A Theoretical Consideration on Helicoidal Girder

By

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A theoretical consideration is given on an arbitrary supported helicoidal girder and an approximate method of analysis is presented, taking into an account the St. Venant's torsion in which the fundamental relationship between the sectional forces and deformations are obtained by use of the Reduction method.

Introduction

Recently a number of curved as well as twisted girders are planned from various structural demands for design of rather complex interchange structures for the urban express highways. The problems for the design of such helicoidal girders seem to refer to two aspects; namely (i) the exact analytical theory to clarify the structural responses under the design loads and (ii) the simple approximate methods for the designer's use. As the first step of analysis some fundamental considerations for structural characteristics of a helicoidal girder are given for the former purpose. Previously, the theoretical studies on this problem have been performed by a number of researchers, such as M.C. Holmes[1)], Y.F. Young & A.C. Scordelis,[2)] F. Baron,[3)] K. Washizu,[4)] et al. Young and Scordelis investigate the heliocidal girder, fixed at ends, subtending a horizontal angle of 180 degrees, with a slope of 30 degrees and compare the theoretical results with the experimental ones[2)]. F. Baron[3)] formulates a general solution of curved structures by use of the matrix method and showed a numerical example. So far the most rigorous investigations on the curved and twisted beam of the cantilever type are presented by K. Washizu[4)]. Based on above mentioned results a consideration is given on the arbitrary supported helicoidal girder and an approximate expression for response is presented.

1. System of Coordinate

The equation of the line of centroids of the cross sections of the girder (this

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line shall be called hereafter the axis of the girder) is expressed as follows.

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}(s) = \boldsymbol{r}(\theta) \\ &= (a\cos\theta)\underline{\boldsymbol{i}} + (a\sin\theta)\underline{\boldsymbol{j}} + (b\theta)\underline{\boldsymbol{k}} \end{aligned} \tag{1.1}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors along the Cartesian coordinate axes, a is the radius of the circle which is the projection of the axis on the plane OXY, b is a coefficient proportional to the pitch of the helicoidal curve and θ is the polar angle in the plane OXY.

It is possible to take the arc S measured from the X axis, on which the terminal point of the axis lies, as the parameter along the curve.

Let $\underline{\xi}_x$, $\underline{\xi}_y$ and $\underline{\xi}_z$ be the unit tangent vector, the unit principal normal vector, and the unit binormal vector of the girder axis respectively.

The relations between them which are well known as the Frenet- Serret formulae of the curve are expressed alternatively as follows

$$\begin{aligned} \frac{d\underline{\xi}_x}{ds} &= \kappa\underline{\xi}_y & \qquad \frac{d\underline{\xi}_x}{d\theta} &= \cos\phi\underline{\xi}_y \\ \frac{d\underline{\xi}_y}{ds} &= -\kappa\underline{\xi}_x + \tau\underline{\xi}_z & \qquad \frac{d\underline{\xi}_y}{d\theta} &= -\cos\phi\underline{\xi}_x + \sin\phi\underline{\xi}_z \\ \frac{d\underline{\xi}_z}{ds} &= -\tau\underline{\xi}_y & \qquad \frac{d\underline{\xi}_z}{d\theta} &= -\sin\phi\underline{\xi}_y \end{aligned} \tag{1.2}$$

where κ and τ are the curvature and the torsion of the girder axis respectively, and are given by

$$\kappa = a(a^2+b^2)^{-1} \qquad \tau = b(a^2+b^2)^{-1} \tag{1.3}$$

ϕ is the angle between the OXY plane and the tangent to the girder axis at any point.

The cosine-directions of the triad of vectors particular to the girder axis with respect to those fixed in space are given in Table-1.

Table 1

	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{\xi}_x$	$-\sin\theta\cos\phi$	$-\cos\theta\cos\phi$	$\sin\phi$
$\underline{\xi}_y$	$-\cos\theta$	$-\sin\theta$	0
$\underline{\xi}_z$	$-\sin\theta\sin\phi$	$-\cos\theta\sin\phi$	$-\cos\phi$

where

$$\cos\phi = a\cdot(a^2+b^2)^{-1/2} \qquad \sin\phi = b\cdot(a^2+b^2)^{-1/2} \tag{1.4}$$

For any cross section of the girder defined by the angle θ, a system of rectangular coordinates with the origin at the centroid O is taken so that they constitute a right-handed system of coordinates in the manner that the axes are so directed that x, y and z coincide respectively with the tangent, the principal normal and the binormal of the girder axis.

The position-vector of any point on the section contour which is denoted by $\boldsymbol{P}$ is expressed as follows, provided that it refers to such a coordinate system.

$$\boldsymbol{P} = y\xi_y + z\xi_z \tag{1.5}$$

and thus the position vector of the same point which is referred to the coordinate system fixed in space and is denoted by $\boldsymbol{R}$ is expressed as follows.

