Sub-harmonic Oscillations in Three-phase Circuit

By

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Sub-harmonic resonance in a three-phase circuit related with a power circuit with series capacitors is studied.

The paper describes that the equations which govern the oscillations in the system are under some restrictions transformed to the form:

$$\frac{dx_k}{d\tau} = \sum_{q=1}^{4} c_{kq}x_q + \varepsilon f_k(x_1, x_2) \qquad (k = 1, 2, 3, 4)$$

and the internal resonance of this system is correlated with the sub-harmonic resonance which occurs in the original three-phase system. An analog computer study on this problem is included.

1. Introduction

When an unloaded or lightly loaded transformer is energized through series capacitors in a power circuit, sub-harmonic resonance may occur. This peculiar circuit behavior depends essentially on the nonlinear characteristic of the transformer. The conditions on the circuit parameters to ensure the absence of 1/3-harmonic resonance have already been presented in the previous publication(1), where the circuit under consideration is a single-phase equivalent circuit.

Analysis of nonlinear oscillations in three-phase circuits is rather laborious and has been presented by several authors[2)~5)]. But, so far as the author knows, papers which deal analytically with the problem of sub-harmonic resonance in a three-phase circuit with series capacitors are few.

The circuit under consideration in this paper is a simplified three-phase circuit, but by this simplification little generality is lost. This paper deals only with the sub-harmonic oscillation of the order 1/3 which may commonly be encountered in the application of series capacitors to a power circuit. But the analytical procedure presented here can also be applied to other types of the nonlinear oscillation in a three-phase circuit.

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2. Fundamental Equations and their Transformation

To simplify matters it is assumed that the system of a transmission line with series capacitors is symmetrical and the impressed voltages are balanced. The system is roughly equivalent to the simplified network as shown in Fig. 1, where the iron-core inductors are connected in delta to render a star-delta connected transformer. We approximate the characteristic of the iron-core inductors by a cubic polynomial in terms of the numbers of flux interlinkage.

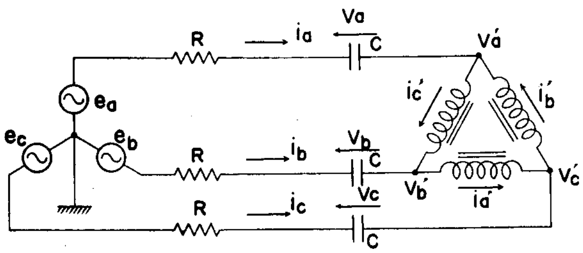

Fig. 1. Three-phase circuit with series capacitors.

The fundamental equations for the circuit shown in Fig. 1 are as follows:

$$\left.\begin{aligned} Ri_a+v_a+v_a' &= e_a = \sqrt{2}E\cos(\omega t+\varphi) \\ Ri_b+v_b+v_b' &= e_b = \sqrt{2}E\cos\left(\omega t+\varphi-\frac{2\pi}{3}\right) \\ Ri_c+v_c+v_c' &= e_c = \sqrt{2}E\cos\left(\omega t+\varphi-\frac{4\pi}{3}\right) \end{aligned}\right\} \tag{1}$$

$$i_a = C\frac{dv_a}{dt}, \quad i_b = C\frac{dv_b}{dt}, \quad i_c = C\frac{dv_c}{dt} \tag{2}$$

$$\left.\begin{aligned} v_b'-v_c' &= R'i_a'+\frac{d\psi_a}{dt} \\ v_c'-v_a' &= R'i_b'+\frac{d\psi_b}{dt} \\ v_a'-v_b' &= R'i_c'+\frac{d\psi_c}{dt} \end{aligned}\right\} \tag{3}$$

$$i_a = i_c'-i_b', \quad i_b = i_a'-i_c', \quad i_c = i_b'-i_a' \tag{4}$$

$$\left.\begin{aligned} i_a' &= c_1\psi_a+c_3\psi_a^3 \\ i_b' &= c_1\psi_b+c_3\psi_b^3 \\ i_c' &= c_1\psi_c+c_3\psi_c^3 \end{aligned}\right\} \tag{5}$$

where

R = equivalent line resistance per phase
C = capacity of the series capacitor
R' = resistance of the iron-core inductor
c_1, c_3 = characteristic constants of the iron-core inductor
i_a, i_b, i_c = currents through the capacitors
i_a', i_b', i_c' = currents through the iron-core inductors
v_a, v_b, v_c = terminal voltages across the capacitors
v_a', v_b', v_c' = voltages at the inductor terminals with respect to ground
ψ_a, ψ_b, ψ_c = numbers of flux interlinkage of the inductors
E, ω, φ = effective value, angular frequency and phase angle of the impressed voltages

In general the equations in terms of the phase (a, b, c) variables are not the most convenient form to solve the problems. The preceding equations are rewritten in terms of zero-phase-sequence-, forward-, and backward-variables as follows: (See Appendix 1)

$$\left.\begin{aligned} Ri_0+v_0+v_0' &= 0 \\ Ri_1+v_1+v_1' &= \sqrt{2}E \\ Ri_2+v_2+v_2' &= \sqrt{2}E \end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned} i_0 &= C\frac{dv_0}{dt} \\ i_1 &= C\frac{dv_1}{dt}+j\omega Cv_1 \\ i_2 &= C\frac{dv_2}{dt}-j\omega Cv_2 \end{aligned}\right\} \qquad (7)$$

$$\left.\begin{aligned} 0 &= R'i_0'+\frac{d\psi_0}{dt} \\ -i\sqrt{3}v_1' &= R'i_1'+\frac{d\psi_1}{dt}+j\omega\psi_1 \\ j\sqrt{3}v_2' &= R'i_2+\frac{d\psi_2}{dt}-j\omega\psi_2 \end{aligned}\right\} \qquad (8)$$

$$i_0 = 0, \quad i_1 = j\sqrt{3}i_1', \quad i_2 = -j\sqrt{3}i_2' \qquad (9)$$

and

$$
\left.\begin{aligned}
i_0' &= c_1\psi_0+c_3\left[\frac{1}{8}\psi_1^3\exp\{j3(\omega t+\psi)\}+\frac{1}{8}\psi_2^3\exp\{-j3(\omega t+\varphi)\}\right.\\
&\qquad\left.+\frac{3}{2}\psi_0\psi_1\psi_2+\psi_0^3\right]\\
i_1' &= c_1\psi_1+c_3\left[\frac{3}{4}\psi_1^3\psi_2+\frac{3}{2}\psi_0\psi_2^2\exp\{-j3(\omega t+\varphi)\}+3\psi_0^2\psi_1\right]\\
i_2' &= c_1\psi_2+c_3\left[\frac{3}{4}\psi_1\psi_2^2+\frac{3}{2}\psi_0\psi_1^2\exp\{j3(\omega t+\varphi)\}+3\psi_0^2\psi_2\right]
\end{aligned}\right\} \tag{10}
$$

