Reliability Considerations on Redundant Systems with Repair

By

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It is well-known that a method of increasing system reliability is the addition of redundancy and/or repair.

In this paper, the reliability models of complex systems with redundancy and/or repair are formulated by birth-death processes and semi-Markov processes. The authors show how the system modification yields the increase in mean time to system failure. They show how the system with repair yields the increase in mean time to system failure compared with the similar system without repair. Finally, they also show that the system reliability depends upon the repair time distributions and that many calculations of the system reliability suggest the significant properties.

1. Introduction

As the engineering techniques are developed, the systems become larger and more complicated. It is a very important problem whether a system can operate with high reliability. It is well-known that a method of increasing system reliability is to make the system redundant and/or repairable. In this paper, we evaluate the reliability of redundant and repairable systems.

The problems concerned with the reliability of complex systems have been discussed by many authors, e.g. Barlow[1], Garg[4], Gaver[5], and others.

Many measures of system reliability have been considered. Hosford[6] has proposed three measures of system reliability as follows:

(i) Pointwise Availability; the probability that the system is operable at time t.

(ii) Interval Availability; the expected fraction of a given interval of time that the system will be able to operate within the tolerances.

(iii) Reliability; the probability that the system will be able to operate without a failure for a given interval of time.

In this paper, we adopt a Mean Time to System Failure (MTSF) as a measure of system reliability. The MTSF represents a mean time until the system is in a state of system first failure. We can give many examples of models where this

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MTSF is an important factor. In these models, occurrence of system failure is considered to be important. We often encounter such systems in recent complicated systems.

At first, let us consider the following general problems. There is a system consisting of many elements. The term element means, for example, computer, electric power supply, communication system, etc. Let us suppose that it is sufficient to use a part of many elements so that the system can perform its function. The system has redundant elements. Each element may sometimes fail. The distribution function of failure time for each element is given by $F(t)$ $(t \geq 0)$. A failed element goes immediately into repair, or forms a queue to be repaired. The distribution function of repair time for each element is given by $G(t)$ $(t \geq 0)$. And if a repair of a failed element is completed, it is put back into an operating state. Thus, we are concerned with the time until the system is in a state of system first failure. The diagram shown in Fig. 1 is useful in illustrating the structure.

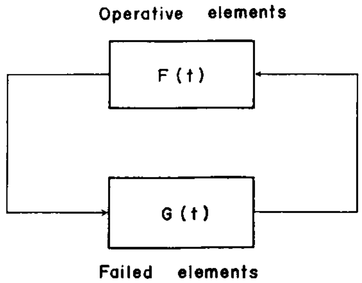

Fig. 1. Diagram illustrating the system.

We will show, especially, three problems arising in these models.

In sections 2 and 3, the first problem is dealt with. We assume that both the failure time distribution $F(t)$ and the repair time distribution $G(t)$ are exponential. Consider a system consisting of n elements and only one repair facility. We analyze this system and consider this as a standard one. So, we investigate to what degree the reliability increases when spare elements are added, or when repair facilities are added on the basis of the standard system.

In section 4, we consider, in detail, "2 out of n" system as a second problem. "2 out of n" system means that the system fails when not less than two of n elements are in the failed state. In particular, the "2 out of 2" system is sometimes called a two element redundant system. We evaluate the MTSF for the system consisting of dissimilar elements and of dissimilar repair facilities. Furthermore, we investigate to what degree the reliability of "2 out of n" system with repair increases compared with a similar system without repair.

In sections 5 and 6, we consider the third problem, i.e., an "m out of n" system with repair, which means that the system fails when not less than m of n elements are in the failed state. We assume, here, that the failure time distribution $F(t)$ is exponential and the repair time distribution $G(t)$ is arbitrary, for this system.

Moreover, we investigate to what degree the reliability of this system increases compared with the standard system with an exponential repair time distribution.

2. Birth-Death Process Models for Complex Systems

Here, we consider the following complex systems. The system consists of $(n+s)$ elements, where at first n elements are operating and s elements are spares. The system has at most r repair facilities being simultaneously available. On the initial condition, n elements are operating at time 0. The system is assumed to fail only when all its elements including spare elements fail. Let us assume that the failure time distribution of each element is exponential with mean $1/\lambda$. That is

$$F(t)=1-\exp(-\lambda t) . \tag{2.1}$$

While, it is assumed that the repair time distribution $G(t)$ of each repair facility is exponential with mean $1/\mu$. That is

$$G(t)=1-\exp(-\mu t) . \tag{2.2}$$

Under these assumptions, it is well-known that the system forms a birth-death process as a mathematical model.

A birth-death process is a stationary Markov process whose state space is the non-negative integers. Now, let us define the state of a birth-death process $\{X_t; t\geq 0\}$. The number of failed elements corresponds to the state of the birth-death process. For example, state i represents that i elements are in the failed state. This system has $(n+s+1)$ states, $i=0, 1, \cdots, (n+s)$.

Let us define the transition probability

$$P_{ij}(t)=P_r\{X(t)=j\,|\,X(0)=i\} \qquad (i, j=0, \cdots, n+s) , \tag{2.3}$$

that the process is in state j at time t starting from state i at time 0.
As t tends to 0, we have

$$P_{ij}(t)=\begin{cases} \lambda_i t+o(t) & (j=i+1) \\ 1-(\lambda_i+\mu_i)t+o(t) & (j=i) \\ \mu_i t+o(t) & (j=i-1) \\ 0 & (\text{otherwise}) . \end{cases} \tag{2.4}$$

where λ_i and μ_i are a birth rate (a failure rate) and a death rate (a repair rate) in state i, respectively. In this system, these rates λ_i's and μ_i's are given in Table 1.

On the basis of the above preparations, $P_{ij}(t)$'s satisfy the following system of differential equations.

Table 1. Birth rates λ_i's and death rates μ_i's for the system.

