}_p(t)||\leq\boldsymbol{\partial}\ , \tag{4–57}$$

over $t\in[0,\ T]$, where $||\ \cdot\ ||$ denotes a element-wise norm,

we obtain the meaningful inequality

$$||\boldsymbol{x}_p-\boldsymbol{x}||\leq\frac{K_1}{K_2}\boldsymbol{\partial}(e^{K_2t}-1)\qquad(0\leq t\leq T)\ , \tag{4–58}$$

where K_1 and K_2 are some constants.

By displacing $\boldsymbol{\partial}$ to $\Delta\boldsymbol{q}$ appearing in previous sections, Eq. (4–58) suggests that the difference $||\boldsymbol{x}_p-\boldsymbol{x}||$ is almost less than $||_q\boldsymbol{y}\boldsymbol{\partial}||$ where $_q\boldsymbol{y}$ is the sensitivity matrix for the system (4–55) with respect to $\boldsymbol{q}$. Hence, the parammeter sensitivity synthesis method may be valid as a compensation for the simplifying loss which occurs in neglecting the slight nonlinear properties of the original.

Example-7

Let us consider the combined system which consists of the simplified linear model;

$$\frac{dx}{at}=ax(t)+bu(t),\quad x(0)=x^0\ , \tag{4–59}$$

for the given nonlinear system;

$$\frac{dx}{dt} = ax(t) + \varepsilon x^3(t) + bu(t), \quad x(0) = x, \tag{4-60}$$

and the sensitivity model of Eq. (4–59);

$$\frac{d_a y}{dt} = a \cdot {}_a y(t) + x(t). \tag{4-61}$$

The optimum control which satisfies the same task in **Ex.-4** is given by Eq. (4–34).

In Fig. 5, we can see that the newly designed control achieves more successfully the task, while the conventional control based only on Eq. (4–59) does not. Rewriting Eq. (4–60) to

$$\frac{dx}{dt} = (a + \varepsilon x^2(t))x(t) + bu(t), \tag{4-62}$$

we can make clear the appropriateness of their results in Fig. 5.

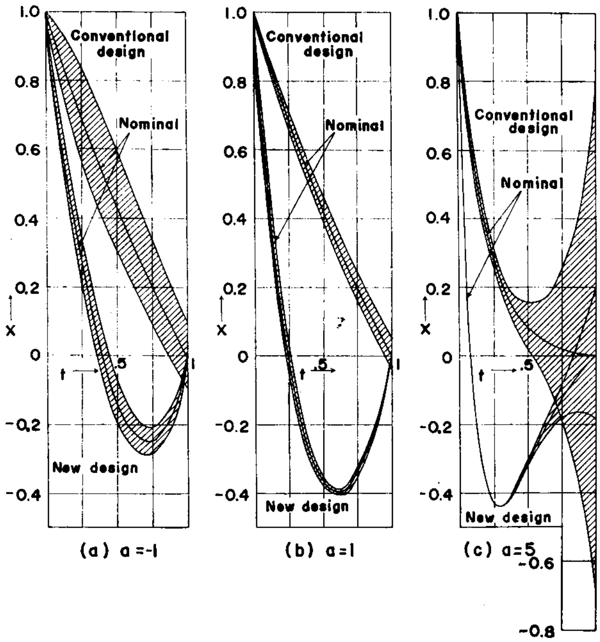

Fig. 5. The applicability of the present method to the samll nonlinear perturbation. (Deviations are multiplied by 10 in (a).)

5. Conclusions

The major purpose of this paper is to provide a buffer between the mathematical development and the engineering one in the open-loop control system design. We pointed out the importance of introducing sensitivity functions as new state variables in the system design process. Then, we were concerned with the controllability of the combined system which consists of the original and its sensitivity model. Though our attentions were restricted only on linear original systems, some interesting controllability properties of the combined system were made clear.

After these fundamental considerations on the controllability, we introduced the concept of sensitivity synthesis into the minimum energy problem with terminal constraints. Several worked examples were given to demonstrate the superiority of the present method to the conventional one against the change or uncertainty of the system parameters. We also showed that in a certain case, the undesirable effect caused by the existing small nonlinearity which is often neglected at the stage of modeling was diminished by similar considerations.

We believe that when the engineers try to design the optimum control system for a given physical system, not for the model of it, the introduction of the sensitivity concept in the initial stage of design will become more and more important.

All numerical calculations were carried out at Kyoto University Computing Center by the KDC-II Computer.

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