On Adaptive Control Processes with Computing Time Delay

By

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In the theory of R. Bellman's adaptive control processes, computing time to yield optimal control signal from data obtained during normal operation is ideally assumed to be zero. There will be, however, cases where the computing time delay may not be neglected.

The subject of this paper is the optimal control policy for adaptive control processes in the case where certain computing time delay is assumed in advance. Functional equations to get such an optimal control policy are described in a fairly general form. These equations are also shown to be applicable to stochastic control processes and deterministic control processes. Two simple examples are presented to illustrate the application of the method.

1. Introduction

There is the theory of adaptive control processes developed by R. Bellman[1] as an approach to optimal control problems under various conditions of uncertainty with regard to underlying physical processes.

The basic idea of this theory may be described as follows: A controlling device is given a facility of estimation, that is, a facility of revising an incomplete information to a more exact one by the use of the data obtained during normal control operation, and taking this facility into account the optimal control policy is determined by means of the technique of dynamic programming. In the theory, computing time to yield optimal control signal from data obtained during control process is ideally assumed to be zero. But a digital computer should generally be used as a main part of the controller in order to realize such an optimal control system that is obtained from this theory. The digital computer has always more or less time delay in computation. Therefore, there will be cases where this computing time delay may not be neglected.

On computing time delay in deterministic control processes, where the object to be controlled is completely known, there is a report by one of the authors and K. Inoue[2]. In the adaptive control processes, this problem becomes more

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important, because more complex computation must be performed to get the optimal control signal.

The purpose of this paper is to discuss the optimal control policy for adaptive control processes in the case where certain computing time delay is assumed in advance.

2. Problem Statement

Let us consider a system described by the vector difference equation

$$x_{i+1} = f_i(x_i, u_i, \alpha, \xi_i) \tag{2.1}$$

where x_i is a state vector, u_i a control signal vector, α a system-parameter vector, and ξ_i a random vector. The subscript i denotes the time $t=i\Delta$, where Δ is the unit increment of time (one sampling period).

The state of the system is observed only through an observation vector y_i which is generated by the following observation equation;

$$y_i = g_i(x_i, \beta, \eta_i) \tag{2.2}$$

where β is a parameter vector in the measuring instrument and η_i is a random vector.

The random vectors ξ_i and η_i account for disturbances whose origin may be internal or external to the system, and they are assumed to be independently and identically distributed with the probability density functions given as $p(\xi_i | \theta_\xi)$ and $p(\eta_i | \theta_\eta)$ where θ_ξ and θ_η are statistical parameter vectors. Furthermore, it is assumed either that at least one of the parameters α, β, θ_ξ or θ_η is unknown, or than (2.1) and (2.2) constitute a partially observable system[3,4].

An optimal control problem of particular interest is formulated as the determination of a control policy which minimizes the expected value of the performance criterion function with N stages to go;

$$W = \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} W_i(x_i, u_{i-1}) \,. \tag{2.3}$$

For problems of this type, the theory of adaptive control processes is applicable.

3. Adaptive Control Processes

In this section, we will discuss the case where the problem posed above is treated by the theory of adaptive control processes under the idealized assumption of no computing time delay. Let us denote by $\mathcal{D}_0$ the a priori information on the initial state x_0 and unknown parameter(s). In the similar way, let $\mathcal{D}_i$ be the information on the state x_i and unknown parameter(s) at time $i\Delta$. $\mathcal{D}_0$ is, for instance, given by the form of a joint probability density function $p(x_0, \theta)$ of x_0 and unknown parameter θ. The information $\mathcal{D}_i$ is constructed from $\mathcal{D}_0$ and the data $\{u_0, u_1, \cdots u_{i-1}; y_1, y_2, \cdots y_i\}$ which are obtained by that time.

With the notations $\mathcal{D}_0$ and $\mathcal{D}_i$, according to the principle of optimality, the determination of the optimal control policy of the problem stated in the previous section is reduced to solving the following functional equations;

$$F_i^*(\mathcal{D}_{i-1}) = \min_{u_{i-1}} E\{W_i + F_{i+1}^*(\mathcal{D}_i) \mid u_{i-1}, \mathcal{D}_{i-1}\}, \qquad (i = 1, 2, \cdots, N) \qquad (3.1)_1$$

$$F_{N+1}^*(\mathcal{D}_N) = 0 \qquad (3.1)_2$$

where $E\{\cdot \mid \cdot\}$ denotes the conditional expectation, $\min_{u_{i-1}}$ represents the minimization of the bracketed term with respect to u_{i-1}, and $F_i^*(\mathcal{D}_{i-1})$ denotes the minimum value. The sequence $\{u_0^*, u_1^*, \cdots, u_{N-1}^*\}$ of u_{i-1} which satisfies (3.1) is the optimal control policy. Optimal control signal u_{i-1}^* is obtained in the form

$$u_{i-1}^* = u_{i-1}^*(\mathcal{D}_{i-1}) . \qquad (3.2)$$

4. Consideration for Computing Time Delay

As stated in Section 1, a digital computer is generally used for calculation of optimal control (3.2). In such a case, some computing time delay is unavoidable, and it causes the loss of control performance. Intending to cope with this loss of control performance, we study in this section the method for determination of the optimal control policy taking into consideration the interposition of certain computing time delay in advance.

