# Flow of Rarefied Gas past a Circular Cylinder at Low Mach Numbers

By

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The two-dimensional flow of a slightly rarefied gas past a circular cylinder at low Mach numbers is considered basing on the B-G-K model of the Boltzmann equation. It is shown that the asymptotic (continuum) field of flow is governed essentially by the Stokes equation in hydrodynamics. The effect of rarefaction makes its appearance in a thin layer, the Knudsen layer, around the cylinder. A kinetic theory analysis of this layer similar to the boundary layer theory in hydrodynamics is put forward to the second approximation. Numerical discussions are made of the structure of the Knudsen layer and of the coefficients of slip and temperature jump. A formula for the drag coefficient of the cylinder is derived and its numerical result is compared favourably with a result of recent experiment.

## 1. Introduction

In recent years, there have been published many studies on rarefied gas flows which are based on the so-called B-G-K model[1] of the Botlzmann equation in the kinetic theory. In particular, if the flow speed is much smaller than the mean thermal speed of the gas molecule, the B-G-K equation can be linearized and thus it becomes possible to treat simple flows with mathematical rigor. Previous studies in this direction deal mainly with one dimensional flows[2-7]. Recently, one of the authors (K.T.) made a study of the stagnation-point flow in two-dimensions[8]. Of particular interest and importance may be the case of flow past a body. In the present study is considered the two-dimensional flow of a slightly rarefied gas past a circular cylinder at low values of the Mach number. There appears in such a flow a thin layer called Knudsen layer around the cylinder where the distribution of the molecular velocity is non-equilibrium. The flow outside this layer may be regarded as the flow of continuum owing to near-equilibrium distribution of the molecular velocity. It is shown that the governing equation of the continuum field remains in essence to be the Stokes equation for

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slow motion of an ordinary viscous fluid. The concept of the Knudsen layer is, in some respects, similar to that of the boundary layer in fluid-dynamics. The former may well be simpler in that there occurs no separation of flow from the body surface. In fact, a kinetic theory analysis similar to the boundary layer theory can be formulated without essential difficulty to the second and higher order approximations. The solution for the flow in the Knudsen layer at each stage of approximation calls for respective correction (e.g. flow slip at the solid boundary) to the solution for the outer (continuum) flow and in practice the inner and outer solutions must be treated alternately on the course to higher approximation. This situation is also similar to the case of the boundary layer. The solution will take on a form of series expansion in the Knudsen number (ratio of the mean free path to the radius of the cylinder). In the present study, the analysis has been actually put forward to the second approximation. The fields of flow velocity, density, temperature together with the coefficients of slip and temperature jump are calculated. A formula for the drag coefficient of the cylinder is also derived and its numerical result is favourably compared with a result of recent experiment. The drag coefficient of a circular cylinder in the limit of free molecular flow is well-known. Quite recently, the present authors have obtained the first order correction to the free molecular flow. Our drag formula seems to be consistent with these results too.

## 2. Fundamental Equations

We consider the flow of a slightly rarefied gas past a circular cylinder with its axis perpendicular to the stream. Let the speed of the undisturbed flow be $Q_\infty$ and the radius of the cylinder $a$. We take rectangular coordinates $(X, Y, Z)$ with $X$-axis parallel to the undisturbed flow and $Z$-axis coinciding with the axis

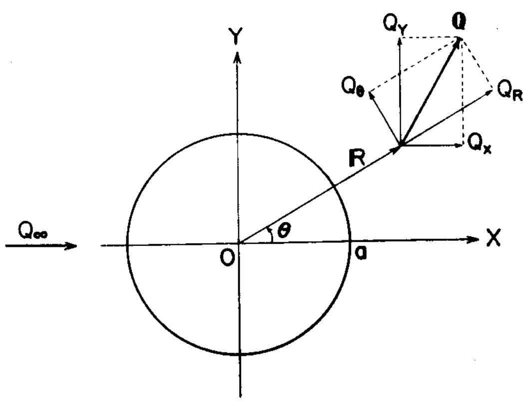


Fig. 1. Configuration and coordinates.

of the cylinder. We shall also refer to the cylindrical coordinates $(R, \theta, Z)$ as shown in figure 1.