A) $r \geq s$

state k	λ_k	μ_k
$k \leq s$	$n\lambda$	$k\mu$
$s \leq k \leq r$	$(n+s-k)\lambda$	$k\mu$
$r \leq k$	$(n+s-k)\lambda$	$r\mu$

B) $r \leq s$

state k	λ_k	μ_k
$k \leq r$	$n\lambda$	$k\mu$
$r \leq k \leq s$	$n\lambda$	$r\mu$
$s \leq k$	$(n+s-k)\lambda$	$r\mu$

$$\begin{cases} P'_{i0}(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \\ P'_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{i,j}(t) + \mu_{j+1} P_{i,j+1}(t) \\ \qquad\qquad (0 \leq i, j \leq n+s-1) . \end{cases} \tag{2.5}$$

Now, we assume that n elements begin to operate simultaneously at time 0. So, an initial condition is given by

$$P_{0j}(0) = \delta_{0j} \qquad (j=0, 1, \cdots, n+s) , \tag{2.6}$$

where

$$\delta_{0j} = \begin{cases} 1 & (j=0) \\ 0 & (j \neq 0) . \end{cases} \tag{2.7}$$

Writing the system of differential equations (2.5) in a matrix form, we have

$$\boldsymbol{P}'(t) = \boldsymbol{A}\boldsymbol{P}(t) , \tag{2.8}$$

where

$$\boldsymbol{P}(t) = [P_{ij}(t)] , \tag{2.9}$$

and

$$\boldsymbol{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots\cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots\cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} . \tag{2.10}$$

Associated with the matrix $\boldsymbol{A}$ is a sequence of polynominal $\{Q_i(x)\}$, defined by the following recurrence relations

$$\begin{cases} Q_0(x) = 1 \\ -xQ_0(x) = -\lambda_0 Q_0(x) + \lambda_1 Q_1(x) \\ -xQ_i(x) = \mu_i Q_{i-1}(x) - (\lambda_i + \mu_i) Q_i(x) + \lambda_i Q_{i+1}(x) \qquad (i>0) . \end{cases} \tag{2.11}$$

Above Eq. (2.11) can be written in a matrix form as

$$\boldsymbol{A}\boldsymbol{Q}(x) = -x\boldsymbol{Q}(x) , \tag{2.12}$$

where

$$^{t}Q(x)=(Q_0(x),\ Q_1(x),\ \cdots)^{*}\ . \tag{2.13}$$

Let $G_{ij}(t)$ be a first passage time distribution of this system from state i to state j. That is, we consider a random variable T at which time this process enters, for the first time, into state j starting from state i at time 0. Then, the probability that T is not greater than t is $G_{ij}(t)$. Let $g_{ij}(s)$ be the Laplace-Stieltjes transform of $G_{ij}(t)$, i.e.,

$$g_{ij}(s)=\int_0^\infty \exp(-st)dG_{ij}(t)\ . \tag{2.14}$$

Then, $g_{ij}(s)$ is given by[10]

$$g_{ij}(s)=Q_i(-s)/Q_j(-s)\ . \tag{2.15}$$

The initial condition is $i=0$, since n elements begin to operate simultaneously at time 0. From the first equation of Eq. (2.11), we have

$$Q_0(-s)=1, \quad \text{for all } s\ . \tag{2.16}$$

Therefore, the Laplace-Stieltjes transform of the first passage time distribution from state 0 to state j is given by

$$g_{0j}(s)=1/Q_j(-s)\ . \tag{2.17}$$

Let E_j be the mean time from state 0, for the first time, to state j. Then, we can easily obtain the following relation.

$$E_j=-Q_j'(-s)]_{s=0}\ . \tag{2.18}$$

We apply this relation to the recurrence relations (2.11) which define a sequence of polynominal $\{Q_i(x)\}$. Differentiating both sides of Eq. (2.11) with respect to x and setting $x=0$, we have the recurrence relations of $E_j's$ as follows:

$$\begin{cases} E_0=0, \quad E_1=1/\lambda_0 \\ \lambda_j E_{j+1}-(\lambda_j+\mu_j)E_j+\mu_j E_{j-1}=1 \qquad (j\geq 1)\ . \end{cases} \tag{2.19}$$

The general solution of these recurrence relations is given by

$$E_j=\sum_{i=0}^{j-1}\frac{1}{\lambda_i\pi_i}\sum_{k=0}^{i}\pi_k\ , \tag{2.20}$$

where

* Superscript t of Q denotes the transpose of the matrix.

$$\pi_0=1, \quad \pi_k=\frac{\lambda_0\lambda_1\cdots\lambda_{k-1}}{\mu_1\mu_2\cdots\mu_k}. \tag{2.21}$$

Consequently, setting $j=n+s$, the MTSF of this system is given by E_{n+s}.

3. The Effects of Repair Facilities and/or Spare Elements

Applying the results given in the preceding section, we can evaluate the system reliability from various viewpoints. In other words, we investigate to what degree the reliability of a system increases by the addition of spare elements and/or repair facilities to the system consisting of n elements and a repair facility.

Let E_n^1 be the MTSF of the system which has n elements, no spare element, $s=0$, and a repair facility, $r=1$. Furthermore, E_{n+s}^r denotes the MTSF of the system consisting of n elements, s spare elements and r repair facilities. We define M as

$$M=E_{n+s}^r/E_n^1. \tag{3.1}$$

This ratio M shows that the ratio of the MTSF of the system consisting of n elements, s spare elements and r repair facilities to that of the system consisting of n elements and a repair facility. Figures 2, 3, 4 and 5 show M versus $1/\rho=\mu/\lambda$ as parameters r and s ($n=3,4,5,6$).

Analysis of these Figures asserts the following properties.