Functional equations to get such an optimal control policy are easily obtained by regarding the problem from the following question, that is, what information or data are available to the controller for calculation of a control signal which is to be applied to the plant at each time instant. For the sake of simplicity, we assume that the computing time delay is $s\Delta$, where s is a positive integer.

In the first place, we consider the case where the computing time delay is the same as one sampling period Δ, that is, $s=1$. In this case, the computation of optimal control ${}_Du_{i-1}^*$, which should be applied to the plant at time $(i-1)\Delta$, must be begun at time $(i-2)\Delta$. Considering this situation from the point of view of the available information, information $\mathcal{D}_{i-1}$ is not available to the controller for determination of ${}_Du_{i-1}^*$ and so the possibly best way is to make use of $\mathcal{D}_{i-2}$ for that. In other words, ${}_Du_{i-1}^*$ should be determined by

$${}_DF_i^*(u_{i-2}, \mathcal{D}_{i-2}) = \min_{u_{i-1}} E\{W_i + {}_DF_{i+1}^*(u_{i-1}, \mathcal{D}_{i-1}) \mid u_{i-1}, u_{i-2}, \mathcal{D}_{i-2}\}, \qquad (i = 2, 3, \cdots, N) \qquad (4.1)_1$$

$${}_DF_{N+1}^*(u_{N-1}, \mathcal{D}_{N-1}) = 0 \qquad (4.1)_2$$

which corresponds to (3.1). From (4.1), optimal control signal ${}_Du_{i-1}^*$ is obtained in the form

$$ {}_Du_{i-1}^* = {}_Du_{i-1}^*(u_{i-2},\ \mathcal{D}_{i-2})\,, \qquad (i=2,\ 3,\cdots,\ N)\,. \tag{4. 2}$$

It is obvious from the principle of optimality that the sequence $\{{}_Du_{i-1}^*\}$ is the optimal control policy in the set U_1 of all control policies in which u_{i-1} is a function of u_{i-2} and $\mathcal{D}_{i-2}$:

$$ U_1 = [\{u_{i-1}\}\,;\ u_{i-1}=u_{i-1}(u_{i-2},\ \mathcal{D}_{i-2}),\ i=2,\ 3,\cdots,\ N]\,. \tag{4. 3}$$

Necessary data to calculate $\mathcal{D}_{i-2}$ are obtained by time $(i-2)\Delta$, and for u_{i-2} in the right hand side of (4.2) we should use ${}_Du_{i-2}^*$ which is calculated by time $(i-2)\Delta$. Therefore it is evident that margin of one unit time interval Δ for computation exists, and the previous assumption is really satisfied. In order that this control policy is realizable, the real time needed for computation of ${}_Du_{i-1}^*$ must be less than or, at worst, equal to one unit time interval Δ.

There remains no time for computation of u_0, so we must predetermine the value of u_0 (for instance, $u_0=0$).

Secondly, we consider the general case where the computing time delay is s unit time intervals ($s\geqq 2$). Through a similar consideration as in the case of one unit time delay, we have

$$ {}_DF_i^*(u_{i-2},\ u_{i-3},\ \cdots,\ u_{i-1-s},\ \mathcal{D}_{i-1-s}) = \min_{u_{i-1}} E\{W_i $$
$$ + {}_DF_{i+1}^*(u_{i-1},\ u_{i-2},\ \cdots,\ u_{i-s},\ \mathcal{D}_{i-s})\,|\,u_{i-1},\ u_{i-2},\ \cdots,\ u_{i-1-s},\ \mathcal{D}_{i-1-s}\}\,, $$
$$ (i=s+1,\ s+2,\ \cdots,\ N) \tag{4. 4$_1$}$$

$$ {}_DF_{N+1}^*(u_{N-1},\ u_{N-2},\ \cdots,\ u_{N-s},\ \mathcal{D}_{N-s}) = 0\,. \tag{4. 4$_2$}$$

Optimal control signal ${}_Du_{i-1}^*$ in this case is obtained in the form

$$ {}_Du_{i-1}^* = {}_Du_{i-1}^*(u_{i-2},\ u_{i-3},\ \cdots,\ u_{i-1-s},\ \mathcal{D}_{i-1-s}), \qquad (i=s+1,\ s+2,\ \cdots,\ N)\,. \tag{4. 5}$$

The sequence $\{{}_Du_{i-1}^*\}$ of (4.5) is the optimal control policy in the set U_{i-1} of all control policies in which u_{i-1} is a function of $u_{i-2},\ u_{i-3},\cdots,\ u_{i-1-s}$ and $\mathcal{D}_{i-1-s}$:

$$ U_s = [\{u_{i-1}\}\,;\ u_{i-1}=u_{i-1}(u_{i-2},\ u_{i-3},\ \cdots,\ u_{i-1-s},\ \mathcal{D}_{i-1-s})\,, $$
$$ i=s+1,\ s+2,\ \cdots,\ N]\,. \tag{4. 6}$$

For $u_{i-2},\ u_{i-3},\ \cdots$, and u_{i-1-s} in the right hand side of (4.5), we should substitute the values of ${}_Du_{i-2}^*$, ${}_Du_{i-3}^*,\cdots$, and ${}_Du_{i-1-s}^*$. For $s\geqq 2$, although data necessary to calculate $\mathcal{D}_{i-1-s}$ and ${}_Du_{i-1-s}^*$ are obtained by the time $(i-1-s)\Delta$, the values of ${}_Du_{i-2}^*$, ${}_Du_{i-3}^*,\cdots$, and ${}_Du_{i-s}^*$ are not available until the time $(i-2)\Delta$, $(i-3)\Delta,\cdots$, and $(i-s)\Delta$ respectively (see Fig. 1). In other words, margine of s unit time