The forward- and backward- variables are always complex conjugates:

$$
\left.\begin{aligned}
&i_2 = i_1^*\,, \quad i_2' = i_1'^*\,, \quad \psi_2 = \psi_1^*\,,\\
&v_2 = v_1^*\,, \quad v_2' = v_1'^*
\end{aligned}\right\} \tag{11}
$$

and, as usual, the asterisk indicates complex conjugate. This complex conjugate relationship leads to the fact that the third equation in each of Eqs. (6) through (10) is superfluous since it is the conjugate of the second one. From the first in Eqs. (6) (7) and (9), we have

$$
i_0 = 0 \tag{12}
$$

$$
v_0 = -v_0' = \text{const.} \tag{13}
$$

Now, in order to simplify, let us introduce new variables x, y, z and τ with the following relations:

$$
\left.\begin{aligned}
&x = j\frac{\alpha_v}{\sqrt{3}}\omega\psi_1\,, \quad y = \alpha_v v_1\,, \quad z = \frac{\alpha_v}{\sqrt{3}}\omega\psi_0\,,\\
&\tau = \omega t+\varphi+\frac{\pi}{6}
\end{aligned}\right\} \tag{14}
$$

and denote:

$$
\left.\begin{aligned}
&e = \alpha_v\sqrt{2}E\,, \quad r = R+\frac{R'}{3}\,, \quad r' = \frac{R'}{3}\,, \quad k = \frac{1}{\omega C}\,,\\
&\bar{c}_1 = \frac{3}{\omega}c_1\,, \qquad \bar{c}_3 = \frac{27}{4\omega^3\alpha_v^2}c_3
\end{aligned}\right\} \tag{15}
$$

where α_v is an arbitrary real constant. Eliminating i_0', i_1', v_1' and i in Eqs. (6) through (10) and substituting Eqs. (11) through (15) into the result obtained, we have

$$
\frac{dz}{d\tau} = -r'f_0(x, z, \tau) \tag{16}
$$

$$\left.\begin{aligned} \frac{dx}{d\tau} &= e-jx-y-rf_1(x,\, z,\, \tau) \\ \frac{dy}{d\tau} &= -jy+kf_1(x,\, z,\, \tau) \end{aligned}\right\} \tag{17}$$

where

$$\begin{aligned} f_0(x,\, z,\, \tau) &= \bar{c}_1 z+\bar{c}_3\left[\frac{1}{6}x^3 \exp(j3\tau)+\frac{1}{6}x^{*3}\exp(-j3\tau)\right. \\ &\qquad \left. +2xx^*z+\frac{4}{3}z^3\right] \end{aligned} \tag{18}$$

$$f_1(x,\, z,\, \tau) = \bar{c}_1 x+\bar{c}_3[x^2x^*+2x^{*2}z\exp(-j3\tau)+4xz^2] \tag{19}$$

Neglecting the resistance of the iron-core inductors, we have from Eq. (16)

$$\frac{dz}{d\tau} = 0 \tag{20}$$

This leads to z=const. which means permanent magnetization, and we assum

$$z = 0 \tag{21}$$

Substituting $z=0$ into Eq. (19), we have from Eq. (17)

$$\left.\begin{aligned} \frac{dx}{d\tau} &= e-jx-y-r\{\bar{c}_1 x+\bar{c}_3 x^2x^*\} \\ \frac{dy}{d\tau} &= -jy+k\{\bar{c}_1 x+\bar{c}_3 x^2x^*\} \end{aligned}\right\} \tag{22}$$

Note that time τ does not appear explicitly in the right-hand sides of Eq. (22). For the state of equilibrium (x_0, y_0), we have

$$\left.\begin{aligned} jx_0+y_0+r(\bar{c}_1+\bar{c}_3 x_0x_0^*)\,x_0 &= e \\ k(\bar{c}_1+\bar{c}_3 x_0x_0^*)\,x_0 &= jy_0 \end{aligned}\right\} \tag{23}$$

Setting

$$\left.\begin{aligned} x_0 &= \rho_0 \exp(j\theta_0) \\ y_0 &= -jk(\bar{c}_1+\bar{c}_3\rho_0^2)\,\rho_0\exp(j\theta_0) \end{aligned}\right\} \tag{24}$$

we have

$$\bar{c}_3^2(r^2+k^2)\,\rho_0^6-2\bar{c}_3\{k-\bar{c}_1(r^2+k^2)\}\,\rho_0^4+\{1-2\bar{c}_1 k+\bar{c}_1^2(r^2+k^2)\}\,\rho_0^2 = e^2 \tag{25}$$

$$\tan\theta_0 = \frac{1}{r}\left\{k-\frac{1}{\bar{c}_1+\bar{c}_3\rho_0^2}\right\} \tag{26}$$

Eq. (25) is of the third degree in ρ_0^2. If $\bar{c}_1>0$ and $\bar{c}_3>0$, all the roots, ρ_0^2's are positive only when,

$$\left.\begin{array}{l}\alpha>\bar{c}_1>0\,, \quad (\alpha-\bar{c}_1)^2>3\beta^2 \\ e_1^2>e^2>e_0^2\end{array}\right\} \tag{27}$$

where

$$\alpha=\frac{k}{r^2+k^2}, \quad \beta=\frac{r}{r^2+k^2} \tag{28}$$

$$\left.\begin{array}{l}e_1^2\\ e_2^2\end{array}\right\}=\frac{(\alpha-\bar{c}_1)\{(\alpha-\bar{c}_1)^2+9\beta^2\}\}\pm\sqrt{\{(\alpha-\bar{c}_1)^2-3\beta^2\}^2}}{\frac{27}{2}\bar{c}_3(\alpha^2+\beta^2)} \tag{29}$$