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{r} + \boldsymbol{P} \\ &= \{(a-y)(\cos\theta + z\sin\phi\sin\theta\}\underline{\boldsymbol{i}} \\ &\quad + \{(a-y)\sin\theta - z\sin\phi\cos\theta\}\underline{\boldsymbol{j}} + (b\theta + z\cos\phi)\underline{\boldsymbol{k}} \\ &\equiv A\cos(\theta-\alpha)\underline{\boldsymbol{i}} + A\sin(\theta-\alpha)\underline{\boldsymbol{j}} + (b\theta + z\cos\phi)\underline{\boldsymbol{k}} \equiv X\underline{\boldsymbol{i}} + Y\underline{\boldsymbol{j}} + Z\underline{\boldsymbol{k}}\end{aligned} \tag{1.6}$$

where

$$A = \{(a-y)^2 + z^2\sin^2\phi\}^{1/2} \qquad \begin{aligned}\cos\alpha &= (a-y)\cdot A^{-1} \\ \sin\alpha &= z\sin\phi\cdot A^{-1}\end{aligned}$$

It is clear from the above expression that any points on the section contour describe helicoidal curves corresponding to an increase of the angle.

Since the Jacobian determinant obtained from above expression does not vanish, any points on the section contour are decided uniquely by the pairs of parameters (θ, y, z) or (s, y, z).

$$J \equiv \frac{\partial(X, Y, Z)}{\partial(\theta, y, z)} = (a^2 + b^2 - ay)(a^2 + b^2)^{-1/2} \neq 0$$

2. Deformation of the Girder Axis

The deformation of the axis is characterized by its rotation which is denoted by φ and by its displacement of which components are denoted by u, v and w in the directions of the x, y and z axes respectively.

And thus the expression of the displacement of the axis is written in vectorial form as

$$\boldsymbol{V} = u(s)\xi_x + v(s)\xi_y + w(s)\xi_z \tag{2.1}$$

The quantities relating directly to the deformed state of the girder will be marked hereafter by an asterisk.

Provided that the relation between the line element of the axis after deformation and one before deformation is given by $ds^*=(1+\varepsilon)ds$ the tangent vector of the axis after deformation is given by

$$\frac{d\boldsymbol{r}^*}{ds^*} \approx (1-\varepsilon)\frac{d\boldsymbol{r}^*}{ds} = (1-\varepsilon)\left(\frac{d\boldsymbol{r}}{ds}+\frac{d\boldsymbol{V}}{ds}\right)$$
$$\approx \xi_x+\left(\kappa u+\frac{dv}{ds}-\tau w\right)\xi_y+\left(\tau v+\frac{dw}{ds}\right)\xi_z \tag{2.2}$$

The square of the length of the line element of the axis after deformation is given by

$$\begin{aligned} ds^{*2} = d\boldsymbol{r}^*\cdot d\boldsymbol{r}^* &= (d\boldsymbol{r}+d\boldsymbol{V})\cdot(d\boldsymbol{r}+d\boldsymbol{V}) \\ &\approx d\boldsymbol{r}\cdot d\boldsymbol{r}+2d\boldsymbol{r}\cdot d\boldsymbol{V} \\ &= ds^2+2ds\xi_x\cdot d\boldsymbol{V} \end{aligned} \tag{2.3}$$

and the increment of the displacement is also given by

$$d\boldsymbol{V} = \frac{d\boldsymbol{V}}{ds}ds = \left\{\left(\frac{du}{ds}-\kappa v\right)\xi_x+\left(\kappa u+\frac{dv}{ds}\right)\xi_y+\frac{dw}{ds}\xi_z\right\}ds \tag{2.4}$$

Substituting this expression in above one, the relation between the square of the length of the line element before deformation and one after deformation can be written as follows

$$ds^{*2} = \left\{1+2\left(\frac{du}{ds}-\kappa v\right)\right\}ds^2 \tag{2.5}$$

from which the expression of ε is given finally by

$$\varepsilon(s) = \frac{ds^*-ds}{ds} \approx \frac{du}{ds}-\kappa v \tag{2.6}$$

and this is called the stretch of the axis.

The infinitesimal rotation vector of the axis caused by deformation, which is denoted by $\underline{\Omega}$ is obtained using the epxression (2.2) as

$$\begin{aligned} \underline{\Omega}(s) &= \varphi\xi_x+\frac{d\boldsymbol{r}}{ds}\times\frac{d\boldsymbol{r}^*}{ds^*} \equiv \Omega_x\xi_x+\Omega_y\xi_y+\Omega_z\xi_z \\ \Omega_x(s) &= \varphi \\ \Omega_y(s) &= -\left(\tau v+\frac{dw}{ds}\right) \\ \Omega_z(s) &= \kappa u+\frac{dv}{ds}-\tau w \end{aligned} \tag{2.7}$$

The rate of rotation vector denoted by $\underline{\psi}$ is expressed as follows

$$
\begin{aligned}
\underline{\psi}(s) &= \frac{d\underline{\Omega}}{ds} \equiv \psi_x\xi_x+\psi_y\xi_y+\psi_z\xi_z \\
\underline{\psi}_x(s) &= \frac{d\Omega_x}{ds}-\kappa\Omega_y \\
&= \frac{d\varphi}{ds}+\kappa\tau v+\kappa\frac{dw}{ds} \\
\psi_y(s) &= \kappa\Omega_x+\frac{d\Omega_y}{ds}-\tau\Omega_z \\
&= \kappa\varphi-\kappa\tau u-2\tau\frac{dv}{ds}+\tau^2 w-\frac{d^2w}{ds^2} \\
\psi_z(s) &= \tau\Omega_y+\frac{d\Omega_z}{ds} \\
&= \kappa\frac{du}{ds}-\tau^2 v+\frac{d^2v}{ds^2}-2\tau\frac{dw}{ds}
\end{aligned}
\tag{2. 8}
$$