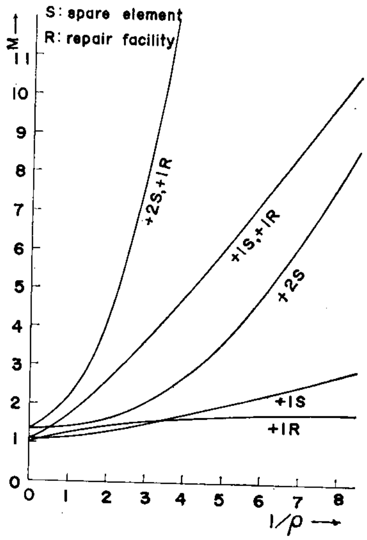

Fig. 2. M as a function of $1/\rho$ for the various system modifications ($n=3$).

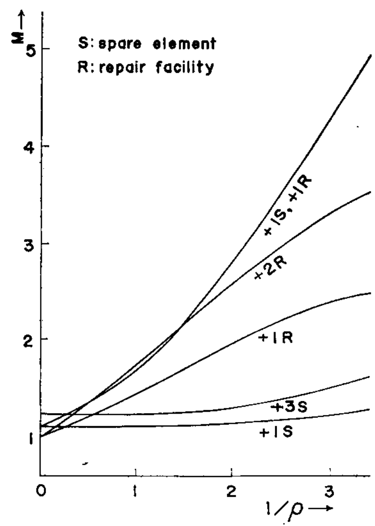

Fig. 3. M as a function of $1/\rho$ for the various system modifications ($n=4$).

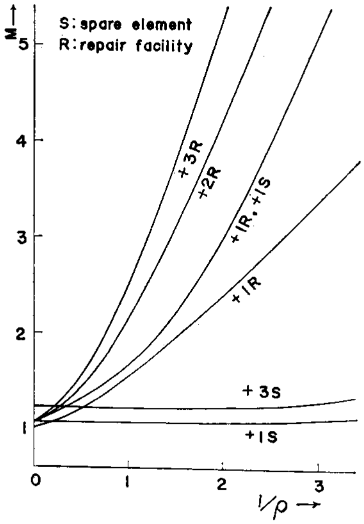

Fig. 4. M as a function of $1/\rho$ for the various system modifications ($n=5$).

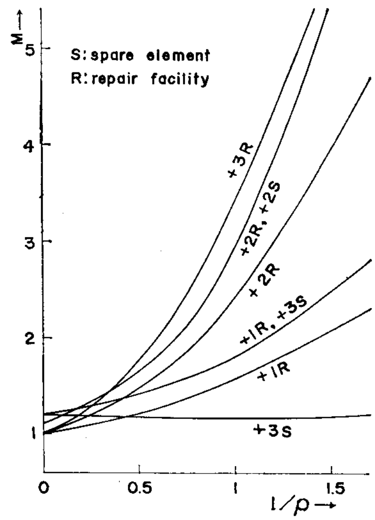

Fig. 5. M as a function of $1/\rho$ for the various system modifications ($n=6$).

(i) If the ratio of repair rate μ to failure rate λ, $1/\rho=\mu/\lambda$, is small, i.e., if many elements fail during a repair time of one element, the addition of spare elements to the system is more effective, in a sense of reliability improvement of the system, than that of repair facilities.

(ii) If the ratio $1/\rho$ is large, i.e., if repairs of many elements are completed during an element failure, the addition of spare elements to the system is more effective than that of repair facilities.

(iii) If the ratio $1/\rho=\mu/\lambda$ lies in a suitable range, the addition of repair facilities to the system is more effective than that of spare elements.

These three properties are valid for any value of n.

As an example, we consider the case of $n=4$. If the ratio $1/\rho=\mu/\lambda$ is smaller than about 0.6 or larger than about 1.4, the addition of a repair facility and a spare element to the system is more effective than that of 2 repair facilities. Moreover,if the ratio $1/\rho=\mu/\lambda$ is smaller than about 1.4 and larger than about 0.6, we can get a reverse conclusion. Thus, we have derived very useful results.

As long as we take an MTSF as a measure of system reliability, these properties are useful. In practice, we have known, from the empirical data, a failure rate λ and a repaire rate μ for an element. Using $1/\rho=\mu/\lambda$, we can determine whether we should add spare elements or repair facilities. Further, it is better to take into account the economical concept, e.g., the personnel expenditure.

Here, we have taken an MTSF as a measure of system reliability, but if we take another measure, for instance, an interval availability, we may change the concept of system reliability itself. Therefore, we have to notice that the properties as we have stated above may not always be valid.

4. "2 out of n" Systems

In the preceding two sections, we assume that both the failure time distribution and the repair time distribution are exponential But, in this and subsequent sections, we assume that the failure time distribution is exponential and the repair time distribution is arbitrary. In practical situations, it is natural that a failure occurs at random. While, the repair time distribution is generally assumed to be an Erlang or regular distribution.

A "2 out of n" system is considered to be in a state of system failure when not less than 2 of n elements are in a failed state. Since the remaining operating elements are overloaded, the system failure occurs. We assume that the system has a repair facility. The repair time distribution of the facility is arbitrary. If the repair time distribution $G(t)$ has its density, we can write

$$G(t)=\int_0^t g(t)dt\,. \tag{4.1}$$

Now, let $h(t)$ define that

$$h(t)=g(t)/[1-G(t)] \qquad (t\geq 0)\,. \tag{4.2}$$

The function $h(t)$ is called a repair rate. $h(t)\Delta t$ can be interpreted as the probability that a repair is completed between t and $t+\Delta t$, if Δt is a short interval. While, using the repair rate $h(t)$, $g(t)$ can be written as follows;

$$g(t)=h(t)\exp\left[-\int_0^t h(t)dt\right]. \tag{4.3}$$

Before the discussion of the general "2 out of n" system, we consider the analysis in detail of a "2 out of 2" system, i.e., a two element redundant system. Consider a system consisting of two dissimilar elements. Let two elements denote $i=1,2$. We assume that each element has an exponential failure time distribution

$$F_i(t)=1-\exp(-\lambda_i t) \qquad (i=1,\,2)\,. \tag{4.4}$$

Furthermore, a repair facility has an arbitrary repair time distribution, but the repair time of each element i is different. For each element, the repair time distribution is supposed that

$$G_i(t)=\int_0^t g_i(t)dt \qquad (i=1,\,2)\,. \tag{4.5}$$

where $g_i(t)$ is the density of $G_i(t)$. And a repair rate of each element can be written as follows;

$$h_i(t)=g_i(t)/[1-G_i(t)] \qquad (i=1, 2) . \tag{4.6}$$

We define the states of the system as follows;

state 0: Both elements are operative.