If the condition (27) is not satisfied, one root is positive and the others are complex conjugate. Then, corresponding to the positive root, a state of equilibrium may exist. The stability of the equilibrium will be investigated in Appendix II. Now, let us introduce new variables u and v by the following equations:

$$\left.\begin{array}{l}x=x_0+u\exp(j\theta_0)\\ y=y_0+v\exp(j\theta_0)\end{array}\right\} \tag{30}$$

where x_0, y_0 and θ_0 are given by Eqs. (24) (25) and (26). Then, Eq. (22) becomes

$$\left.\begin{array}{l}\dfrac{du}{d\tau}=-ju-v-\dfrac{r}{2}(m_3+m_1)\,u-\dfrac{r}{2}(m_3-m_1)\,u^* \\ \qquad -r\bar{c}_3(\rho_0u+2\rho_0u^*+uu^*)\,u \\ \dfrac{dv}{d\tau}=-jv+\dfrac{k}{2}(m_3+m_1)\;u+\dfrac{k}{2}(m_3-m_1)\,u^* \\ \qquad +k\bar{c}_3(\rho_0u+2\rho_0u^*+uu^*)\,u\end{array}\right\} \tag{31}$$

where

$$m_3=\bar{c}_1+3\bar{c}_3\rho_0^2\,, \quad m_1=\bar{c}_1+\bar{c}_3\rho_0^2 \tag{32}$$

3. Solution of Transformed Equations

In this section, we look for a periodic solution of Eq. (31) when r and $\bar{c}_3$ are sufficiently small quantities. If we set

$$\left.\begin{array}{l}u=x_1+jx_2\\ v=x_3+jx_4\end{array}\right\} \tag{33}$$

where x_1, x_2, x_3 and x_4 are real variables, we have from Eq. (31)

$$\left.\begin{aligned}
\frac{dx_1}{d\tau} &= -rm_3x_1+x_2-x_3-r\bar{c}_3\{\rho_0(3x_1^2+x_2^2)+(x_1^2+x_2^2)x_1\} \\
\frac{dx_2}{d\tau} &= -x_1-rm_1x_2-x_4-r\bar{c}_3\{2\rho_0x_1x_2+(x_1^2+x_2^2)x_2\} \\
\frac{dx_3}{d\tau} &= km_3x_1+x_4+k\bar{c}_3\{\rho_0(3x_1^2+x_2^2)+(x_1^2+x_4^2)x_2\} \\
\frac{dx_4}{d\tau} &= km_1x_2-x_3+k\bar{c}_3\{2\rho_0x_1x_1+(x_1^2+x_2^2)x_2\}
\end{aligned}\right\} \tag{34}$$

We turn, for a time, to a discussion of the character of the generating system of Eq. (34). If $r=0$ and $\bar{c}_3=0$, Eq. (34) becomes

$$\left.\begin{aligned}
\frac{dx_1}{d\tau} &= x_2-x_3\,, & \frac{dx_2}{d\tau} &= -x_1-x_4 \\
\frac{dx_3}{d\tau} &= km_3x_1+x_4\,, & \frac{dx_4}{d\tau} &= km_1x_2-x_3
\end{aligned}\right\} \tag{35}$$

where m_1 and m_3 remain unchanged. The solution of Eq. (35) is related to the nature of the roots of its characteristic equation:

$$\lambda^4+\{2+k(m_1+m_3)\}\lambda^2+(1-km_1)(1-km_3) = 0 \tag{36}$$

If $km_1<1<km_3$ holds, Eq. (36) has a pair of imaginary roots and two real roots, whereas if either $km_1>1$ or $km_3<1$ holds, Eq. (36) has imaginary roots only, that is, the system of Eq. (35) has two periodic solutions.

We consider the later case where the roots of Eq. (36) are $\pm j\omega_1$ and $\pm j\omega_2$. Three cases are possible: (1) when $\omega_2=\nu\omega_1$ where ν is an integer, that is, the exact internal resonance; (2) when $\omega_2\cong\nu\omega_1$, that is, the neighborhood of the internal resonance; and (3) when $\omega_2\neq\nu\omega_1$, the non-internal-resonance. We will consider the cases (1) and (2). We introduce h_3 and h_3 and rewrite Eq. (34) as

$$\frac{dx_k}{d\tau} = \sum_{q=1}^{4} c_{kq}x_q+\varepsilon f_k(x_1, x_2)\,, \qquad (k = 1, 2, 3, 4) \tag{37}$$

or

$$\left.\begin{aligned}
\frac{dx_1}{d\tau} &= x_2-x_3+\varepsilon f_1(x_1, x_2) \\
\frac{dx_2}{d\tau} &= -x_1-x_4+\varepsilon f_2(x_1, x_2) \\
\frac{dx_3}{d\tau} &= h_3x_1+x_4+\varepsilon f_3(x_1, x_2) \\
\frac{dx_4}{d\tau} &= h_1x_2-x_3+\varepsilon f_4(x_1, x_2)
\end{aligned}\right\} \tag{38}$$

where

$$\left.\begin{aligned} \varepsilon f_1(x_1, x_2) &= -rm_3x_1+r\bar{c}_3\{\rho_0(3x_1^2+x_2^2)+(x_1^2+x_2^{22})\,z_1\} \\ \varepsilon f_2(x_1, x_2) &= -rm_1x_2-r\bar{c}_3\{2\rho_0x_1x_2+(x_1^2+x_2^2)\,x_2\} \\ \varepsilon f_3(x_1, x_2) &= (km_3-k_3)\,x_1+k\bar{c}_3\{\rho_0(3x_1^2+x_2^2)+(x_1^2+x_2^2)\,x_1\} \\ \varepsilon f_1(x_2, x_1) &= (km_1-h_2)\,x_4+k\bar{c}_3\{2\rho_0x_1x_2+(x_1^2+x_2^2)\,x_2\} \end{aligned}\right\} \tag{39}$$

The reason why h_1 and h_3 are introduced is the following: If $\varepsilon=0$, Eq. (38) becomes

$$\left.\begin{aligned} \frac{dx_1}{d\tau} &= x_2-x_3\,, \quad & \frac{dx_2}{d\tau} &= -x_1-x_4\,, \\ \frac{dx_3}{d\tau} &= h_3x_3+x_4\,, \quad & \frac{dx_4}{d\tau} &= h_1x_2-x_3 \end{aligned}\right\} \tag{40}$$

Then, we denote the roots of the characteristic equation of Eq. (40) as $\pm j\omega_1$ and $\pm j\omega_2$ where $\omega_2>\omega_1$, and we determine the values of h_1 and h_3 to satisfy the relation $\omega_2=\nu\omega_1$.