3. Deformation of the Cross Section

The displacement of the cross section based on the Bernoulli-Navier hypothesis (which is the same as the law of plane sections) is given in vectorial form as

$$\dot{\boldsymbol{V}}_{(1)} = \boldsymbol{V}+\underline{\Omega}\times\boldsymbol{P} \equiv \bar{u}\xi_x+\bar{v}\xi_y+\bar{w}\xi_z \tag{3. 1}$$

where

$$
\begin{aligned}
\bar{u} &= u-\Omega_z y+\Omega_y z \\
&= u-\left(\kappa u+\frac{dv}{ds}-\tau w\right)y-\left(\tau v+\frac{dw}{ds}\right)z \\
\bar{v} &= v-\Omega_x z = v-\varphi z \\
\bar{w} &= w+\Omega_x y = w+\varphi y
\end{aligned}
$$

The warping displacement is considered as below. Using Table-1 and next relations between a, b and κ, τ

$$a = \kappa(\kappa^2+\tau^2)^{-1} \qquad b = \tau(\kappa^2+\tau^2)^{-1}$$

the radius-vector of a point on the axis which issues from the origin fixed in space is expressed by using unit vectors particular to the axis as

$$\boldsymbol{r}(s) = \tau^2(\kappa^2+\tau^2)^{-1}s\xi_x-\kappa(\kappa^2+\tau^2)^{-1}\xi_y+\kappa\tau(\kappa^2+\tau^2)^{-1}s\xi_z \tag{3. 2}$$

and also the position-vector of a point on the section contour is expressed in the same manner as

$$\begin{aligned}\boldsymbol{R}(s, y, z) &= \boldsymbol{r}(s)+\boldsymbol{P}(s, y, z)\\ &= \tau^2(\kappa^2+\tau^2)^{-1}s\xi_x - \{\kappa(\kappa^2+\tau^2)^{-1}-y\}\xi_y + \{\kappa\tau(\kappa^2+\tau^2)^{-1}s+z\}\xi_z \end{aligned} \tag{3. 3}$$

by using (1.2), the base vectors are obtained as follows.

$$\begin{aligned} \boldsymbol{g}_x &\equiv \frac{\partial \boldsymbol{R}}{\partial s} = (1-\kappa y)\xi_x - \tau z \xi_y + \tau y \xi_z \\ \boldsymbol{g}_y &\equiv \frac{\partial \boldsymbol{R}}{\partial y} = \xi_y \\ \boldsymbol{g}_z &\equiv \frac{\partial \boldsymbol{R}}{\partial z} = \xi_z \end{aligned} \tag{3. 4}$$

where $\boldsymbol{g}_x$ is the vector tangent to a generator of the girder, and unit vector in this direction which is denoted by $\boldsymbol{e}_x$ is

$$\boldsymbol{e}_x = (\boldsymbol{g}_x \cdot \boldsymbol{g}_x)^{-1/2} \cdot \boldsymbol{g}_x \equiv g_{xx}^{-1/2} \boldsymbol{g}_x$$

where $g_{xx}{}^{-1/2}$ is obtained from equation (3.4) as

$$\begin{aligned} g_{xx}^{-1/2} &= \{(1-\kappa y)^2 + \tau^2(y^2+z^2)\}^{-1/2} \\ &\approx (1-\kappa y)^{-1} - \frac{1}{2}\tau^2(y^2+z^2)(1-\kappa y)^{-3} \end{aligned} \tag{3. 5}$$

The unit vector tangent to a generator of the girder and the partial derivatives of this vector with respect to coordinates s, y and z must be calculated for the sake of the estimation of strain components.

$$\begin{aligned} \boldsymbol{e}_x &\approx \xi_x - \tau z(1-\kappa y)^{-1}\xi_y + \tau y(1-\kappa y)^{-1}\xi_z \\ \frac{\partial \boldsymbol{e}_x}{\partial s} &\approx \kappa\tau z(1-\kappa y)^{-1}\xi_x + \{\kappa - \tau^2 y(1-\kappa y)^{-1}\}\xi_y - \tau^2 z(1-\kappa y)^{-1}\xi_z \\ \frac{\partial \boldsymbol{e}_x}{\partial y} &\approx -\kappa\tau z(1-\kappa y)^{-2}\xi_y + \tau(1-\kappa y)^{-2}\xi_z \\ \frac{\partial \boldsymbol{e}_x}{\partial z} &\approx -\tau(1-\kappa y)^{-1}\xi_y \end{aligned} \tag{3. 6}$$

4. The estimation of Strain Component

Let $\boldsymbol{V}_{(2)}$ be the vectorial expression of a warping displacement of the girder, it can be assumed using unit vector $\boldsymbol{e}_x$ as

$$\tilde{\boldsymbol{V}}_{(2)} = U(s, y, z)\boldsymbol{e}_x(s, y, z) \tag{4. 1}$$

Since the total displacement of the section is

$$\tilde{\boldsymbol{V}} = \tilde{\boldsymbol{V}}_{(1)} + \tilde{\boldsymbol{V}}_{(2)}$$

the position-vector of the section contour after deformation is given by

$$\boldsymbol{R}^* = \boldsymbol{R}+\tilde{\boldsymbol{V}}$$

Strain components of the girder are obtained by definition from the difference between the square of the length of a line element after deformation and one before deformation,

$$\begin{aligned} d\tilde{S}^{*2}-d\tilde{S}^2 &= d\boldsymbol{R}^*\cdot d\boldsymbol{R}^*\cdot -d\boldsymbol{R}\cdot d\boldsymbol{R} \\ &= (d\boldsymbol{R}+d\tilde{\boldsymbol{V}})\cdot(d\boldsymbol{R}+d\tilde{\boldsymbol{V}})-d\boldsymbol{R}\cdot d\boldsymbol{R} \\ &\approx 2d\boldsymbol{R}\cdot d\tilde{\boldsymbol{V}} \equiv 2d\boldsymbol{R}\cdot\underline{\gamma}\cdot d\boldsymbol{R} \end{aligned} \tag{4. 2}$$