state 1: Element 1 is down and element 2 is operative.

state 2: Element 2 is down and element 1 is operative.

state 3: Both elements are in a failed state and this state is considered to be a system failure.

The state transition diagram of this system can be given in Fig. 6.

Define that

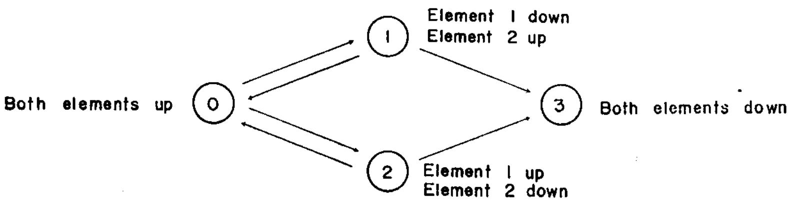

Fig. 6. The state transition diagram of a two element redundant system.

$P_i(t)$; the probability that the system is in state i at time t where $i=0$, 3.

$P_i(t,x)$; the probability that the system is in state i at time t and has been there between $t-x$ and t, where $i=1$, 2.

Using these probabilities and Eqs. (4.4) and (4.6), we have the following system of difference equations.

$$P_0(t+\Delta t)=P_0(t)[1-(\lambda_1+\lambda_2)\Delta t]+\int_0^\infty P_1(t, x)h_1(x)dx\Delta t$$
$$+\int_0^\infty P_2(t, x)h_2(x)dx\Delta t+o(\Delta t) . \tag{4.7}$$

$$P_1(t+\Delta t, x+\Delta t)=P_1(t, x)[1-h_1(t)\Delta t](1-\lambda_2\Delta t)+o(\Delta t) . \tag{4.8}$$

$$P_2(t+\Delta t, x+\Delta t)=P_2(t, x)[1-h_2(t)\Delta t](1-\lambda_1\Delta t)+o(\Delta t) . \tag{4.9}$$

$$P_3(t+\Delta t) = \int_0^\infty P_1(t, x)[1-h_1(x)\Delta t](\lambda_2\Delta t)dt$$
$$+\int_0^\infty P_2(t, x)[1-h_2(x)\Delta t](\lambda_1\Delta t)dx+P_3(t)+o(\Delta t) . \tag{4.10}$$

When $\Delta t\rightarrow 0$, Eqs.(4.7)-(4.10) become

$$\frac{dP_0(t)}{dt}+(\lambda_1+\lambda_2)P_0(t)=\int_0^\infty P_1(t, x)\, h_1(x)\, dx+\int_0^\infty P_2(t, x)h_2(x)dx\,, \tag{4.11}$$

$$\frac{\partial P_1(t, x)}{\partial x}+\frac{\partial P_1(t, x)}{\partial t}=-[\lambda_2+h_1(x)]P_1(t, x)\,, \tag{4.12}$$

$$\frac{\partial P_2(t, x)}{\partial x}+\frac{\partial P_2(t, x)}{\partial t}=-[\lambda_1+h_2(x)]P_2(t, x)\,, \tag{4.13}$$

$$\frac{dP_3(t)}{dt}=\lambda_2\int_0^\infty P_1(t, x)dx+\lambda_1\int_0^\infty P_2(t, x)dx\,. \tag{4.14}$$

If we assume that both elements are simultaneously operative, the initial conditions are given by

$$\left.\begin{array}{l} P_0(0)=1, \quad P_3(0)=0, \\ P_1(0, x)=0, \quad P_2(0, x)=0, \\ P_1(t, 0)=\lambda_1 P_0(t), \quad P_2(t, 0)=\lambda_2 P_2(t)\,. \end{array}\right\} \tag{4.15}$$

Let the Laplace transforms of $P_0(t)$, $P_1(t, x)$, $P_2(t, x)$, and $P_3(t)$ denote, respectively,

$$L_0(s)=\int_0^\infty \exp(-st)P_0(t)dt\,, \tag{4.16}$$

$$L_1(s, x)=\int_0^\infty \exp(-st)P_1(t, x)dt\,, \tag{4.17}$$

$$L_2(s, x)=\int_0^\infty \exp(-st)P_2(t, x)dt\,, \tag{4.18}$$

$$L_3(s)=\int_0^\infty \exp(-st)P_3(t)dt\,. \tag{4.19}$$

Solving the system of integro-differential-difference equations (4.11)-(4.14) with respect to $L_0(s)$, $L_1(s, x)$, $L_2(s, x)$, and $L_3(s)$ under the initial conditions (4.15), we have (see Appendix I)

$$L_0(s)=\frac{1}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\,, \tag{4.20}$$

$$L_1(s, x)=\frac{\lambda_1 \exp[-(s+\lambda_2)x] \exp\left[-\int_0^x h_1(x)\, dx\right]}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\,, \tag{4.21}$$

$$L_2(s, x)=\frac{\lambda_2 \exp[-(s+\lambda_1)x] \exp\left[-\int_0^x h_2(x)\, dx\right]}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\,, \tag{4.22}$$

$$L_3(s,x)=\frac{1}{s}\left\{\frac{\lambda_1\lambda_2}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\cdot\frac{1-L_{G_2}(s+\lambda_1)}{s+\lambda_1}\right.$$
$$\left.+\frac{\lambda_1\lambda_2}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\cdot\frac{1-L_{G_1}(s+\lambda_2)}{s+\lambda_2}\right\}. \tag{4.23}$$

where

$$L_{G_1}(s)=\int_0^\infty \exp(-st)dG_1(t), \tag{4.24}$$

$$L_{G_2}(s)=\int_0^\infty \exp(-st)dG_2(t). \tag{4.25}$$

The inverse Laplace transforms of these equations (4.20)-(4.23) give the pointwise availabilities of this system.