In case (1), (km_3-h_3) and (km_1-h_1) are equal to zero, whereas in case (2) they are not equal to zero but sufficiently small. We consider Eq. (38) as a nearly linear system. If $\varepsilon=0$ in Eq. (38), we have the generating system (40) whose solution is

$$x_k^{(0)} = a\varphi_k\,e^{j\psi}+a\varphi_k^*\,e^{-j\psi}+(b+jc)\,\chi_k\,e^{j\psi\nu}+(b-jc)\,\chi_k^*\,e^{-j\psi\nu} \tag{41}$$

where

$$\psi = \omega_1\tau\,, \quad \nu\psi = \omega_2\tau \tag{42}$$

and a, b and c are constants. Furthermore

$$\begin{aligned} &\varphi_1 = \varphi\,, \quad \varphi_2 = j\mu_1\varphi\,, \quad \varphi_3 = j(\mu_1-\omega_1)\,\varphi\,, \quad \varphi_4 = (\omega_1\mu_1-1)\,\varphi\,, \\ &\chi_1 = \chi\,, \quad \chi_2 = j\mu_2\chi\,, \quad \chi_3 = j(\mu_2-\omega_2)\,\chi\,, \quad \chi_4 = (\omega_2\mu_2-1)\,\chi\,. \end{aligned} \tag{43}$$

φ and χ being constants, and

$$\mu_1 = \frac{2\omega_1}{\omega_1^2+(1-h_1)}\,, \quad \mu_2 = \frac{2\omega_2}{\omega_2^2+(1-h_3)} \tag{44}$$

Now, we will follow the Bogoliubov-Mitropolsky mthod[6]. Note that in the non-internal-resonance case, there is no relation between the phases of two oscillations; here, on the contrary, this relation plays an important role.

We have the first approximation of the solution of Eq. (38):

$$x_k = a\varphi_k e^{j\psi} + a\varphi_k{}^* e^{-j\psi} + (b-jc) x_k e^{j\nu\psi} + (b-jc) x_k{}^* e^{-j\nu\psi} \qquad (k = 1, 2, 3, 4) \tag{45}$$

where

$$\left.\begin{aligned} \frac{da}{d\tau} &= \varepsilon A_1(a, b, c) , \quad \frac{db}{d\tau} = \varepsilon B_1(a, b, c) \\ \frac{dc}{d\tau} &= \varepsilon C_1(a, b, c) , \quad \frac{d\psi}{d\tau} = \omega_1 + \varepsilon D_1(a, b, c) \end{aligned}\right\} \tag{46}$$

A_1, B_1, C_1 and D_1 can be determined by the similar method to that in Bogoliubov-Mitropolsky's text. (See Appendix III) Eq. (46) cannot be integrated in a closed form. However, some qualitative aspects of the solution can be establised. There are two principal forms of stationary oscillations: (a) those which correspond to the singular point of Eq. (46) and (b) oscillations corresponding to the periodic solution. The singular points are given by the solution of the set of equations:

$$\varepsilon A_1 = 0 , \quad \varepsilon B_1 = 0 , \quad \varepsilon C_1 = 0 \tag{47}$$

The stability of these singular points can be investigated by the variational equations:

$$\frac{d\xi_k}{d\tau} + \sum_{q=1}^{3} a_{kq}\xi_q = 0 \qquad (k = 1, 2, 3) \tag{48}$$

where

$$\left.\begin{aligned} a_{11} &= -\left\{\frac{\partial}{\partial a}(\varepsilon A_1)\right\}_{(a_0, b_0, c_0)} , \quad a_{12} = -\left\{\frac{\partial}{\partial b}(\varepsilon A_1)\right\}_{(a_0, b_0, c_0)} , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \quad a_{33} = -\left\{\frac{\partial}{\partial c}(\varepsilon C_1)\right\}_{(a_0, b_0, c_0)} \end{aligned}\right\} \tag{49}$$

(a_0, b_0, c_0) being singular point. If every root of the characteristic equation of Eq. (48) has a negative real part the singular point (a_0, b_0, c_0) is stable and the system operates in a internal resonance condition. There exists an oscillation:

$$x_k = a_0\varphi_k e^{j\psi} + a_0\varphi_k{}^* e^{-j\psi} + (b_0+jc_0) x_k e^{j\nu\psi} + (b_0-jc_0) x_k{}^* e^{-j\xi\psi} \qquad (k = 1, 2, 3, 4) \tag{50}$$

where

$$\psi = (\omega_1 + \Delta)\tau , \quad \Delta = \varepsilon D_1(a_0, b_0, c_0) \tag{51}$$

The periodic solution of Eq. (46) will be correlated with a quasi-periodic solution of Eq. (38).

In the case where the resistance of the inductors, R' is not equal to zero but sufficiently small, we may have a nonautonomous nearly linear system:

$$\frac{dx_k}{d\tau}=\sum_{q=1}^{5} c_{kq}x_q+\varepsilon f_k(\tau, x_1, x_2, x)\ , \qquad (k=1, 2, 3, 4, 5) \tag{62}$$

corresponding to Eq. (37). We may then proceed in a similar procedure to the above. A detailed study will be given in a later paper.

4. 1/3-harmonic Oscillation

We shall consider the case where the 1/3-harmonics may occur in the original system.
Setting

$$h_1=h_3=\frac{1}{9} \tag{53}$$

in Eq. (40), we obtain

$$\omega_1=\frac{2}{3}\ , \quad \omega_2=\frac{4}{3}\ ;\ \nu=2 \tag{54}$$

Denoting the stable singular point of Eq. (46) by (a_0, b_0, c_0) we have the first approximate solution in the form:

$$\left.\begin{aligned} x_1 &= 2a_0\cos\psi+2b_0\cos 2\psi-2c_0\sin 2\psi \\ x_2 &= -2a_0\sin\psi-2b_0\sin 2\psi-2c_0\cos 2\psi \\ z_3 &= -\frac{2}{3}a_0\sin\psi+\frac{2}{3}b_0\sin 2\psi+\frac{2}{3}c_0\cos 2\psi \\ x_4 &= -\frac{2}{3}a_0\cos\psi+\frac{2}{3}b_0\cos 2\psi-\frac{2}{3}c_0\sin 2\psi \end{aligned}\right\} \tag{55}$$

where

$$\psi=\left(\frac{2}{3}+\Delta\right)\tau+\varphi_0 \tag{56}$$

φ_0 being a constant. From Eqs. (14), (24), (30) and (33), we have

$$\left.\begin{aligned} \psi_1 &= -j\frac{\sqrt{3}}{\omega\alpha_v}[\rho_0+2a_0\exp(-j\psi)+2(b_0-jc_0)\exp(-j2\psi)]\exp(j\theta_0) \\ v_1 &= -j\frac{1}{\alpha_v}\Big[k(\bar{\tau}_1+\bar{\tau}_3\rho_0^2)\rho_0+\frac{2}{3}a_0\exp(-j\psi) \\ &\qquad -\frac{2}{3}(b_0-jc_0)\exp(-j2\psi)\Big]\exp(j\theta_0) \end{aligned}\right\} \tag{57}$$