While from (3.4) the increment of position-vector $d\boldsymbol{R}$ is given by the formulae and from (3.1) the increment of the displacement vector based on the law of plane section is written as

$$\begin{aligned} d\tilde{\boldsymbol{V}}_{(1)} = &\left\{\left(\frac{\partial\tilde{u}}{\partial s}-\kappa\tilde{v}\right)ds+\frac{\partial\tilde{u}}{\partial y}dy+\frac{\partial\tilde{u}}{\partial z}dz\right\}\xi_x \\ &+\left\{\left(\frac{\partial\tilde{v}}{\partial s}+\kappa\tilde{u}-\tau\tilde{w}\right)ds+\frac{\partial\tilde{v}}{\partial z}dz\right\}\xi_y \\ &+\left\{\left(\frac{\partial\tilde{w}}{\partial s}+\tau\tilde{v}\right)ds+\frac{\partial\tilde{w}}{\partial v}dy\right\}\xi_z \end{aligned} \tag{4. 3}$$

and from (4.1) the increment of warping displacement vector is given by the formulae

$$\begin{aligned} d\tilde{\boldsymbol{V}}_{(2)} = &\left(\frac{\partial U}{\partial s}ds+\frac{\partial U}{\partial y}dy+\frac{\partial U}{\partial z}dz\right)\boldsymbol{e}_x \\ &+\left(\frac{\partial \boldsymbol{e}_x}{\partial s}ds+\frac{\partial \boldsymbol{e}_x}{\partial y}dy+\frac{\partial \boldsymbol{e}_x}{\partial z}dz\right)U \end{aligned} \tag{4. 4}$$

Considering these increments of vectors and (3.6), strain components γ_{ij} $(i, j=x, y, z)$ are expressed in terms of (2.6), (2.8) as

$$\begin{aligned} \gamma_{xx} &\approx (1-\kappa y)^{-1}\varepsilon+\tau^2(y^2+z^2)(1-\kappa y)^{-2}\psi_x+(1-\kappa y)^{-1}z\psi_y \\ &\qquad -(1-\kappa y)^{-1}y\psi_z+(1-\kappa y)^{-1}\frac{\partial U}{\partial s} \\ \gamma_{xy} &\approx \frac{1}{2}\left\{-(1-\kappa y)^{-1}z\psi_x-\tau z(1-\kappa y)^{-2}\frac{\partial U}{\partial s}+(1-\kappa y)\frac{\partial}{\partial y}[(1-\kappa y)^{-1}U]\right\} \\ \gamma_{xz} &\approx \frac{1}{2}\left\{(1-\kappa y)^{-1}y\psi_x+\tau y(1-\kappa y)^{-2}\frac{\partial U}{\partial s}+(1-\kappa y)\frac{\partial}{\partial z}[(1-\kappa y)^{-1}U]\right\} \\ \gamma_{yy} &\approx -\tau z\frac{\partial}{\partial y}\{(1-\kappa y)^{-1}U\} \end{aligned} \tag{4. 5}$$

$$\gamma_{yz} \approx \frac{1}{2}\left\{\tau y\frac{\partial}{\partial y}[(1-\kappa y)^{-1}U]-\tau z\frac{\partial}{\partial z}[(1-\kappa y)^{-1}U]\right\}$$

$$\gamma_{zz} \approx \tau y\frac{\partial}{\partial z}\{(1-\kappa y)^{-1}U\}$$

but these expressions are derived by taking into account the following relation between the line element in the direction of vector ξ_x which issues from any point on section contour and that of the girder axis before deformation

$$d\bar{S}_{(\xi_x)} = \{(1-\kappa y)^2+\tau^2(y^2+z^2)\}^{1/2}ds \approx (1-\kappa y)ds \tag{4.6}$$

Strain components γ_{yy}, γ_{yz} and γ_{zz} appear from the fact that the tangent to a generator of the girder is not in accord in general with that of the girder axis as it is clear from (3.6), and so is the second term of the strain components γ_{xy} and γ_{xz}.

Ordinally rampway is characteried as

$$\tau y,\ \tau z \ll (1-\kappa y) \approx 1$$

therefore, provided that τy and τz can be neglected compared with $(1-\kappa y)$ and so the terms with τy and τz can be dropped out, the strain components are approximately expressed as follows,

$$\begin{aligned}
\gamma_{xx} &\approx (1-\kappa y)^{-1}\varepsilon+(1-\kappa y)^{-1}z\psi_y-(1-\kappa y)^{-1}y\psi_z+(1-\kappa y)^{-1}\frac{\partial U}{\partial s}\\
\gamma_{xy} &\approx \frac{1}{2}\left\{-(1-\kappa y)^{-1}z\psi_x+(1-\kappa y)\frac{\partial}{\partial y}[(1-\kappa y)^{-1}U]\right\}\\
\gamma_{xz} &\approx \frac{1}{2}\left\{(1-\kappa y)^{-1}y\psi_x+(1-\kappa y)\frac{\partial}{\partial z}[(1-\kappa y)^{-1}U]\right\}
\end{aligned} \tag{4.7}$$

and more roughly they are reduced to the expression by cylindrical coordinates of which Z axis is inclined by the angle ϕ with respect to vertical axis in a sense fixed in space at any cross section of the girder.