It is assumed in the present study that the gas obeys the B-G-K model of the Boltzmann equation. Let $\boldsymbol{V}=(V_X, V_Y, V_Z)$ be the velocity of a molecule, $F(\boldsymbol{V}, X, Y)$ the distribution function of the velocity and $N$ the number density of molecules. Since the mean fields are steady and two-dimensional in the present problem, the B-G-K equation may be written as

$$V_X \frac{\partial F}{\partial X} + V_Y \frac{\partial F}{\partial Y} = AN(F_e - F)\,, \tag{1}$$

where $A$ is a constant, $AN$ the collision frequency. The function $F_e$ is the local equilibrium distribution given by

$$F_e = N\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m}{2kT}(\boldsymbol{V}-\boldsymbol{Q})^2\right\}, \tag{1a}$$

where $\boldsymbol{Q}$ is the flow velocity, $T$ the absolute temperature, $m$ the mass of a molecule and $k$ the Boltzmann constant. The mean values $N$, $\boldsymbol{Q}$, $T$ and the pressure $P$ are defined as

$$\left.\begin{aligned} N &= \int F d\boldsymbol{V}, \quad N\boldsymbol{Q} = \int \boldsymbol{V} F d\boldsymbol{V}, \\ 3(k/m)NT &= \int (\boldsymbol{V}-\boldsymbol{Q})^2 F d\boldsymbol{V}, \quad P = kNT, \end{aligned}\right\} \tag{2}$$

where $d\boldsymbol{V}=dV_X\, dV_Y\, dV_Z$ and integrations are extended to the whole $\boldsymbol{V}$-space.

We assume throughout this study that the undisturbed flow speed $Q_\infty$ is small compared with the mean thermal speed of the gas molecule (small Mach number), that the mean free path $l$ of the molecule is small in comparison with the radius $a$ of the cylinder (small Knudsen number), and that the Mach number is small compared with the Knudsen number (small Reynolds number). It is further assumed that the temperature of the gas in the undisturbed flow is equal to the temperature $T_0$ of the cylinder. Then, the distribution of the molecular velocity at any point will be close to that of equilibrium with zero mean velocity, number density $N_\infty$ of the undisturbed flow and temperature $T_0$. Deviations from that equilibrium are seen to be of the order $S$

$$S = Q_\infty / C_m\,, \quad C_m = [2(k/m)T_0]^{1/2}\,, \tag{3}$$

where $C_m$ is the most probable speed of the molecule†. Thus, we write

† The speed ratio $S$ is related to the Mach number $M$ of the undisturbed flow as $S=(\kappa/2)^{1/2}M$, $\kappa$ being the adiabatic exponent, since the speed of sound is $[\kappa(k/m)T_0]^{1/2}=(\kappa/2)^{1/2}C_m$.

$$\left.\begin{aligned} &F = N_\infty(\pi C_m^2)^{-3/2}e^{-(V/C_m)^2}(1+S\Phi)\ , \\ &N = N_\infty(1+S\Sigma)\ , \quad T = T_0(1+S\Theta)\ , \\ &P = P_\infty(1+S\Xi)\ , \quad P_\infty = kN_\infty T_0\ . \end{aligned}\right\} \tag{4}$$

Inserting (4) in (1), (2) and neglecting the terms of $O(S^2)$†, we may obtain a linearized system of the fundamental equations as follows:

$$K\left(u\frac{\partial\Phi}{\partial x}+v\frac{\partial\Phi}{\partial y}\right) = -\Phi+\Sigma-\frac{3}{2}\Theta+2\hat{\boldsymbol{Q}}\cdot\boldsymbol{v}+\Theta v^2\ , \tag{5}$$

$$\left.\begin{aligned} &\Sigma = \int E\Phi d\boldsymbol{v}\ , \quad \hat{\boldsymbol{Q}} = \int \boldsymbol{v}E\Phi d\boldsymbol{v}\ , \\ &\frac{3}{2}(\Sigma+\Theta) = \int v^2 E\Phi d\boldsymbol{v}\ , \quad \Xi = \Sigma+\Theta\ , \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} &(x, y) = (X, Y)/a\ , \\ &\boldsymbol{v} = (u, v, w) = \boldsymbol{V}/C_m\ , \quad \hat{\boldsymbol{Q}} = (\hat{Q}_x, \hat{Q}_y) = \boldsymbol{Q}/Q_\infty\ , \\ &E = \pi^{-3/2}e^{-v^2}\ , \quad K = C_m/(AN_\infty a) = (\sqrt{\pi}/2)(l/a)\ , \end{aligned}\right\} \tag{7}$$

$l$ being the mean free path.