Then, we have the α- and β-components:

$$
\left.
\begin{aligned}
\psi_\alpha &= \bar{\Phi}_1 \sin(\omega t+\gamma_1)+\bar{\Phi}_2 \sin\left(\overline{\frac{1}{3}-\Delta\omega t+\gamma_2)}\right) \\
&\quad -\bar{\Phi}_3 \sin\left(\overline{\frac{1}{3}+\Delta\omega t+\gamma_3}\right) \\
\psi_\beta &= \bar{\Phi}_1 \cos(\omega t+\gamma_1)-\bar{\Phi}_2 \cos\left(\overline{\frac{1}{3}-\Delta\omega t+\gamma_2}\right) \\
&\quad -\bar{\Phi}_3 \cos\left(\overline{\frac{1}{3}+\Delta\omega t+\gamma_3}\right)
\end{aligned}
\right\} \tag{58}
$$

where

$$
\bar{\Phi}_1 = \frac{\sqrt{3}}{\alpha_v\omega}\rho_0, \quad \bar{\Phi}^2 = \frac{2\sqrt{3}}{\alpha_v\omega}a_0, \quad \bar{\Phi}_3 = \frac{2\sqrt{3}}{\alpha_v\omega}\sqrt{b_0^2+c_0^2} \tag{59}
$$

$$
\left.
\begin{aligned}
\gamma_1 &= \varphi+\theta_0, \quad \gamma_2 = \varphi+\theta+\varphi_0 \\
\gamma_3 &= 2\varphi_0-\varphi-\theta_0-\tan^{-1}\left(\frac{c_0}{b_0}\right)
\end{aligned}
\right\} \tag{60}
$$

From Eq. (21), $\psi_0=0$. Therefore, we have the phase components of the flux interlinkage:

$$
\left.
\begin{aligned}
\psi_a &= \bar{\Phi}_1 \sin(\omega t+\gamma_1)+\bar{\Phi}_2 \sin\left(\frac{1}{3}\omega t+\gamma_2\right)-\bar{\Phi}_3 \sin\left(\frac{1}{3}\omega t+\gamma_3\right) \\
\psi_b &= \bar{\Phi}_1 \sin\left(\omega t+\gamma_1-\frac{2\pi}{3}\right)+\bar{\Phi}_2 \sin\left(\frac{1}{3}\omega t+\gamma_2-\frac{2\pi}{3}\right) \\
&\quad -\bar{\Phi}_3 \sin\left(\frac{1}{3}\omega t+\gamma_3+\frac{2\pi}{3}\right) \\
\psi_c &= \bar{\Phi}_1 \sin\left(\omega t+\gamma_1+\frac{2\pi}{3}\right)+\bar{\Phi}_2 \sin\left(\frac{1}{3}\omega t+\gamma_2+\frac{2\pi}{3}\right) \\
&\quad -\bar{\Phi}_3 \sin\left(\frac{1}{3}\omega t+\gamma_3-\frac{2\pi}{3}\right)
\end{aligned}
\right\} \tag{61}
$$

setting the sufficiently small value Δ equal to zero.

If the initial values of the terminal voltages across the capacitors are equal to zero, $v_0=0$ from Eq. (13).

Therefore, we have the phase components:

$$
\left.
\begin{aligned}
v_a &= V_1 \sin(\omega t+\gamma_1)+V_2 \sin\left(\frac{1}{3}\omega t+\gamma_2\right)+V_3 \sin\left(\frac{1}{3}\omega t+\gamma_3\right) \\
v_b &= V_1 \sin\left(\omega t+\gamma_1-\frac{2\pi}{3}\right)+V_2 \sin\left(\frac{1}{3}\omega t+\gamma_2-\frac{2\pi}{3}\right) \\
&\quad +V_3 \sin\left(\frac{1}{3}\omega t+\gamma_3+\frac{2\pi}{3}\right)
\end{aligned}
\right\} \tag{62}
$$

$$v_c = V_1 \sin\left(\omega t+\gamma_1+\frac{2\pi}{3}\right)+V_2 \sin\left(\frac{1}{3}\omega t+\gamma_2+\frac{2\pi}{3}\right)$$
$$+V_3 \sin\left(\frac{1}{3}\omega t+\gamma_3-\frac{2\pi}{3}\right)$$

where

$$V_1 = \frac{1}{\alpha_v}k(\bar{c}_1+\bar{c}_3\rho_0^2)\rho_0\,, \quad V_2 = \frac{2}{3\alpha_v}a_0\,, \quad V_3 = \frac{2}{3\alpha_v}\sqrt{b_0^2+c_0^2} \tag{63}$$

γ_1, γ_2 and γ_3 being given by Eq. (60).

In Eqs. (61) and (62) we see that the first and the second term of the right-hand sides forms a positive-phase sequence respectively whereas the third term forms a negative-phase sequence.

That is, the 1/3-harmonic oscillation consists of two components: one a positive-phase sequence, and the other a negative-phase sequence.

If Δ is not exactly equal to zero, we also see that the 1/3-harmonic oscillation with a time-varying amplitude may occur.

Such unsymmetrical 1/3-harmonic oscillations were observed on the Suriko-Shingu transmission line (See Appendix V).

5. Numerical Example and Analog Computer Study

We consider an example, where $e=0.4$, $\bar{c}_1=0$, $\bar{c}_3k=0.3$ and $\bar{c}_3r=0.1$.