And thus the warping displacement U can be assumed to take the form of the product of a basic transverse distribution (the unit warping function) and rate of twist.

$$U = -\omega\psi_x$$

5. Equilibrium Equation

Let $\boldsymbol{f}$ be the stress vector acting upon the section contour of which components are σ_{xx}, σ_{xy} and σ_{xz} along the axes, that is

$$\boldsymbol{f} = \sigma_{xx}\xi_x+\sigma_{xy}\xi_y+\sigma_{xz}\xi_z$$

Using stress vector $\boldsymbol{f}$, the force vector $\boldsymbol{F}$ (with the components N, Q_y and Q_z along the axes x, y and z) and the moment vector $\boldsymbol{H}$ (with the components T, M_y and M_z along the same axes) acting upon the cross section are obtained by definition as follows.

$$\begin{aligned} \boldsymbol{F} &= \int \boldsymbol{f}\, dy\, dz \equiv N\xi_x + Q_y\xi_y + Q_z\xi_z \\ \boldsymbol{H} &= \int (\boldsymbol{P}\times\boldsymbol{f})\, dy\, dz \equiv T\xi_x + M_y\xi_y + M_z\xi_z \end{aligned} \tag{5. 1}$$

Let $\boldsymbol{G}(\tilde{\theta})$ be the external force vector acting upon the section $\theta=\tilde{\theta}$ with the components G_x, G_y and G_z along the axes x, y and z respectively and $\boldsymbol{K}(\tilde{\theta})$ be the external moment vector given by its component K_x, K_y and K_z along the same axes, that is

$$\begin{aligned} \boldsymbol{G}(\tilde{\theta}) &\equiv G_x\xi_x + G_y\xi_y + G_z\xi_z \\ \boldsymbol{K}(\tilde{\theta}) &\equiv K_x\xi_x + K_y\xi_y + K_z\xi_z \end{aligned}$$

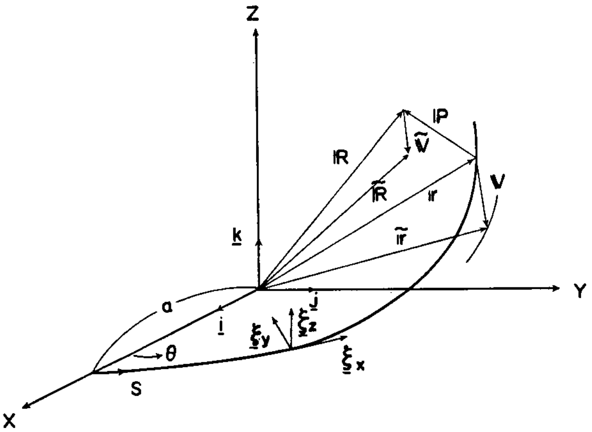

Fig. 1. Coordinate Systemand Description of Description of Deformations.

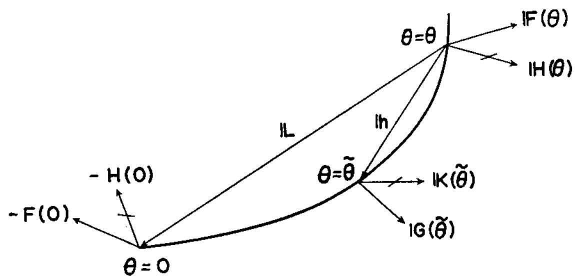

Fig. 2. Equilibrium of Forces.

The internal force vector and moment vector in any cross section $\theta=\theta$ are found using that of the cross section $\theta=0$ and external ones from usual equilibrium conditions of the girder, considered as a rigid three-dimensional body.

Referring to Fig. 2, the vector equations of equilibrium are written by

$$\begin{aligned} &\boldsymbol{F}(\theta)+\boldsymbol{G}(\tilde{\theta})\mathfrak{U}(\theta-\tilde{\theta})-\boldsymbol{F}(0)=0 \\ &\boldsymbol{H}(\theta)-\boldsymbol{H}(0)+\boldsymbol{K}(\tilde{\theta})\mathfrak{U}(\theta-\tilde{\theta})+\boldsymbol{L}\times(-\boldsymbol{F}(0)) \\ &\qquad+\boldsymbol{h}\times\boldsymbol{G}(\tilde{\theta})\mathfrak{U}(\theta-\tilde{\theta})=0 \end{aligned} \tag{5. 2}$$

where $\mathfrak{U}(\theta-\tilde{\theta})$ is the unit step function defined by

$$\mathfrak{U}(\theta-\tilde{\theta})=\begin{cases}1 & \text{for} \quad \theta>\tilde{\theta} \\ 0 & \text{for} \quad \theta>\tilde{\theta}\end{cases}$$

$\boldsymbol{L}$ is a position vector with the components L_x, L_y and L_z which is drawn from the point on the axis considered to the origin of the axis and $\boldsymbol{h}$ is a position vector given by its components h_x, h_y and h_z which issues from the same point to the application point of external force and moment.

The relations between the triads of vectors particular to the axis defined by $\theta=\theta_1$ and those defined by $\theta=\theta_2$ are needed in order to reduce the forees and the moments in any cross section to those in the section considered.