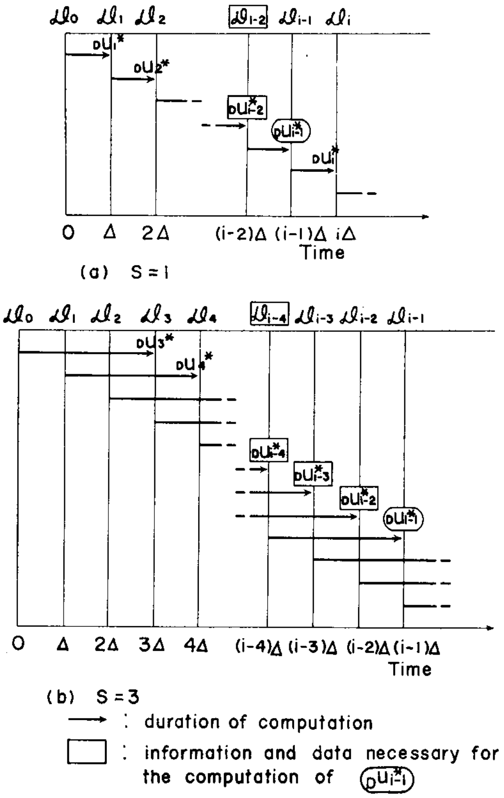

Fig. 1. Diagram showing the progress of computation of controls.

intervals which are necessary for calculation of ${}_{D}u_{i-1}^{*}$ is not given in perfect form. This is a very different point between the case $s=1$ and the case $s\geqq 2$. The following point must be also noticed. In the case $s\geqq 2$, as will be understood from Fig. 1 (b), a digital computer must compute s control signals in parallel fashion, namely, it should have an ability of multi-processing.

However, in the case $s=1$, no such consideration is necessary and the computer need not have the ability of multi-processing. Therefore, it is most desirable to design a control system so that $s=1$, either by setting sampling period Δ adequately or by using suitable computer as a controller.

We can easily specialize the idea and the formulation so as to apply them to stochastic control processes and deterministic control processes, that is, to the dynamic optimization problems of usual sampled-data control systems. See Appendix concerning the application to deterministic control processes.

5. Discussion

As stated in the previous section, it is most desirable to design a control system so that $s=1$, so we will confine ourselves to the case $s=1$.

Let us concentrate our attention on the following two cases:

Case N: The case where the optimal control policy $\{u_{i-1}^*\}$ of (3.2) is employed, which is derived by assuming the idealized condition of no computing time delay, but in reality, there exists computing time delay $\tau_1 (\tau_1 \leqq \Delta)$ for computation of u_{i-1}^*. Control signal is applied at the first sampling instant after the computation is carried out. In other words, control signal is applied one sampling period later than the time assumed theoretically.

Case D: The case where the optimal control policy $\{_D u_{i-1}^*\}$ of (4.2) is employed, which is derived by taking into consideration the interposition of the computing time delay Δ in advance. It is postulated that the real time interval τ_2 necessary for computation of control signal $_D u_{i-1}^*$ is also less than or at most equal to Δ.

From the viewpoint of control performance, which is expressed quantitatively by the expected value of the performance criterion (2.3), we can suppose that $\tau_1 = \tau_2 = \Delta$. Therefore, it is intuitively evident that the control performance in Case D is better than or at worst equal to that of Case N. But as will be understood from the comparison of (4.2) and (3.2), control policy $\{_D u_{i-1}^*\}$ is generally more complicated than $\{u_{i-1}^*\}$ and so computing time necessary for the former is usually longer than for the latter, that is, $\tau_1 \leqq \tau_2$. If $\{_D u_{i-1}^*\}$ is more complicated than certain degree compared with $\{u_{i-1}^*\}$, we should either make the sampling interval Δ longer, or use another type of computer whose computing speed is larger. In these cases, it is necessary to consider the performance losses caused by these design changes. That is to say, for each individual case, we must investigate which control policy is to be employed, $\{u_{i-1}^*\}$ or $\{_D u_{i-1}^*\}$, taking these performance losses into consideration.

Now let us compare Case N with Case D from the side of structure of the control policy in some detail. In Case N, control signal u_{i-2}^*, which is calculated from (3.2), is in reality applied to the system at time $(i-1)\Delta$ because of computing time delay Δ. Denoting this control signal by $_N u_{i-1}$, we have

$$_N u_{i-1} = u_{i-2}^*(\mathcal{D}_{i-2}) \equiv {_N u_{i-1}}(\mathcal{D}_{i-2}) \,, \qquad (i = 2, 3, \cdots, N) \,. \tag{5.1}$$

From (5.1), we see that $\{_N u_{i-1}\}$ has the same form as the control ploicy in which u_{i-1} is a function only of $\mathcal{D}_{i-2}$ and not of u_{i-2}. So $\{_N u_{i-1}\}$ can be regarded as a special control policy contained in U_1. Since $\{_D u_{i-1}^*\}$ is the optimal in U_1, the control performance of $\{_N u_{i-1}\}$ is never better than that of $\{_D u_{i-1}^*\}$. Moreover, except for the case where ${}_Du_{i-1}^*$ is a function only of $\mathscr{D}_{i-2}$ and not of u_{i-2}, it can not happen that

$$ {}_Du_{i-1}^* = {}_Nu_{i-1}\,, \qquad (i=2,3,\cdots,N)\,, \tag{5.2}$$

in other words, the control performance of $\{{}_Du_{i-1}^*\}$ is always better than that of $\{{}_Nu_{i-1}\}$.