The solution of Eqs. (25) and (26) corresponding to the state of equilibrium is $\rho_0=0.423$ and $\theta_0=-89°$, respectively. Hence, the roots of the characteristic equation (36) become

$$\pm j\omega_1 = \pm j\,0.672 \quad \text{and} \quad \pm j\omega_2 = \pm j\,1.327\,.$$

Then, in this case $\omega_2 \cong 2\omega_1$ and $\omega_1 \cong \frac{2}{3}$

By Eq. (47), the singular points (a_0, b_0, c_0) are determined. They are

$$(1)\begin{cases} a_0 = 0.144 \\ b_0 = -0.0945 \\ c_0 = 0.0234 \end{cases} \qquad (2)\begin{cases} a_0 = -0.144 \\ b_0 = -0.0945 \\ c_0 = 0.0234 \end{cases}$$

$$(3)\begin{cases} a_0 = 0.0343 \\ b_0 = -0.000844 \\ c_0 = 0.0240 \end{cases} \qquad (4)\begin{cases} a_0 = -0.0343 \\ b_0 = -0.000844 \\ c_0 = 0.0240 \end{cases}$$

$$(5)\quad a_0 = b_0 = c_0 = 0$$

From the variational equation (48), we see that the singular points (3) and (4) are unstable and the others are stable. The singular points (1) and (2) are correlated with the 1/3-harmonic oscillations in the original system. An analog computer is used to obtain the solution of the circuit analysed in the preceding. In this study we compute with the α- and β-variables assuming that $\bar{c}_1=0$, $R'=0$ and $\varphi=0$. (See Appendix IV). The results are given in Fig. 2 where 1/3-harmonic

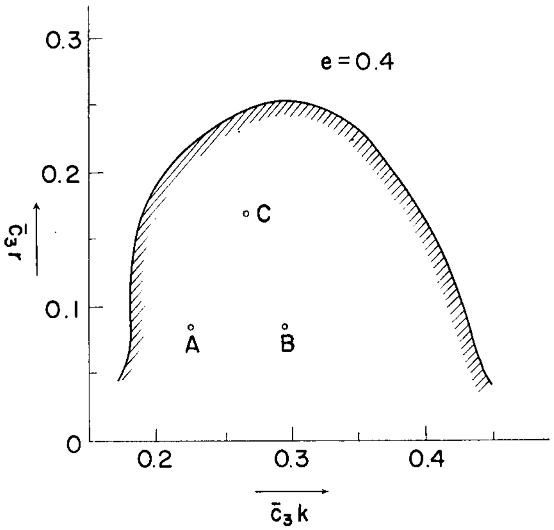

Fig. 2. Region in which 1/3-harmonic oscillation is sustained.

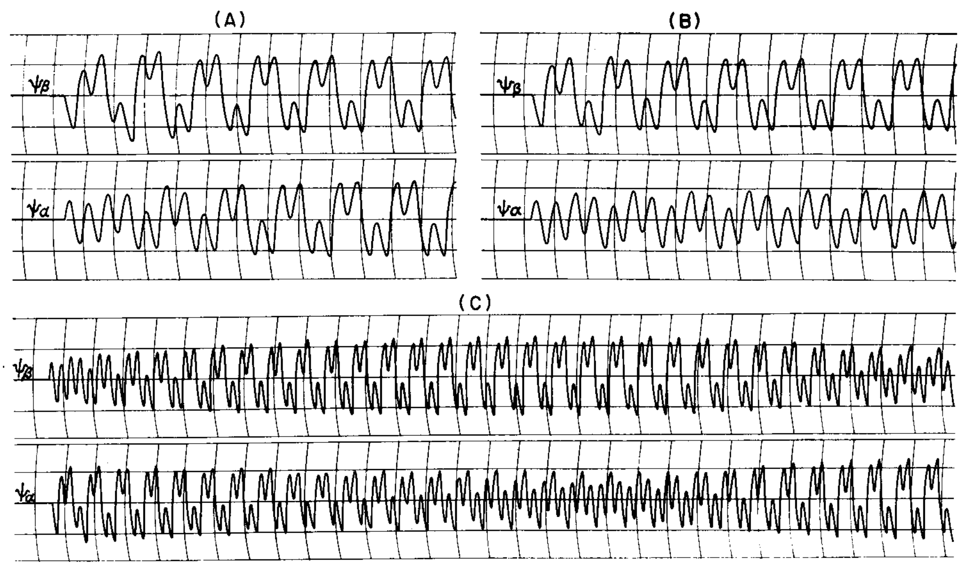

Fig. 3. Wave forms of 1/3-harmonic oscillation.

resonance may occur in the shaded region and smaller resistance results in an almost periodic or a non-periodic oscillation.

Typical wave forms when 1/3-harmonic occurs are shown in Fig. 3 (A), (B) and (C). The parameters corresponding to these are indicated by the points A, B and C in Fig. 2.

6. Conclusion

An analysis of generation of sub-harmonic resonance in a three-phase circuit has been presented. The original differential equations of the circuit are transformed under certain restrictions to the following differential equations:

$$\frac{dd_k}{d\tau}=\sum_{q=1}^{4}c_{kq}x_q+\varepsilon f_k(x_1,\, x_2)\,,\qquad (k=1,\,2,\,3,\,4)$$

By the Bogoliubov-Mitropolsky's method we can analize the internal resonance which is correlated with 1/3-harmonic resonance in the original system. We show that the 1/3-harmonic components consist of a positive- and a negative-phase-sequence components with time-varying amplitudes in general. The analytical results are compared with the results obtained by analog computer.

Further work on the analysis of the three-phase circuit where the other nonlinear oscillations may occur is proceeding.

Acknowledgment

The author wishes to espress sincere gratitude to Professor Emertus Shigenori Hayashi for his valuable guidance and generous encouragement. Thanks are also due to Mr. Ryoichi Kato for his assistance in analog computations.