Taking into account of the Table-1, they are expressed in the form of matrix as follows.

$$\{\xi(\theta_2)\}^t=[\boldsymbol{A}(\theta_2-\theta_1)]\{\xi(\theta_1)\}^t \tag{5. 3}$$

The elements of the matrix $[\boldsymbol{A}]$ which are function of angle θ are given by

$$\begin{aligned} a_{11}&=\cos^2\phi\cos\theta+\sin^2\phi & a_{21}&=-\cos\phi\sin\theta & a_{31}&=\sin\phi\cos\phi(1-\cos\theta) \\ a_{21}&=\cos\phi\sin\theta & a_{22}&=\cos\theta & a_{32}&=-\sin\phi\sin\theta \\ a_{13}&=\sin\phi\cos\phi(1-\cos\theta) & a_{23}&=\sin\phi\sin\theta & a_{33}&=\sin^2\phi\cos\theta+\cos^2\phi \end{aligned} \tag{5. 4}$$

From the above expressions and the fact that matrix $[\boldsymbol{A}]$ is a orthogonal one, the following relations are obtained.

$$[\boldsymbol{A}(-\theta)]=[\boldsymbol{A}(\theta)]^{-1}=[\boldsymbol{A}(\theta)]^t \tag{5. 5}$$

Considering these relations, the ones needed for the translation of the force and moment vector are given by

$$\begin{aligned} &\{\xi(0)\}\{\boldsymbol{F}(0)\}^t=\{\xi(\theta)\}[\boldsymbol{A}(\theta)]\{\boldsymbol{F}(0)\}^t & &\{\xi(0)\}\{\boldsymbol{H}(0)\}^t=\{\xi(\theta)\}[\boldsymbol{A}(\theta)]\{\boldsymbol{H}(0)\}^t \\ &\{\xi(\tilde{\theta})\}\{\boldsymbol{G}(\tilde{\theta})\}^t=\{\xi(\theta)\}[\boldsymbol{A}(\theta-\tilde{\theta})]\{\boldsymbol{G}(\tilde{\theta})^t\} & &\{\xi(\tilde{\theta})\}\{\boldsymbol{K}(\tilde{\theta})\}=\{\xi(\theta)\}[\boldsymbol{A}(\theta-\tilde{\theta})]\{\boldsymbol{K}(\theta)\}^t \end{aligned}$$

The position vector $\boldsymbol{h}$ is expressed in the form of matrix by taking into account of the Figs. 1, 2 and Table-1,

$$\boldsymbol{h} = \boldsymbol{r}(\tilde{\theta}) - \boldsymbol{r}(\theta) \equiv \kappa^{-1} \sec\phi \{\xi(\theta)\}\{\bar{\boldsymbol{h}}(\theta-\tilde{\theta})\}^t \tag{5.6}$$

where

$$\bar{h}_x = -\{\cos^2\phi \sin(\theta-\tilde{\theta}) + (\theta-\tilde{\theta})\sin^2\phi\}$$
$$\bar{h}_y = \{1-\cos(\theta-\tilde{\theta})\}\cos\phi$$
$$\bar{h}_z = \{\sin(\theta-\tilde{\theta}) - (\theta-\tilde{\theta})\}\cos\phi \sin\phi$$

In the same manner, the position vector $\boldsymbol{L}$ is written by

$$\boldsymbol{L} = \boldsymbol{r}(0) - \boldsymbol{r}(\theta) = \kappa^{-1}\sec\phi\{\xi(\theta)\}\{\bar{\boldsymbol{h}}(\theta)\}^t \tag{5.7}$$

Introducing a new matrix defined by

$$[\bar{\boldsymbol{h}}] \equiv \begin{bmatrix} 0 & -\bar{h}_z & \bar{h}_y \\ \bar{h}_z & 0 & -\bar{h}_x \\ -\bar{h}_y & \bar{h}_x & 0 \end{bmatrix} \tag{5.8}$$

the vector products of the vector $\bar{\boldsymbol{h}}$ and $\boldsymbol{G}$, $\boldsymbol{L}$ and $\boldsymbol{F}(0)$ are expressed as follows.

$$\boldsymbol{h}\times\boldsymbol{G}(\tilde{\theta}) = \{\xi(\theta)\}[\bar{\boldsymbol{h}}(\theta-\tilde{\theta})][\boldsymbol{A}(\theta-\tilde{\theta})]\{\boldsymbol{G}(\tilde{\theta})\}^t\kappa^{-1}\sec\phi$$
$$\boldsymbol{L}\times\boldsymbol{F}(0) = \{\xi(\theta)\}[\bar{\boldsymbol{h}}(\theta)][\boldsymbol{A}(\theta)]\{\boldsymbol{F}(0)\}^t\kappa^{-1}\sec\phi$$

For the sake of simplicity, the product of matrices $[\boldsymbol{h}]\sec\phi$ and $[\boldsymbol{A}]$ are replaced by new matrix $[\boldsymbol{B}]$ of which are written as follows.