Multiplying both sides of (1) by $m$, $m\boldsymbol{V}$, $\frac{1}{2}m\boldsymbol{V}^2$, and integrating throughout the $\boldsymbol{V}$-space, we obtain the transport equations of the mass, momentum and energy, which may be written in terms of $\Phi$ as

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\int uE\Phi d\boldsymbol{v}+\frac{\partial}{\partial y}\int vE\Phi d\boldsymbol{v} = 0\ , \\ &\frac{\partial}{\partial x}\int \begin{pmatrix} u^2 \\ uv \end{pmatrix} E\Phi d\boldsymbol{v}+\frac{\partial}{\partial y}\int \begin{pmatrix} uv \\ v^2 \end{pmatrix} E\Phi d\boldsymbol{v} = 0\ , \\ &\frac{\partial}{\partial x}\int uv^2 E\Phi d\boldsymbol{v}+\frac{\partial}{\partial y}\int vv^2 E\Phi d\boldsymbol{v} = 0\ . \end{aligned}\right\} \tag{8}$$

## 3. Asymptotic Theory

We are considering the case $K \ll 1$ in the present study, in which the left-hand side of (5) is small in comparison with the right-hand side except in a thin layer around the cylinder, the Knudsen layer, where the gradient of $\Phi$ may become large. Thus, we may assume outside the Knudsen layer the next expansion:

$$\Phi \sim \Phi_m = \sum_{i=0}^{m} K^i \Psi_i\ , \qquad m = 0, 1, 2 \cdots . \tag{9}$$

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† The perturbation $S\Phi$ may become large for large $|\boldsymbol{V}|$, when however the exponential factor in $F$ and $F_e$ tends strongly to zero.

Inserting this in (5) and equating the same order terms in $K$ on both sides, we have the result†

$$\left.\begin{aligned} \Psi_0 &= \Sigma - \frac{3}{2}\Theta + 2\hat{\boldsymbol{Q}}\cdot\boldsymbol{v} + \Theta v^2, \\ \Psi_i &= (-1)^i\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^i \Psi_0, \qquad i = 0, 1, 2\cdots. \end{aligned}\right\} \tag{9a}$$

It is convenient to treat the cases $m=2n$ and $m=2n+1$ separately. We shall deal here with the case $m=2n$ only. The other case may be treated quite similarly. From (9) and (9a), we have

$$\Phi_{2n} = \Psi_0 + \sum_{j=1}^{n}\left\{-K^{2j-1}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)^{2j-1} + K^{2j}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)^{2j}\right\}\Psi_0. \tag{9b}$$

Substituting this in (8) and performing integrations with respect to $\boldsymbol{v}$, we have the following equations:

$$\frac{\partial\hat{Q}_x}{\partial x} + \frac{\partial\hat{Q}_y}{\partial y} - \sum_{j=1}^{n} K^{2j-1}\frac{(2j-1)!!}{2^j}\Delta^j\left\{\Sigma + j\Theta - (2j+1)K\left(\frac{\partial\hat{Q}_x}{\partial x}+\frac{\partial\hat{Q}_y}{\partial y}\right)\right\} = \mathrm{O}(K^{2n+1}), \tag{10a}$$

$$\begin{aligned} &\sum_{j=1}^{n+1} K^{2j-2}\frac{(2j-1)!!}{2^{j-1}}\Delta^{j-1}\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}(\Sigma + j\Theta) \\ &\quad -\sum_{j=1}^{n} K^{2j-1}\frac{(2j-1)!!}{2^{j-1}}\Delta^{j-1}\left\{\Delta\begin{pmatrix}\hat{Q}_x\\ \hat{Q}_y\end{pmatrix} + (2j)\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}\left(\frac{\partial\hat{Q}_x}{\partial x}+\frac{\partial\hat{Q}_y}{\partial y}\right)\right\} = \mathrm{O}(K^{2n+1}), \end{aligned} \qquad \begin{matrix}(10\mathrm{b})\\(10\mathrm{c})\end{matrix}$$