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Appendix I

The transformation from three-phase variables to 0-, α- and β-variables is defined as follows:

$$\left.\begin{aligned} w_0 &= \frac{1}{3}(w_a+w_b+w_c) \\ w_\alpha &= \frac{1}{3}(2w_a-w_b-w_c) \\ w_\beta &= \frac{1}{\sqrt{3}}(w_b-w_c) \end{aligned}\right\} \tag{I.1}$$

or inversely

$$\left.\begin{aligned} w_a &= w_0+w_\alpha \\ w_b &= w_0-\frac{1}{2}w_\alpha+\frac{\sqrt{3}}{2}w_\beta \\ w_c &= w_0-\frac{1}{2}w_\alpha-\frac{\sqrt{3}}{2}w_\beta \end{aligned}\right\} \tag{I.2}$$

Using this transformation and the equations given by Eqs. (1) through (5) we have the following set of equatiots:

$$\left.\begin{aligned} Ri_0+v_0+v_0' &= 0 \\ Ri_\alpha+v_\alpha+v_\alpha' &= \sqrt{2}E\cos(\omega t+\varphi) \\ Ri_\beta+v_\beta+v_\beta' &= \sqrt{2}E\sin(\omega t+\varphi) \end{aligned}\right\} \tag{I.3}$$

$$i_0 = C\frac{dv_0}{dt}, \quad i_\alpha = C\frac{dv_\alpha}{dt}, \quad i_\beta = C\frac{dv_\beta}{dt} \tag{I.4}$$

$$\left.\begin{aligned} 0 &= \frac{d\psi_0}{dt}+R'i_0 \\ \sqrt{3}v_\beta' &= \frac{d\psi_\alpha}{dt}+R'i_\alpha \\ -\sqrt{3}v_\alpha' &= \frac{d\psi_\beta}{dt}+R'i_\beta \end{aligned}\right\} \tag{I.5}$$

$$i_0 = 0, \quad i_\alpha = -\sqrt{3}i_\beta, \quad i_\beta = \sqrt{3}i_\alpha' \tag{I.6}$$

$$\left.\begin{aligned} i_0' = c_1\psi_0+c_3\Big\{&\frac{1}{4}\psi_\alpha^3-\frac{3}{4}\psi_\alpha\psi_\beta^3+\frac{3}{2}\psi_\alpha^2\psi_0 \\ &+\frac{3}{2}\psi_\beta^2\psi_0+\psi_0^3\Big\} \end{aligned}\right\} \tag{I.7}$$

$$\left.\begin{aligned} i_{\alpha}' &= c_1\psi_1+c_3\left\{\frac{3}{4}\psi_{\alpha}^3+\frac{3}{4}\psi_{\alpha}\psi_{\beta}^2+\frac{3}{2}\psi_{\alpha}^2\psi_0 \right. \\ &\qquad \left. -\frac{3}{2}\psi_{\beta}^2\psi_0+3\psi_{\alpha}\psi_0^2\right\} \\ i_{\beta}' &= c_1\psi_{\beta}+c_3\left\{\frac{3}{4}\psi_{\alpha}^2\psi_{\beta}+\frac{3}{4}\psi_{\beta}^3-3\psi_{\alpha}\psi_{\beta}\psi_0 \right. \\ &\qquad \left. +3\psi_{\beta}\psi_0^2\right\} \end{aligned}\right|$$

The transformation which relates the forward- and the backward-variables, w_1 and w_2, to α- and β-variables, $w_{\alpha}/$ and w_{β}, is defined as follows:

$$\left.\begin{aligned} w_1 &= (w_{\alpha}+jw_{\beta})\exp(-j\theta) \\ w_2 &= (w_{\alpha}-j_1w_{\beta})\exp(j\theta) \end{aligned}\right\} \tag{I.8}$$

or inversely

$$\left.\begin{aligned} w_{\alpha} &= \frac{1}{2}\left\{w_1\exp(j\theta)+w_2\exp(-j\theta)\right\} \\ w_{\beta} &= \frac{1}{2}\left\{w_1\exp(j\theta)-w_2\exp(-j\theta)\right\} \end{aligned}\right\} \tag{I.9}$$

In this study $\theta=\omega t+\varphi$

Therefore, the equations given by Eqs. (I.3) through (I.7) are rewritten in terms of the zero-phase-sequence-, the forward- and the backward-variables as Eqs. (6) through (10).

Appendix II

If we, in Eq. (34), replace x_1, x_2, x_3 and x_4 by ξ_1, ξ_2, ξ_3 and ξ_4, respectively and assume that they are sufficiently small to be able to neglect their higher powers, we have the variational equations corresponding to the constant solution:

$x_1=x_2=x_3=x_4=0$, that is, $x=x_0$, $y=y_0$:

$$\left.\begin{aligned} \frac{d\xi_1}{d\tau} &= -rm_3\xi_1+\xi_2-\xi_3, & \frac{d\xi_2}{d\tau} &= -\xi_1-rm_1\xi_2-\xi_4 \\ \frac{d\xi_3}{d\tau} &= km_3\xi_1+\xi_{4\,1} & \frac{d\xi_4}{d\tau} &= km_1\xi_2-\xi_3 \end{aligned}\right\} \tag{II.1}$$

In order that every solution of the characteristic equation of this system has a negative real part, it is necessary and sufficient that

$$1+m_1m_3(r^2+k^2)>k(m_1+m_3) \tag{II.2}$$

If all the roots ρ_0^2's are positive, that is, the condition (27) is satisfied, the condition (II, 2) may be written in the form

$$(\rho_0^2-\rho_{01}^2)(\rho_0^2-\rho_{02}^2)>0 \qquad \text{(II.3)}$$

where

$$\left.\begin{matrix}\rho_{01}^2\\ \rho_{02}^2\end{matrix}\right\} = \frac{2(\alpha-\tau_1)\pm\sqrt{(\alpha-\tau_1)^2-3\beta^2}}{3\tau_3} \qquad \text{(II.4)}$$

This shows that the equilibrium corresponding to the middle root of the three positive roots is asymptotically unstable. From Eq. (25), the condition (II, 3) may be written in the form

$$d(e^2)/d(\rho_0^2)>0 \qquad \text{(II.5)}$$

This shows that the eqilibrium is stable under the circumstance where the amplitude ρ_0 increases with the increasing impressed voltage e.

Appendix III

We can assume that for $\varepsilon \neq 0$ in Eq. (38), we have a relation

$$x_k = x_k^{(0)}(a, b, c, \psi)+\varepsilon x_k^{(1)}(a. b, v, \psi)+\varepsilon^2 x_k^{(2)}(a, b, c, \psi)+\cdots \qquad \text{(III.1)}$$

where $x_k^{(0)}$, $x_k^{(2)}$, $\cdots$ are certain periodic functions. As to a, b, c and ψ themselves, we will try to determine them from the equations

$$\left.\begin{aligned}\frac{da}{d\tau} &= \varepsilon A_1(a, b, c)+\varepsilon^2 A_2(a, b, c)+\cdots\\ \frac{db}{d\tau} &= \varepsilon B_1(a, b, c)+\varepsilon^2 B_2(a, b, c)+\cdots\\ \frac{dc}{d\tau} &= \varepsilon C_1(a, b, c)=\varepsilon^2 C_2(a, b, c)+\cdots\\ \frac{d\psi}{d\tau} &= \omega_1+\varepsilon D_1(a, b, c)+\varepsilon^2 D_2(a, b, c)+\cdots\end{aligned}\right\} \qquad \text{(III.2)}$$