6. Examples

To illustrate the development of the preceding sections, two examples will be considered here. All variables in this section are assumed to be scalar quantities.

Example 1

The system to be considered here is described by the difference equation

$$x_{i+1} = x_i + u_i + \alpha + \xi_i \tag{6.1}$$

$$y_i = x_i \tag{6.2}$$

and the criterion function is

$$W = \sum_{i=1}^{N} (a_i x_i^2 + b_i u_{i-1}^2) \tag{6.3}$$

where $\{\xi_i\}$ is an independent Gaussian sequence, with mean zero and variance v_ξ. α is an unknown constant with the a priori density function

$$p(\alpha) = \frac{1}{\sqrt{2\pi v_0}} \exp\left\{-\frac{(\alpha-m_0)^2}{2v_0}\right\} \equiv N_\alpha(m_0, v_0)\,. \tag{6.4}$$

Let the initial state x_0 be known.

We make some preparations. Firstly, the revising procedure of the a priori density function of α by Bayesian formula is investigated. The a posteriori density function, given u_0 (at time $t=0$) and $y_1=x_1$ (at time $t=\Delta$), is

$$p(\alpha\,|\,x_1, u_0) = \frac{p(x_1\,|\,\alpha, u_0)p_0(\alpha)}{p(x_1\,|\,u_0)} \equiv p_1(\alpha)\,. \tag{6.5}$$

Using (6.1), (6.3) and (6.4), (6.5) is reduced to

$$\begin{aligned} p_1(\alpha) &= N_\alpha\left(\frac{v_0(x_1-x_0-u_0)+v_\xi m_0}{v_0+v_\xi}\,,\ \frac{v_0 v_\xi}{v_0+v_\xi}\right) \\ &\equiv N_\alpha(m_1, v_1)\,. \end{aligned} \tag{6.6}$$

This $p_1(\alpha)$ is interpreted as the revised a priori density for α obtained by the estimation operation at time $t=\Delta$. In the same way, the revised a priori density $p_i(\alpha)$ at any time instant $t=i\Delta$ turns out to be also Gaussian. Denoting the mean and variance of $p_i(\alpha)$ as m_i and v_i, the following equations are obtained.

$$m_i = \frac{v_{i-1}(x_i - x_{i-1} + u_{i-1}) + v_\xi m_{i-1}}{v_{i-1} + v_\xi} \tag{6.7}$$

$$v_i = \frac{v_{i-1} v_\xi}{v_{i-1} + v_\xi} . \tag{6.8}$$

Secondly, contents of $\mathscr{D}_i$ for the system under investigation are examined. Information $\mathscr{D}_0$ at time $t=0$ is composed of (6.4) and the value of x_0, and $\mathscr{D}_i$ of $p_i(\alpha)$ and the observed value x_i. Since $p_i(\alpha)$ has sufficient statistics m_i and v_i, $\mathscr{D}_i$ can be represented by three values x_i, m_i and v_i.

Now we can proceed to the main step; solving the functional equations.

Firstly, we consider the ideal case where no computing time delay exists. From (3.1)

$$F_i^*(x_{i-1}, m_{i-1}, v_{i-1}) = \min_{u_{i-1}} E\{a_i x_i^2 + b_i u_{i-1}^2 + F_{i+1}^*(x_i, m_i, v_i) \, | \, u_{i-1}, x_{i-1}, m_{i-1}, v_{i-1}\}, \qquad (i=1, 2, \cdots, N) \tag{6.9$_1$}$$

$$F_{N+1}^*(x_N, m_N, v_N) = 0 . \tag{6.9$_2$}$$

Optimal control policy is obtained by solving the above equation backwardly, beginning with $i=N$. For $i=N$, we get

$$\begin{aligned} &F_N^*(x_{N-1}, m_{N-1}, v_{N-1}) \\ &= \min_{u_{N-1}} E\{a_N x_N^2 + b_N u_{N-1}^2 \, | \, u_{N-1}, x_{N-1}, m_{N-1}, v_{N-1}\} \\ &= \min_{u_{N-1}} E\{a_N(x_{N-1} + u_{N-1} + \alpha + \xi_{N-1})^2 + b_N u_{N-1}^2 \, | \, u_{N-1}, x_{N-1}, m_{N-1}, v_{N-1}\} \\ &= \min_{u_{N-1}} \{a_N(x_{N-1}^2 + m_{N-1}^2 + v_{N-1} + v_\xi + 2x_{N-1} m_{N-1}) \\ &\quad + 2a_N(x_{N-1} + m_{N-1}) u_{N-1} + (a_N + b_N) u_{N-1}^2\} . \end{aligned} \tag{6.10}$$