$$\begin{aligned}
b_{11} &= \cos\phi\sin\phi\{2(1-\cos\theta)-\theta\sin\theta\} \\
b_{21} &= \sin\phi(\sin\theta-\theta\cos\theta) \\
b_{12} &= -\sin\phi(\sin\theta-\theta\cos\theta) \\
b_{22} &= \theta\sin\theta-\tan\phi \\
b_{13} &= (\cos^2\phi-\sin^2\phi)(1-\cos\theta)+\theta\sin\theta\sin^2\phi \\
b_{23} &= \cos\phi\sin\theta+\theta\cos\theta\tan\phi\sin\phi \\
b_{31} &= (\cos^2\phi-\sin^2\phi)(1-\cos\theta)+\theta\sin\theta\sin^2\phi \\
b_{32} &= -\cos\phi\sin\theta-\theta\cos\theta\tan\phi\sin\phi \\
b_{33} &= -2\cos\phi\sin\phi(1-\cos\theta)+\theta\sin\theta\tan\phi\sin^2\phi
\end{aligned} \tag{5.9}$$

Using the previously obtained relations and the dimensionless internal and external forces and moments defined by

$$\begin{aligned}
\boldsymbol{F}(\theta) &\equiv \kappa^2 EJ_y\bar{\boldsymbol{F}}(\theta) & \boldsymbol{H}(\theta) &\equiv \kappa EJ_y\bar{\boldsymbol{H}}(\theta) \\
\boldsymbol{G}(\tilde{\theta}) &\equiv \kappa^2 EJ_y\bar{\boldsymbol{G}}(\tilde{\theta}) & \boldsymbol{K}(\tilde{\theta}) &\equiv \kappa EJ_y\bar{\boldsymbol{K}}(\tilde{\theta})
\end{aligned} \tag{5.10}$$

in which EJ_y is the moment of inertia with respect to y axis.

The matrix equation of equilibrium are finally expressed as follows.

$$\begin{aligned}\{\bar{F}(\theta)\}^t &= [A(\theta)]\{\bar{F}(0)\}^t-[A(\theta-\bar{\theta})]\{\bar{G}(\bar{\theta})\}^t\mathfrak{U}(\theta-\bar{\theta})\\ \{\bar{H}(\theta)\}^t &= [A(\theta)\{\bar{H}(0)\}^t-[A(\theta-\bar{\theta})[\{\bar{K}(\bar{\theta})\}^t\mathfrak{U}(\theta-\bar{\theta})\\ &\quad+[B(\theta)]\{\bar{F}(0)\}^t[B(\theta-\bar{\theta})]\{\bar{G}(\bar{\theta})\}^t\mathfrak{U}(\theta-\bar{\theta})\end{aligned} \tag{5.11}$$

6. Fundamental Differential Equation for the Stress Resultants and Displacements

From equation (4.7), the position of the coordinate origin is determined by the following two orthogonal relations

$$\int_F(1-\kappa y)^{-1}ydF=0 \qquad \int_F(1-\kappa y)^{-1}zdF=0 \tag{6.1}$$

and the position of the pole of the sectorial areas, in other words, the shear center is determined by the two orthogonal ones

$$\int_F(1-\kappa y)^{-1}y\omega dF=0 \qquad \int_F(1-\kappa y)^{-1}z\omega dF=0 \tag{6.2}$$

and the position of the origin of the sectorial areas is determined by the following equation

$$\int_F(1-\kappa y)^{-1}\omega dF=0 \tag{6.3}$$

but these points are substituted for simplicity by those determined by cylindrical coordinate system.

For brevity, the non-dimensional forms of the rate of rotations and displacements which are marked similarly by a bar are introduced and are given by the following relations respectively

$$\bar{\varPsi}\equiv\kappa^{-1}\varPsi \qquad \bar{V}\equiv\kappa V \tag{6.4}$$

According to (2.8), (4.7) and (5.1), and considering the three orthogonal relations (6.1), (6.2) and (6.3), and neglecting the terms with τ^2 in (2.5), and taking into account the relation $\frac{d}{ds}=\kappa\sec\phi\frac{d}{d\theta}\left(\frac{d}{d\theta}\right.$ which is denoted hereafter by the symbol $\left.D\right)$, the non-dimensional components of the rate of rotation $\bar{\varPsi}_y$, $\bar{\varPsi}_z$ and the stretch of the axis ε (which has as a matter of course no dimension) are expressed in terms of the non-dimensional components of the stress resultants as follows

$$\varepsilon\approx\sec\phi D\bar{u}-\bar{v}=\delta\bar{N} \tag{6.5}$$

$$\bar{\varPsi}_y\approx\varphi-\tan\phi\cdot\bar{u}-2\tan\phi\sec\phi D\bar{v}-\sec^2\phi D^2\bar{w}=\alpha\bar{M}_y+\beta M_z \tag{6.6}$$

$$\bar{\varPsi}_z\approx\sec\phi D\bar{u}+\sec^2\phi D^2\bar{v}-2\tan\phi\sec\phi D\bar{w}=\beta(\bar{M}_y+\gamma\bar{M}_z) \tag{6.7}$$

where α, β, γ and δ are non-dimensional section constants defined by

$$\begin{aligned} &\alpha \equiv \frac{J_y \cdot J_z / J_{yz}^2}{(J_y \cdot J_z / J_{yz})^2 - 1} \qquad \beta \equiv \frac{J_y / J_{yz}}{(J_y \cdot J_z / J_{yz})^2 - 1} \\ &\gamma \equiv J_y / J_{yz} \qquad\qquad\quad \delta \equiv \kappa^2 J_y / F \end{aligned} \tag{6.8}$$

And the torque referring to the shear center is expressed approximately in terms of $\bar{\Phi}_x$ as follows

$$\bar{\Phi}_x \approx \sec\phi D\varphi + \tan\phi v + \sec\phi D\bar{w} \approx \nu(\mu^2 - D^2)(\bar{T} + \bar{z}_s \bar{Q}_y - \bar{y}_s \bar{Q}_z) \tag{6.9}$$

where ν and μ^2 are non-dimensional section constants defined by

$$\nu \equiv \frac{J_y}{\kappa^2 C_\omega} \qquad \mu^2 \equiv \frac{G J_T}{\kappa^2 E C_\omega} \tag{6.10}$$

in which J_T and C_ω are torsion constant and warping constant respectively.