Following the method in the Bogoliubov-Mitropolsky's text, we have

$$\left.\begin{aligned}A_1+jaD_1 &= \{\sum_k \bar{\varphi}_k{}^*\Phi_k{}^{(1)}\}/\{\sum_k \bar{\varphi}_k{}^*\varphi_k\}\\ B_1+jC_1+j\nu(b+c)D_1 &= \{\sum_k \bar{\chi}_k{}^*\Phi_k{}^{(\nu)}\}/\{\sum_k \bar{\chi}_k{}^*\chi_k\}\end{aligned}\right\} \qquad \text{(III.3)}$$

where

$$\left.\begin{aligned} \Phi_k^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} f_k(x_1^{(0)}, \cdots)\, e^{-j\psi}\, d\psi \\ \Phi_k^{(\nu)} &= \frac{1}{2\pi}\int_0^{2\pi} f_k(x_1^{(0)}, \cdots)\, e^{-j\nu\psi}\, d\psi \end{aligned}\right\} \qquad \text{(III.4)}$$

$\bar{\varphi}_k$ and $\bar{\chi}_k$ being the solutions of the adjoint system of Eq. (40). From Eq. (III, 3) A_1, B_1, C_1 and D_1 are obtained in a closed form. For $\nu=2$, they are

$$\left.\begin{aligned} \varepsilon A_1 &= -a\{A_{11}+A_{12}b-A_{13}c+A_{14}a^2+A_{15}(b^2+c^2)\} \\ \varepsilon B_1 &= -B_{11}b-B_{12}c-B_{13}a^2-B_{14}bc-B_{16}c^2-B_{16}a^2b \\ &\quad -B_{17}a^2c-B_{18}(b^2+c^2)\,b-B_{19}(b^2+c^2)\,c \\ \varepsilon C_1 &= C_{11}b-C_{12}c+C_{13}a^2+C_{14}b^2+C_{15}bc+C_{16}a^2b \\ &\quad -C_1{}^La_7{}^2c+C_{18}(b^2+c^2)\,b-C_{19}(b^2+c^2)\,c \\ \varepsilon D_1 &= -D_{11}-D_{12}b-D_{13}c-D_{14}a^2-D_{15}(b^2+c^2) \end{aligned}\right\} \qquad \text{(III.5)}$$

$A_{11}, \cdots, B_{11}, \cdots, C_{11}, \cdots, D_{11}, \cdots$ being constants.

Appendix IV

Under the assumption $\psi_0=0$ and with negligible small R', Eqs. (I, 3) through (I, 7) may be written in the following:

$$\left.\begin{aligned} Ri_\alpha+v_\alpha+\frac{d\phi_\alpha}{dt} &= \sqrt{2}\,E\cos(\omega t+\varphi) \\ Ri_\beta+v_\beta+\frac{d\phi_\beta}{dt} &= \sqrt{2}\,E\sin(\omega t+\varphi) \end{aligned}\right\} \qquad \text{(IV.1)}$$

$$i_\alpha = C\frac{dv_\alpha}{dt}, \quad i_\beta = C\frac{dv_\beta}{dt} \qquad \text{(IV.2)}$$

$$\left.\begin{aligned} i_\alpha &= 3c_1\phi_\alpha+\frac{27}{4}c_3(\phi_\alpha{}^2+\phi_\beta{}^2)\,\phi_\alpha \\ i_\beta &= 3c_1\phi_\beta+\frac{27}{4}c_3(\phi_\alpha{}^2+\phi_\beta{}^2)\,\phi_\beta \end{aligned}\right\} \qquad \text{(IV.3)}$$

$$i_0' = \frac{3}{4}\sqrt{3}\,c_3(3\phi_\alpha{}^1-\phi_\beta{}^2)\,\phi_\beta \qquad \text{(IV.4)}$$

where

$$\phi_\alpha = -\frac{1}{\sqrt{3}}\psi_\beta, \quad \phi_\beta = \frac{1}{\sqrt{3}}\psi_\alpha$$

Eqs. (IV,1), (IV,2) and (IV,3) are instrumented on the analog computer.

Appendix V

In Nov. 1953, a sub-harmonic oscillation was observed on the Suriko- Shingu transmission line when an unloaded transformer bank had been energized through series capacitors[8]. The wave forms of the oscillation are shown in Fig. V.

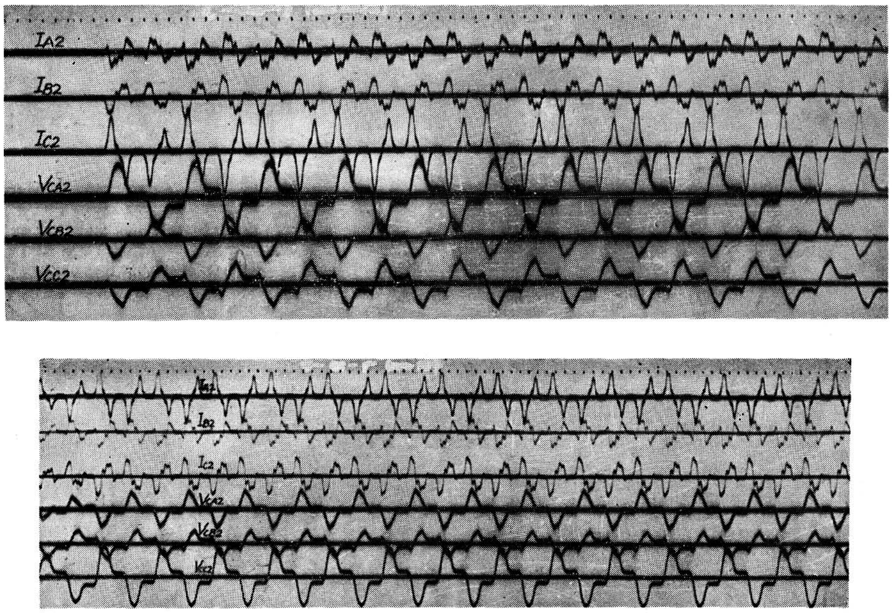

Fig. V. Waves forms of 1/3-harmonic oscillation.
I_{A2}, I_{B2}, I_{C2}=currents through the capacitors.
V_{CA2}, V_{CB2}, V_{CC2}=Terminal voltages across the capacitors.