Thus u_{N-1}^* and F_N^* are

$$u_{N-1}^* = -\frac{a_N}{a_N + b_N}(x_{N-1} + m_{N-1}) \tag{6.11$_1$}$$

$$F_N^* = \frac{b_N(a_N + b_N)}{a_N}(u_{N-1}^*)^2 + a_N(v_{N-1} + v_\xi) . \tag{6.11$_2$}$$

In an analogous fashion, we get

$$u_{i-1}^* = -(c_{1,i-1} x_{i-1} + c_{2,i-1} m_{i-1}) \tag{6.12}$$

$$F_i^* = c_{3,i-1}(u_{i-1}^*)^2 + c_{4,i-1} m_{i-1}^2 + \sum_{j=i-1}^{N-1} (c_{5,j} v_j + c_{6,j} v_{j-1} + c_{7,j} v_\xi) , \qquad (i = 2, 3, \cdots, N) \tag{6.13}$$

where coefficients $c_{1,i} \sim c_{7,i}$ are calculated from the following recursive relations.

$$c_{1,i-1} = \frac{a_i + c_{1,i}^2 c_{3,i}}{a_i + b_i + c_{1,i}^2 c_{3,i}} \tag{6.14$_1$}$$

$$c_{2,i-1} = \frac{a_i + c_{1,i}^2 c_{3,i} + c_{1,i} c_{2,i} c_{3,i}}{a_i + b_i + c_{1,i}^2 c_{3,i}} \tag{6.14$_2$}$$

$$c_{3,i-1} = \frac{b_i(a_i + b_i + c_{1,i}^2 c_{3,i})}{a_i + c_{1,i}^2 c_{3,i}} \tag{6.14$_3$}$$

$$c_{4,i-1} = c_{4,i} + \frac{a_i c_{2,i}^2 c_{3,i}}{a_i + c_{1,i}^2 c_{3,i}} \tag{6.15$_1$}$$

$$c_{5,i-1} = a_i + (c_{1,i} + c_{2,i})^2 c_{3,i} + c_{4,i} \tag{6.15$_2$}$$

$$c_{6,i-1} = -(c_{2,i}^2 c_{3,i} + c_{4,i}) \tag{6.15$_3$}$$

$$c_{7,i-1} = a_i + c_{1,i}^2 c_{3,i} \tag{6.15$_4$}$$

and the terminal condition

$$c_{1,N} = c_{2,N} = c_{3,N} = c_{4,N} = 0\,. \tag{6.16}$$

The optimal control u_{i-1}^* is calculated from (6.7), (6.8), (6.12), (6.14), and (6.16).

Secondly, we will obtain optimal control policy for the case where one unit time delay for computation is taken into consideration. From (4.1),

$$\begin{aligned} {}_DF_i^*(u_{i-2}, x_{i-2}, m_{i-2}, v_{i-2}) &= \min_{u_{i-1}} E\{a_i x_i^2 + b_i u_{i-1}^2 \\ &\quad + {}_DF_{i+1}^*(u_{i-1}, x_{i-1}, m_{i-1}, v_{i-1}) \,|\, u_{i-1}, u_{i-2}, x_{i-2}, m_{i-2}, v_{i-2}\}\,, \\ &\qquad (i = 2, 3, \cdots, N) \end{aligned} \tag{6.17$_1$}$$

$${}_DF_{N+1}^*(u_{N-1}, x_{N-1}, m_{N-1}, v_{N-1}) = 0\,. \tag{6.17$_2$}$$

For $i=N$

$$\begin{aligned} &{}_DF_N^*(u_{N-2}, x_{N-2}, m_{N-2}, v_{N-2}) \\ &= \min_{u_{N-1}} E\{a_N x_N^2 + b_N u_{N-1}^2 \,|\, u_{N-1}, u_{N-2}, x_{N-2}, m_{N-2}, v_{N-2}\} \\ &= \min_{u_{N-1}} E\{a_N(x_{N-2} + u_{N-1} + u_{N-2} + 2\alpha + \xi_{N-1} + \xi_{N-2})^2 \\ &\quad + b_N u_{N-1}^2 \,|\, u_{N-1}, u_{N-2}, x_{N-2}, m_{N-2}, v_{N-2}\} \\ &= \min_{u_{N-1}} \{a_N(x_{N-2}^2 + u_{N-2}^2 + 4m_{N-2}^2 + 4v_{N-2} + 2v_\xi \\ &\quad + 2x_{N-2}u_{N-2} + 4m_{N-2}u_{N-2} + 4x_{N-2}m_{N-2}) \\ &\quad + 2a_N(x_{N-2} + u_{N-2} + 2m_{N-2})u_{N-1} + (a_N + b_N)v_{N-1}^2\} \end{aligned} \tag{6.18}$$

and so ${}_Du_{N-1}^*$ and ${}_DF_N^*$ are

$${}_Du_{N-1}^* = -\frac{a_N}{a_N + b_N}(x_{N-2} + u_{N-2} + 2m_{N-2}) \tag{6.19$_1$}$$

$${}_DF_N^* = \frac{b_N(a_N + b_N)}{a_N}({}_Du_{N-1}^*)^2 + 2a_N(2v_{N-2} + v_\xi)\,. \tag{6.19$_2$}$$