From equations (6.5), (6.6), and (6.7), the differential equation with respect to $\bar{w}$ is obtained as follows.

$$\begin{aligned} &D\{D^2 + (1 + 3\sin^2\phi)\}(D^2 - \mu^2)\bar{w} \\ &= -\cos^3\phi \cdot \nu \cdot (\bar{T} + \bar{z}_s \bar{Q}_y - \bar{y}_s \bar{Q}_z) - \cos^2\phi \cdot D(D^2 - \mu^2)(\alpha \bar{M}_y + \beta \bar{M}_z) \\ &\quad -2\cos^2\phi \sin\phi \cdot \beta (D^2 - \mu^2)(\bar{M}_y + \gamma \bar{M}_z) + \cos^2\phi \sin\phi \cdot \delta (D^2 - \mu^2)\bar{N} \end{aligned} \tag{6.11}$$

and the solution of this differential equation is

$$\begin{aligned} \bar{w} = &\, C_1 + C_2 \sin(1 + 3\sin^2\phi)\theta + C_3 \cos(1 + 3\sin^2\phi)\theta \\ &+ C_4 \sinh \mu\theta + C_5 \cosh \mu\theta \\ &- \cos^3\phi \cdot \nu \cdot L_3^{(0)}(\bar{T} + \bar{z}_s \bar{Q}_y - \bar{y}_s \bar{Q}_z) - \cos^2\phi \cdot L_1^{(0)} \bar{M}_y \\ &- 2\cos^2\phi \sin\phi \cdot \beta\gamma \cdot L_2^{(0)} \bar{M}_z + \cos^2\phi \sin\phi \cdot \delta \cdot L_2^{(0)} \bar{N} \end{aligned} \tag{6.12}$$

where $C_1 \sim C_5$ are integration constants and $L_1^{(0)}$, $L_2^{(0)}$ and $L_3^{(0)}$ are integration operators defined by

$$\begin{aligned} &L_1^{(0)} = \frac{1}{D^2 + 1 + 3\sin^2\phi} \qquad L_2^{(0)} = \frac{1}{D(D^2 + 1 + 3\sin^2\phi)} \\ &L_3^{(0)} = \frac{1}{D(D^2 + 1 + 3\sin^2\phi)(D^2 - \mu^2)} \end{aligned} \tag{6.13}$$

and the terms with these operators give the particular solution in which the stress resultants at the section $\theta = 0$ are included which are as yet unknown.

Similarly the differential equation with respect to v is

$$D^2 v + (1 - \sin^2\phi) v = \cos^2\phi \cdot \beta\gamma \bar{M}_z - \cos^2\phi \cdot \delta \bar{N} + 2\sin\phi D\bar{w} \tag{6.14}$$

and the solution of this differential equation is

$$\bar{v} = C_6 \sin(1-\sin^2\phi)\theta + C_7 \cos(1-\sin^2\phi)\theta + \cos^2\phi\cdot\beta\gamma L_4^{(0)}\bar{M}_z - \cos^2\phi\cdot\delta L_4^{(0)}\bar{N} + 2\sin\phi\cdot L_4^{(0)}(D\bar{w}) \quad (6.15)$$

where

$$L_4^{(0)} = \frac{1}{D^2+1-\sin^2\phi} \quad (6.16)$$

By substituting (6.12) into (6.14), $\bar{v}$ is expressed by integration constants $C_2 \sim C_7$ and the stress resultants at the section $\theta=0$.

From (6.5), $\bar{u}$ is written by

$$\bar{u} \approx \cos\phi D^{-1}(\bar{v}+\delta\bar{N}) + C_8 \quad (6.17)$$

and introducing (6.15) into (6.17) $\bar{u}$ is expressed by the integration constants $C_2 \sim C_8$ and the stress resultants at the section.

From (6.6) φ is given by

$$\varphi \approx \tan\phi\cdot\bar{u} + 2\tan\phi\sec\phi D\bar{v} + \sec^2\phi D^2\bar{w} + \alpha\bar{M}_y + \beta\bar{M}_z \quad (6.18)$$

and expressed by the same unknowns as $\bar{u}$ by substituting (6.12), (6.15) and (6.17) into (6.16).

Finally the warping moment is given by

$$\bar{M}_\omega \approx D\bar{\varphi}_x \approx \sec\phi D^2\bar{M}_y + 2\tan\phi\cdot\beta\gamma D\bar{M}_z - \tan\phi\cdot\delta D\bar{N} + \sec^3\phi D^2(D^2+1+3\sin^2\phi)\bar{w} \quad (6.19)$$

Using these equations the unknowns are determined by imposing the proper boundary conditions at the both ends of the girder.

Conclusions

In this analysis an approximate solutions for arbitrary supported helicoidal girder are presented. The expressions in this paper should be more simplified for design purposes and although it also calls for detailed numerical studies the fundamental characteristics of helicoidal girders are clarified in this paper.

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