While, the MTSF for the system is defined as

$$\mathrm{MTSF}=\int_0^\infty P_A(t)dt, \tag{4.26}$$

provided that

$$\lim_{t\to\infty} tP_A(t)<\infty, \tag{4.27}$$

where P_A(t) indicates the probability that the system is in an operative state. For this system, using the above $P_3(t)$, we have

$$\mathrm{MTSF}=\int_0^\infty [1-P_3(t)]dt. \tag{4.28}$$

Using Eq. (4.23), we have (see Appendix II)

$$\mathrm{MTSF}=\frac{1+\dfrac{\lambda_1}{\lambda_2}[1-L_{G_1}(\lambda_2)]+\dfrac{\lambda_2}{\lambda_1}[1-L_{G_2}(\lambda_1)]}{\lambda_1[1-L_{G_1}(\lambda_2)]+\lambda_2[1-L_{G_2}(\lambda_1)]}. \tag{4.29}$$

If we put, in particular, $\lambda_1=\lambda_2=\lambda$ and $G_1(t)=G_2(t)=G(t)$ into Eq. (4.29), the MTSF becomes

$$\mathrm{MTSF}=1/\lambda+1/2\lambda[1-L_G(\lambda)], \tag{4.30}$$

which corresponds to Eq. (8) of Liebowitz[11].

The MTSF for a "2 out of n" system is given by Downton[3] as follows;

$$\mathrm{MTSF}=1/(n-1)\lambda+1/n\lambda[1-L_G((n-1)\lambda)], \tag{4.31}$$

where the failure time of each element obeys the identical distribution

$$F(t)=1-\exp(-\lambda t), \tag{4.32}$$

and the repair time distribution $G(t)$ is arbitrary. $L_G((n-1)\lambda)$ is the Laplace-Stieltjes transform of $G(t)$ with substituting $s=(n-1)\lambda$.

The MTSF for a "2 out of n" system without repair is given by

$$\text{MTSF}=1/(n-1)\lambda+1/n\lambda . \tag{4.33}$$

This derivation is easily obtained by substituting $L_G((n-1)\lambda)=0$.

Let M, called an improvement factor[11] for a "2 out of n" system, define that

$$M=\frac{1/(n-1)+1/n[1-L_G((n-1)\lambda)]}{1/(n-1)+1/n} . \tag{4.34}$$

This can be interpreted as the ratio of the MTSF for a "2 out of n" system with repair to that system without repair. The improvement factor is a measure of degree how the reliability of "2 out of n" system with repair increases compared with a similar system without repair.

Here, we give some repair time distributions with mean $1/\mu$ and their Laplace-Stieltjes transforms.

(i) Exponential distribution

$$G(t)=1-\exp(-\mu t) . \tag{4.35}$$

$$L_G(s)=\mu/(\mu+s) . \tag{4.36}$$

(ii) k-Erlang distribution

$$G(t)=\int_0^t k\mu(k\mu t)^{k-1}\exp(-k\mu t)dt/(k-1)! . \tag{4.37}$$

$$L_G(s)=[k\mu/(k\mu+s)]^k . \tag{4.38}$$

(iii) Regular (constant repair) distribution

$$G(t)=\begin{cases} 0 & 0\leq t<1/\mu \\ 1 & 1/\mu\leq t . \end{cases} \tag{4.39}$$

$$L_G(s)=\exp(-s/\mu) . \tag{4.40}$$

(iv) Uniform distribution

$$G(t)=\begin{cases} 0 & 0\leq t<(R-k)/\mu \\ \dfrac{\mu}{2k}\left(t-\dfrac{R-k}{\mu}\right) & (R-k)/\mu\leq t<(R+k)/\mu \\ 1 & (R+k)/\mu\leq t . \end{cases} \tag{4.41}$$

$$L_G(s)=\frac{1}{2ks}\{\exp[-s(R-k)/\mu]-\exp[-s(R+k)/\mu]\}\ . \qquad (4.42)$$

Applying these distributions to Eq. (4.34), M is a function of $\rho=\lambda/\mu$.

Table 2 shows the calculated results of M as a function of $\rho=\lambda/\mu$ for the various distributions. Further, Fig. 7 shows the improvement factor M as a function of ρ. This Figure suggests that the asymptotic forms of M have

$$M\approx\frac{1/(n-1)+1/n(n-1)\rho}{1/(n-1)+1/n}\qquad(\rho\to0)\ , \qquad (4.43)$$

$$M\approx1\quad(\rho\to\infty)\ . \qquad (4.44)$$

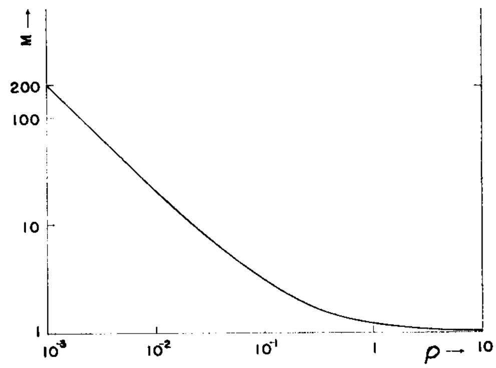

Fig. 7. M as a function of ρ for "2 out of 3" system (exponential case).