In an analogous fashion, we get

$$ {}_{D}u_{i-1}^{*} = -\{c_{1,i-1}(x_{i-2}+u_{i-2}+m_{i-2})+c_{2,i-1}m_{i-2}\} \tag{6.20} $$

$$ {}_{D}F_{i}^{*} = c_{3,i-1}({}_{D}u_{i-1}^{*})^{2}+c_{4,i-1}m_{i-2}^{2}+\sum_{j=i-1}^{N-1}(c_{8,j}v_{j}+c_{9,j}v_{j+1}+c_{10,j}v_{\xi}) , \quad (i=2, 3, \cdots, N) \tag{6.21} $$

where coefficients $c_{8,i}$, $c_{9,i}$ and $c_{10,i}$ are determined from

$$ c_{8,i} = 4a_{i}+(2c_{1,i}+c_{2,i})^{2}c_{3,i}+c_{4,i} \tag{6.22$_1$} $$

$$ c_{9,i} = -(c_{1,i}+c_{2,i})^{2}c_{3,i}-c_{4,i} \tag{6.22$_2$} $$

$$ c_{10,i} = 2a_{i}+c_{1,i}^{2}c_{3,i} \, . \tag{6.22$_3$} $$

Comparing ${}_{D}u_{i-1}^{*}$ of (6.20) with u_{i-1}^{*} of (6.12), it can be shown that ${}_{D}u_{i-1}^{*}$ is obtained from ${}_{D}u_{i-1}^{*}$ by substituting $(x_{i-2}+u_{i-2}+m_{i-2})$ into x_{i-1}, and m_{i-2} into m_{i-1}. This may be interpreted as follows; when computing time delay exists, x_{i-1} and m_{i-1} are not available for computation of u_{i-1}. Therefore, the control policy is obtained which uses the estimated values $(x_{i-2}+u_{i-2}+m_{i-2})$ and m_{i-2} instead of x_{i-1} and m_{i-1} respectively. For Ceas N (see Section 5), from (6.12), control ${}_{N}u_{i-1}$ which is applied to the controlled system at time $(i-1)\Delta$, is

$$ {}_{N}u_{i-1} = -(c_{1,i-2}x_{i-2}+c_{2,i-2}m_{i-2}) \, . \tag{6.23} $$

The expected value of the criterion function for Case N can also be calculated easily.

Fig. 2 shows the expected values of the performance criterion for the following three cases:

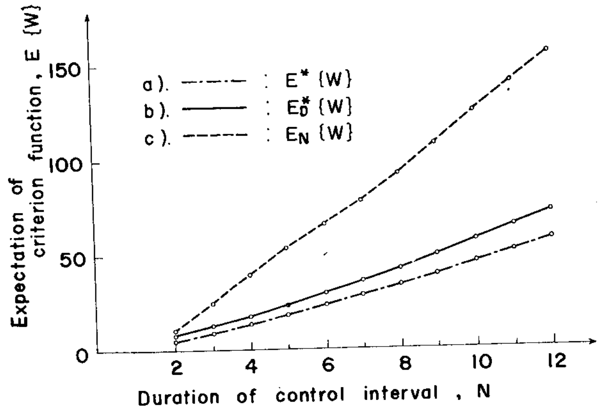

Fig. 2. Evaluation of control performance of the system in Example 1.

a) The idealized case where no computing time delay exists. Control is performed by the policy {(6.12)}; $E^*\{W\}$.
b) The case where the policy {(6.20)} is employed, taking into account the interposition of the computing time delay of one sampling period. (Case D); $E\{_D{}^*W\}$.
c) The case where control is performed by the policy {(6.12)} but, in reality, computing time delay of one sampling period exists. (Case N); $E_N\{W\}$.

Constants are chosen as

$$a_i = b_i = 1\,, \qquad (i = 1, 2, \cdots, N)$$
$$x_0 = 1\,, \quad m_0 = 0\,, \quad v_0 = 1, \quad v_\xi = 1 \tag{6.24}$$

and for the cases b) and c) we take u_0 as

$$_Du_0{}^* = {}_Nu_0 = 0\,.$$

It can easily be understood that a fairly large improvement of the control performance is acquired by the use of the optimal control policy which takes the interposition of the computing time delay into account.

In Fig. 3 is shown the degree of improvement of the control performance which is defined by

$$\frac{E_N\{W\} - E_D{}^*\{W\}}{E_N\{W\}} \times 100\ (\%).$$

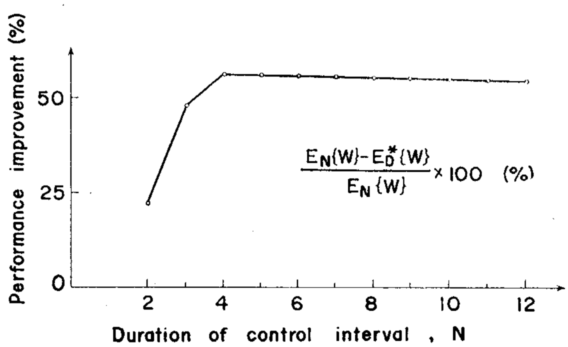

Fig. 3. Percent ratio control performance improvement in Example 1.

Example 2

The second example is a very special one which satisfies (5.2) in the previous section.