The facts can be verified by the well-known expansions of the Laplace-Stieltjes transform $L_G((n-1)\lambda)$, i.e.,

$$\begin{aligned}L_G((n-1)\lambda)&=1-(n-1)\lambda/\mu+o(\lambda)\\&\approx1-(n-1)\rho\qquad(\rho\to0)\ .\end{aligned} \qquad (4.45)$$

and

$$L_G((n-1)\lambda)=\frac{f_0}{(n-1)\lambda}+\frac{f_1}{(n-1)^2\lambda^2}+\cdots\qquad(\rho\to\infty)\ . \qquad (4.46)$$

where f_0, f_1, $\cdots$ denote the intercept, slope of the repair time density at the origin.

5. Semi-Markov Process Models for Complex Systems

In the preceding section, we have discussed "2 out of n" systems. In this section, we shall discuss the general "m out of n" systems. A method of differential-difference equations is effective for the analysis of a "2 out of n" system, but an application of semi-Markov processes[14,15] is relevant to an "m out of n" system.

An "m out of n" system with which we are concerned is composed of n elements and a repair facility. If not less than m of n elements are in a failed state, the system is considered to be in a state of system failure. Because the remaining elements, not greater than $(n-m)$ elements, are overloaded, the system can be considered to fail.. We assume that the failure time distribution of each element is exponential, i.e.,

Table 2. Improvement factor M(ratio of MTSF with repair to MTSF without repair) for a "2 out of 3" system as a function of ρ.

ρ	Exponential	2-Erlang	5-Erlang	Regular	Uniform $a=0.8/\mu$ $b=1.2/\mu$	Uniform $a=0.5/\mu$ $b=1.5/\mu$
0.001	201.00	200.90	200.84	200.79	200.66	200.66
0.002	101.00	100.90	100.84	101.00	100.85	100.81
0.003	67.67	67.57	67.51	67.47	67.45	67.49
0.004	51.00	50.90	50.84	51.00	50.80	50.82
0.005	41.00	40.90	40.84	41.00	40.81	40.82
0.006	34.33	34.23	34.17	34.13	34.14	34.15
0.007	29.57	29.47	29.41	29.37	29.37	29.39
0.008	26.00	25.90	25.84	25.80	25.80	25.82
0.009	23.22	23.12	23.06	23.02	23.02	23.04
0.010	21.00	20.90	20.84	20.80	20.80	20.82
0.020	11.00	10.90	10.84	10.80	10.80	10.82
0.030	7.67	7.57	7.51	7.47	7.47	7.48
0.040	6.00	5.90	5.84	5.80	5.80	5.82
0.050	5.00	4.90	4.84	4.80	4.81	4.82
0.060	4.33	4.24	4.18	4.14	4.14	4.15
0.070	3.86	3.76	3.70	3.66	3.66	3.68
0.080	3.50	3.40	3.35	3.31	3.31	3.32
0.090	3.22	3.13	3.07	3.03	3.03	3.04
0.100	3.00	2.90	2.85	2.81	2.81	2.82
0.200	2.00	1.91	1.85	1.81	1.82	1.83
0.300	1.67	1.58	1.52	1.49	1.49	1.50
0.400	1.50	1.42	1.36	1.33	1.33	1.34
0.500	1.40	1.32	1.27	1.23	1.24	1.25
0.600	1.33	1.26	1.21	1.17	1.17	1.19
0.700	1.29	1.21	1.16	1.13	1.13	1.15
0.800	1.25	1.18	1.13	1.10	1.10	1.12
0.900	1.22	1.15	1.11	1.08	1.08	1.09
1.000	1.20	1.13	1.09	1.06	1.06	1.08
2.000	1.10	1.05	1.02	1.01	1.01	1.01
3.000	1.07	1.03	1.01	1.00	1.00	1.00
4.000	1.05	1.02	1.00	1.00	1.00	1.00
5.000	1.04	1.01	1.00	1.00	1.00	1.00
6.000	1.03	1.01	1.00	1.00	1.00	1.00
7.000	1.03	1.01	1.00	1.00	1.00	1.00
8.000	1.03	1.01	1.00	1.00	1.00	1.00
9.000	1.02	1.00	1.00	1.00	1.00	1.00

$$F(t)=1-\exp(-\lambda t)\ . \tag{5.1}$$

and the repair time distribution $G(t)$ of the facility is arbitrary.

Downton[3] has shown the Laplace-Stieltjes transform of the first passage time distribution, i.e., the time to system failure, for the general "m out of n" system. In particular, the MTSF for a "3 out of n" system is

$$\text{MTSF}=1/(n-1)\lambda+1/(n-2)\lambda+c_3/n\lambda d_3\ , \tag{5.2}$$

where

$$c_3=1-L_G[(n-1)\lambda]+L_G[(n-2)\lambda]\ , \tag{5.3}$$

$$d_3=1-(n-1)L_G[(n-2)\lambda]+(n-2)L_G[(n-1)\lambda]\ . \tag{5.4}$$

Here, we investigate to what degree the reliability of a "3 out of n" system, in which the repair time distribution $G(t)$ is k-Erlang increases comparing with the same system in which the repair time distribution is exponential. Using the Laplace-Stieltjes transform of exponential distribution (4.36), the following equation holds

$$L_G(n\lambda)=1/(1+n\rho)\ . \tag{5.5}$$

While, for k-Erlang distribution we have

$$L_G(n\lambda)=[k/(k+n\rho)]^k\ . \tag{5.6}$$

Let M define that

$$M=\frac{\text{MTSF for "3 out of } n\text{" system with } k\text{-Erlang repair time distribution}}{\text{MTSF for "3 out of } n\text{" system with exponential repair time distribution}}\ . \tag{5.7}$$

Figures 8, 9, and 10 show M for various k-Erlang distributions and constant repair time distribution as a function of ρ in the case of $n=3,4$ and 5, respectively.