Consider a system similar to that of Example 1.

$$x_{i+1} = u_i + \alpha + \xi_i \tag{6.25}$$

$$y_i = x_i$$

$$W = \sum_{i=1}^{N} (x_d - x_i)^2 \tag{6.26}$$

where x_d is a given constant (desired signal) and ξ_i is the same as in Example 1. The parameter α is again an unknown constant with the a priori probability density of (6.4). Corresponding to (6.7) and (6.8), we get

$$m_i = \frac{v_{i-1}(x_i - u_{i-1}) + v_\xi m_{i-1}}{v_{i-1} + v_\xi} \tag{6.27}$$

$$v_i = \frac{v_{i-1} v_\xi}{v_{i-1} + v_\xi} \tag{6.28}$$

and so $\mathcal{D}_i$ is represented by three values x_i, m_i, and v_i.

Functional equations (6.9) and (6.17) are, therefore, available also to this problem.

Firstly, for the case of no computing time delay, we get from (6.9)

$$u_{i-1}^* = x_d - m_{i-1}\,. \tag{6.29}$$

Secondly, for the case of one computing time delay, from (6.17)

$${}_Du_{i-1}^* = x_d - m_{i-2}\,. \tag{6.30}$$

By the use of (6.27) and (6.28), the above equation can be written as

$${}_Du_{i-1}^* = x_d - \frac{\sum_{j=1}^{i-2} (x_j - u_{j-1}) + m_0\left(\frac{v_\xi}{v_0}\right)}{i-2+\left(\frac{v_\xi}{v_0}\right)}\,, \qquad (i = 2, 3, \cdots, N) \qquad (6.31)_2$$

$${}_Du_1^* = x_d - m_0\,. \qquad (6.31)_2$$

As can be seen from (6.31), ${}_Du_{i-1}^*$ has in this example a form which does not contain u_{i-2} as a variable. So we can predict that $\{{}_Nu_{i-1}\}$ may coincide with $\{{}_Du_{i-1}^*\}$, that is, there is a possibility for Case D and Case N to give the same control performance. Indeed from (6.29)

$${}_Nu_{i-1} = u_{i-2}^* = x_d - m_{i-2} \tag{6.32}$$

and so

$${}_Du_{i-1}^* = {}_Nu_{i-1}\,, \qquad (i = 2, 3, \cdots, N)\,. \tag{6.33}$$

If we take ${}_Du_0^* = {}_Nu_0$, Case D and Case N really show the same control performance. Here, we must note that, when x_d takes a different value for each stage, (6.33) is no longer satisfied.

7. Conclusions

In this paper we have discussed the problem of obtaining the optimal control policy for adaptive control processes in the case where certain computing time delay is assumed in advance. Functional equations to get such optimal control policy have been derived in a fairly general form by regarding the problem from the viewpoint of available information or data for the calculation of each control signal.

It has been shown that these equations can also be applied to deterministic control processes as well as stochastic ones.

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Appendix

In this appendix, we will consider deterministic control processes with computing time delay. For deterministic control processes whose characteristics are known perfectly, (4.4) can be simplified as follows. It is sufficient to consider only the value of x_i as the information $\mathscr{D}_i$, and the operation of statistical expectation $E\{\cdot\,|\,\cdot\}$ can be taken off. Therefore,

$$\begin{aligned} {}_DF_i^*(u_{i-2}, u_{i-3}, \cdots, u_{i-1-s}, x_{i-1-s}) &= \min_{u_{i-1}} \{W_i^s + {}_DF_{i+1}^*(u_{i-1}, u_{i-2}, \cdots, u_{i-s}, x_{i-s})\}\,, \\ &\qquad (i = s+1, s+2, \cdots, N) \qquad (\text{A. }1)_1 \\ {}_DF_{N+1}^*(u_{N-1}, u_{N-2}, \cdots, u_{N-s}, x_{N-s}) &= 0 \qquad (\text{A. }1)_2 \end{aligned}$$

where W_i^s denotes W_i expressed by variables $u_{i-1}, u_{i-2}, \cdots, u_{i-1-s}$, and x_{i-1-s}. From (A.1) we get

$${}_Du_{i-1}^* = {}_Du_{i-1}^*(u_{i-2}, u_{i-3}, \cdots, u_{i-1-s}, x_{i-1-s})\,. \qquad (\text{A. }2)$$

As an example, let us consider a system described by the linear difference equation

$$x_{i+1} = \Phi_i x_i + G_i u_{i-1} \tag{A. 3)$_1$ }$$

$$y_i = x_i\ , \qquad \text{(A. 3)}_2$$

under a quadratic criterion function

$$W = \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} (x_i^T Q_i x_i + u_{i-1}^T R_i u_{i-1}) \qquad \text{(A. 4)}$$

where x_i and u_i are n- and r-dimensional column vectors, Φ_i and G_i are $n \times n$ and $n \times r$ matrixes respectively, and Q_i and R_i are $n \times n$ and $r \times r$ positive semi-definite symmetrical matrixes respectively. Especially R_N is assumed positive definite. The subscript ^{T} denotes transpose of vector or matrix.

This problem has been solved in reference [2] using a little different method. It will be shown that the solution by the method developed in this paper agrees with that of [2].