These Figures suggest the useful properties concerned with repair time distributions. These properties are summarized below.

(i) It has been believed that, if the failure time is exponentially distributed, the most effective repair time distribution is also exponential. But from these Figures, we know, if ρ is small, a k-Erlang distribution (including regular distribution) is more effective than the exponential distribution. This fact is valid for $n=3,4$ and 5. It is estimated to be valid for arbitrary n ($n\geq 0$). In particular, M tends to 2 as $\rho\to 0$.

(ii) For $n=3$, when ρ is around $\rho_0=0.768$, M is nearly constant 0.987,

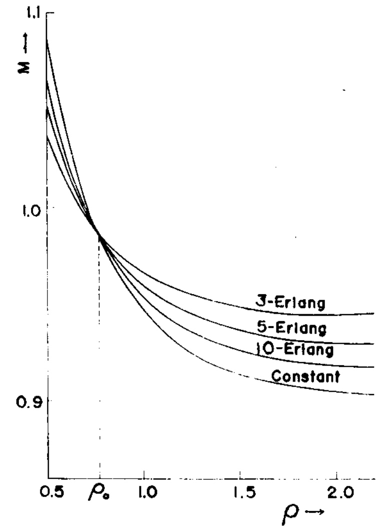

Fig. 8. M as a function of ρ for "3 out of 3" system.

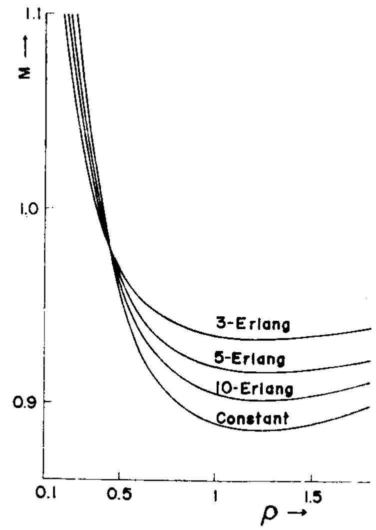

Fig. 9. M as a function of ρ for "3 out of 4" system.

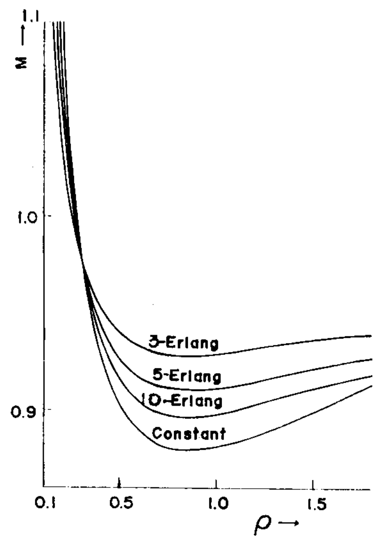

Fig. 10. M as a function of ρ for "3 out of 5" system.

independent of a parameter k of k-Erlang distribution. Therefore, it is better that we adopt a k-Erlang distribution with a parameter k as large as possible for ρ smaller than ρ_0 when the repair time distribution is a k-Erlang distribution. For ρ larger than ρ_0, we can get a converse conlusion. This remarkable property is vaild for arbitrary integer n $(n \geqslant 3)$. But, the larger n is, the smaller ρ_0 is.

These two properties should be proved by the following asymptotic behavior.

Here, we develop the asymptotic behavior of M. First, for a small ρ, the Laplace-Stieltjes transform of an arbitrary distribution function $G(t)$ is approximated by

$$L_G(n\lambda)=1-n\rho+\frac{n^2\lambda^2R_2}{2}+o(\lambda^2) \qquad (\lambda\to 0)\,, \tag{5.8}$$

where R_2 is the second moment at the origin, and

$$L_G(n\lambda)=\frac{f_0}{n\lambda}+\frac{f_1}{n^2\lambda^2}+o\left(\frac{1}{\lambda^2}\right) \qquad (\lambda\to\infty)\,. \tag{5.9}$$

In particular, if $G(t)$ is an exponential distribution, $L_G(n\lambda)$ is given by

$$L_G(n\lambda)=1-n\rho+o(\lambda) \qquad (\lambda\to 0)\,, \tag{5.10}$$

and

$$L_G(n\lambda)=1/n\lambda+o(1/\lambda) \qquad (\lambda\to\infty)\,. \tag{5.11}$$

Therefore, M is approximated, for $n=3$, by

$$M\approx\frac{6\rho^2}{6\rho^2+\rho+1}\left(1+\frac{1+\rho}{3\lambda^2R_2}\right) \qquad (\rho\to 0)\,, \tag{5.12}$$

and

$$M\approx\frac{11+4f_0}{3\,(1+2\rho)} \qquad (\rho\to\infty)\,. \tag{5.13}$$

Furthermore, if $G(t)$ is a k-Erlang distribution, M is approximated by

$$M\approx\frac{6\rho^2}{6\rho^2+\rho+1}+\frac{2}{6\rho^2+\rho+1}\cdot\frac{1+\rho}{1+1/k}\,. \tag{5.14}$$

Therefore, we have

$$\lim_{\substack{\rho\to 0\\ k\to\infty}} M=2\,. \tag{5.15}$$

Further, as $\lambda\to 0$ for $k_1>k_0$, it is evident from Eq. (5.14) that

$$M(k_1)>M(k_0)\,. \tag{5.16}$$

For a large λ, M is approximated by

$$M\approx\frac{6\rho}{11\rho+4}\left\{\frac{11}{6}+\left(\frac{1}{\rho}\right)^k-\frac{2}{3}\left(\frac{1}{2\rho}\right)^k\right\}\,. \tag{5.17}$$

Thus, we have

$$\lim_{\substack{\rho\to\infty \\ k\to\infty}} M=1 . \tag{5.18}$$

As $\rho\to\infty$ for $k_1>k_0$, we have from a simple calculation

$$M(k_1)<M(k_0) . \tag{5.19}$$

Therefore, the two properties stated above are justified from the asymptotic behavior.