Let us express x_i by variables $u_{i-1}, u_{i-2}, \cdots, u_{i-s}$ and x_{i-s};

$$\begin{aligned} x_i &= \Phi_{i-1} x_{i-1} + G_{i-1} u_{i-1} \\ &= \Phi_{i-1}(\Phi_{i-2} x_{i-2} + G_{i-2} u_{i-2}) + G_{i-1} u_{i-1} \\ &= \{\prod_{k=i-s}^{i-1} \Phi_k\} x_{i-s} + \sum_{k=i-s}^{i-1} \{\prod_{j=k+1}^{i-1} \Phi_j\} G_k u_k\ . \end{aligned} \qquad \text{(A. 5)}$$

Denoting x_i given in (A.5) by $X(i, s)$;

$$X(i, s) = \{\prod_{k=i-s}^{i-1} \Phi_k\} x_{i-s} + \sum_{k=i-s}^{i-1} \{\prod_{j=k+1}^{i-1} \Phi_j\} G_k u_k \qquad \text{(A. 6)}$$

we get

$$x_i = \Phi_{i-1} X(i-1, s) + G_{i-1} u_{i-1} \qquad \text{(A. 7)}$$

and W_i is expressed by $u_{i-1}, u_{i-2}, \cdots, u_{i-1-s}$ and x_{i-1-s},

$$\begin{aligned} W_i &= x_i^T Q_i x_i + u_{i-1}^T R_i u_{i-1} \\ &= \{\Phi_{i-1} X(i-1, s) + G_{i-1} u_{i-1}\}^T Q_i \{\Phi_{i-1} X(i-1, s) + G_{i-1} u_{i-1}\} \\ &\qquad + u_{i-1}^T R_i u_{i-1}\ . \end{aligned} \qquad \text{(A. 8)}$$

Now let us solve (A.1). Assume that ${}_DF_i^*$ and ${}_Du_{i-1}^*$ have the forms

$${}_DF_i^* = X^T(i-1, s) P_{i-1} X(i-1, s) \qquad \text{(A. 9)}$$

$${}_Du_{i-1}^* = D_{i-1} X(i-1, s) \qquad \text{(A. 10)}$$

where P_{i-1} and D_{i-1} are $n \times n$ and $r \times r$ matrixes respectively. Using (A.8) and (A.9), (A.1) can be rewritten as

$$\begin{aligned} {}_DF_i^* = \min_{u_{i-1}} \{ & X^T(i-1, s) L_{1,i-1} X(i-1, s) \\ & + 2u_{i-1}^T L_{2,i-1} X(i-1, s) + u_{i-1}^T (L_{3,i-1} + R_i) u_{i-1}\}\ , \end{aligned} \qquad \text{(A. 11)}$$

where

$$L_{1,i-1} = \Phi_{i-1}^T(P_i+Q_i)\Phi_{i-1} \tag{A. 12)$_1$}$$

$$L_{2,i-1} = G_{i-1}^T(P_i+Q_i)\Phi_{i-1} \quad \text{(A. 12)}_2$$

$$L_{3,i-1} = G_{i-1}^T(P_i+Q_i)G_{i-1}\,. \quad \text{(A. 12)}_3$$

From (A.11),

$$_Du_{i-1}^* = -(L_{3,i-1}+R_i)^{-1}L_{2,i-1}X(i-1,\, s) \quad \text{(A. 13)}_1$$

$$_DF_i^* = X^T(i-1,\, s)(L_{1,i-1}+L_{2,i-1}^T D_{i-1})X(i-1,\, s)\,. \quad \text{(A. 13)}_2$$

Then D_{i-1} and P_{i-1} must have the form

$$D_{i-1} = -(L_{3,i-1}+R_i)^{-1}L_{2,i-1} \quad \text{(A. 14)}_1$$

$$P_{1-i} = L_{1,i-1}+L_{2,i-1}^T D_{i-1} \quad \text{(A. 14)}_2$$

in order to satisfy (A.9), (A.10), and (A.13).

Summing up the result, optimal control is

$$_Du_{i-1}^* = D_{i-1}X(i-1,\, s)\,, \qquad (i = s+1,\, s+2,\, \cdots,\, N) \quad \text{(A. 15)}$$

where

$$X(i-1,\, s) = \{\prod_{k=i-1-s}^{i-2} \Phi_k\}x_{i-1-s}+\sum_{k=i-1-s}^{i-2}\{\prod_{j=k+1}^{i-2}\Phi_j\}G_k u_k \quad \text{(A. 16)}$$

and the feedback matrix D_{i-1} is calculated recursively by

$$L_{1,i-1} = \Phi_{i-1}^T(P_i+Q_i)\Phi_{i-1} \quad \text{(A. 17)}_1$$

$$L_{2,i-1} = G_{i-1}^T(P_i+Q_i)\Phi_{i-1} \quad \text{(A. 17)}_2$$

$$L_{3,i-1} = G_{i-1}^T(P_i+Q_i)G_{i-1} \quad \text{(A. 17)}_3$$

$$D_{i-1} = -(L_{3,i-1}+R_i)^{-1}L_{2,i-1} \quad \text{(A. 17)}_4$$

$$P_{i-1} = L_{1,i-1}+L_{2,i-1}^T D_{i-1} \quad \text{(A. 17)}_5$$

under the terminal condition

$$P_N = 0\,. \quad \text{(A. 18)}$$

Especially for $s=1$, that is, for the case where the computing time delay is one sampling period,

$$_Du_{i-1}^* = D_{i-1}\{\Phi_{i-2}x_{i-2}+G_{i-2}u_{i-2}\}\,. \quad \text{(A. 19)}$$

This result, (A.15)~(A.19), agrees with that of reference [2].