6. Conclusion

Reliability analysis of complex systems is important and yet difficult. In this paper, we have discussed the redundant systems with repair.

In the case of exponential-failure-exponential-repair, Birth-Death Processes are applied to the complex systems as mathematical models. An elegant treatment of these processes has been given by Karlin and McGregor. This treatment as we have stated in Section 2 is useful in the practical situations.

While, in the case of exponential-failure-general-repair, Semi-Markov Processes are relevant to the systems. But, in particular, a method of Dieffrential-Difference Equations is intuitive and comprehensive to analyze a "2 out of n" system.

Many numerical calculations of the system reliability as we have stated in this paper are useful. Further, taking account of many factors, these results are helpful in designing the optimal redundant systems.

Acknowledgement

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Appendix I

Taking the Laplace transforms of Eqs. (4.11)-(4.14) and using the initial conditions (4.15), we have

$$L_0(s)\cdot(s+\lambda_1+\lambda_2)=1+\int_0^\infty L_1(s, x)h_1(x)dx + \int_0^\infty L_2(s, x)h_2(x)dx , \tag{A.1}$$

$$\frac{\partial L_1(s, x)}{\partial x}+[s+\lambda_2+h_1(x)]L_1(s, x)=0 , \tag{A.2}$$

$$\frac{\partial L_2(s, x)}{\partial x}+[s+\lambda_1+h_2(x)]L_2(s, x)=0 , \tag{A.3}$$

$$sL_3(s)=\lambda_2\int_0^\infty L_1(s,\ x)\,dx+\lambda_1\int_0^\infty L_2(s,\ x)dx\,. \tag{A.4}$$

Solving Eqs. (A.2) and (A. 3), we have

$$L_1(s,\ x)=C_1\exp\left[-(s+\lambda_2)x\right]\exp\left[-\int_0^x h_1(x)\,dx\right], \tag{A.5}$$

$$L_2(s,\ x)=C_2\exp\left[-(s+\lambda_1)x\right]\exp\left[-\int_0^x h_2(x)\,dx\right], \tag{A.6}$$

where C_1 and C_2 are arbitrary constants. Using the initial conditions (4.15), we have

$$C_1=L_1(s,\ o)=\lambda_1 L_0(s)\,, \tag{A.7}$$

$$C_2=L_2(s,\ o)=\lambda_2 L_0(s)\,. \tag{A.8}$$

Then, we have

$$L_1(s,\ x)=\lambda_1 L_0(s)\exp\left[-(s+\lambda_2)x\right]\exp\left[-\int_0^x h_1(x)dx\right], \tag{A.9}$$

$$L_2(s,\ x)=\lambda_2 L_0(s)\exp\left[-(s+\lambda_1)x\right]\exp\left[-\int_0^x h_2(x)dx\right]. \tag{A.10}$$

Substituting Eqs. (A.9) and (A.10) into Eq. (A.1) and using the relation (4.3), we have

$$L_0(s)=\frac{1}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\,. \tag{A.11}$$

Substituting Eq. (A. 11) into Eqs. (A. 9) and (A.10), we have

$$L_1(s,\ x)=\frac{\lambda_1\exp\left[-(s+\lambda_2)x\right]\exp\left[-\int_0^x h_1(x)dx\right]}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\,, \tag{A.12}$$

$$L_2(s,\ x)=\frac{\lambda_2\exp\left[-(s+\lambda_1)x\right]\exp\left[-\int_0^x h_2(x)\,dx\right]}{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\,. \tag{A.13}$$

Substituting Eqs. (A.12) and (A.13) into (A.4), we obtain

$$L_3(s)=\frac{\lambda_1\lambda_2}{s\{s+\lambda_1[1-L_{G_1}(s+\lambda_2)]+\lambda_2[1-L_{G_2}(s+\lambda_1)]}\times \left\{\frac{1-L_{G_2}(s+\lambda_1)}{s+\lambda_1}+\frac{1-L_{G_1}(s+\lambda_2)}{s+\lambda_2}\right\}. \tag{A.14}$$

Appendix II

By the definition (4.28) of MTSF, we have

$$\mathrm{MTSF} = \lim_{T\to\infty} B(T)\ , \tag{A.15}$$

where

$$B(T) = \int_0^T [1-P_3(t)]dt\ . \tag{A.16}$$

Taking the Laplace transform of both sides of Eq. (A.16), we have

$$\begin{aligned} L_B(s) &= \int_0^\infty \exp\,(-sT) \int_0^T [1-P_3(t)]dtdT \\ &= \frac{1}{s}\left[\frac{1}{s} - L_3(s)\right]. \end{aligned} \tag{A.17}$$

Using the final value theorem of the Laplace transforms, we have

$$\begin{aligned} \lim_{T\to\infty} B(T) &= \lim_{s\to 0} sL_B(s) \\ &= \lim_{s\to 0}\left[\frac{1}{s} - L_3(s)\right] \\ &= \frac{1+\frac{\lambda_1}{\lambda_2}[1-L_{G_1}(\lambda_2)] + \frac{\lambda_2}{\lambda_1}[1-L_{G_2}(\lambda_1)]}{\lambda_1[1-L_{G_1}(\lambda_2)]+\lambda_2[1-L_{G_2}(\lambda_1)]}\ . \end{aligned} \tag{A.18}$$

Thus, we obtain

$$\mathrm{MTSF} = \frac{1+\frac{\lambda_1}{\lambda_2}\left[1-L_{G_1}(\lambda_2)\right] + \frac{\lambda_2}{\lambda_1}[1-L_{G_2}(\lambda_1)]}{\lambda_1[1-L_{G_1}(\lambda_2)]+\lambda_2[1-L_{G_2}(\lambda_1)]}\ . \tag{A.19}$$

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