$$\begin{aligned} &\frac{5}{2}\left(\frac{\partial\hat{Q}_x}{\partial x}+\frac{\partial\hat{Q}_y}{\partial y}\right) - \sum_{j=1}^{n} K^{2j-1}\frac{(2j-1)!!}{2^{j+1}}(2j+3)\Delta^j\{\Sigma + (j+1)\Theta\} \\ &\qquad + \sum_{j=1}^{n} K^{2j}\frac{(2j+1)!!}{2^{j+1}}(2j+5)\Delta^j\left(\frac{\partial\hat{Q}_x}{\partial x}+\frac{\partial\hat{Q}_y}{\partial y}\right) = \mathrm{O}(K^{2n+1}), \end{aligned} \tag{10d}$$

where $\Delta \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ and $(2j-1)!! = (2j-1)(2j-3)\cdots 3\cdot 1$. Differentiating (10b), (10c) by $x, y$ respectively and adding, we get

$$\sum_{j=1}^{n+1} K^{2j-1}\frac{(2j-1)!!}{2^j}\Delta^j(\Sigma + j\Theta) = \sum_{j=1}^{n} K^{2j}\frac{(2j+1)!!}{2^j}\Delta^j\left(\frac{\partial\hat{Q}_x}{\partial x}+\frac{\partial\hat{Q}_y}{\partial y}\right) + \mathrm{O}(K^{2n+2}).$$

Using this in (10a) we obtain

$$\frac{\partial\hat{Q}_x}{\partial x} + \frac{\partial\hat{Q}_y}{\partial y} = \mathrm{O}(K^{2n+1}), \tag{11}$$

and so

† It is assumed here and below that differentiation with respect to $x$ and $y$ does not amplify the order of magnitude of the quantity in the asymptotic field.

$$\sum_{j=1}^{n+1} K^{2j-1}\frac{(2j-1)!!}{2^j}\Delta^j(\Sigma+j\Theta) = O(K^{2n+2}) . \tag{12}$$

Owing to (11), the energy equation (10d) reduces to

$$\sum_{j=1}^{n} K^{2j-1}\frac{(2j-1)!!}{2^{j+1}}(2j+3)\Delta^j\{\Sigma+(j+1)\Theta\} = O(K^{2n+1}) . \tag{13}$$

Operating $\Delta$ to this and rewriting the result by means of (12) we have

$$\Delta\Sigma = \Delta\left\{-1+\sum_{j=1}^{n} K^{2j}\frac{(2j-1)!!}{2^{j-1}}\Delta^j\right\}\Theta+O(K^{2n+1}) . \tag{14}$$

On substituting of (14) in (13) and rearranging, we get the equation:

$$\begin{aligned}\frac{5}{4}K\Delta\Theta+\sum_{j=2}^{n} K^{2j-1}2^{-j}\Big\{&\frac{1}{2}j(2j+3)(2j-1)!! \\ &+\sum_{i=1}^{j-1}(2i+3)(2i-1)!!(2j-2i-1)!!\Big\}\Delta^j\Theta = O(K^{2n+1}) .\end{aligned} \tag{15}$$

From this it follows that

$$K\Delta\Theta = O(K^{2n+1}) , \tag{16}$$

while (14), (16) yield

$$K\Delta\Sigma = O(K^{2n+1}) . \tag{17}$$

Further, if we take $\Delta$ of (10b) and (10c) and use (11), (16), (17), we have

$$\sum_{j=1}^{n} K^{2j-1}\frac{(2j-1)!!}{2^{j-1}}\Delta^{j+1}\begin{pmatrix}\hat{Q}_x\\ \hat{Q}_y\end{pmatrix} = O(K^{2n+1}) .$$

This equation results, as before, in

$$K\Delta^2\begin{pmatrix}\hat{Q}_x\\ \hat{Q}_y\end{pmatrix} = O(K^{2n+1}) . \tag{18}$$

By means of (11), (16), (17) and (18), momentum equations (10b) and (10c) become as follows:

$$\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}(\Sigma+\Theta)-K\Delta\begin{pmatrix}\hat{Q}_x\\ \hat{Q}_y\end{pmatrix} = O(K^{2n+1}) . \tag{19}$$

Now, if we take the coefficient of viscosity $\mu$ to be

$$\mu = kT_0/A , \tag{20}$$

equations (11), (16), (17) and (19) are seen to agree with the Stokes' equations in hydrodynamics. Thus, it turns out that Stokes' equations govern the asymptotic (or continuum) field to any order of approximation in $K$. It can be shown in a similar manner that $\Phi_{2n}$ given by (9b) fulfills equations (6) of the hydrodynamic quantities within errors of $O(K^{2n+1})$, provided that the Stokes' equations mentioned above are satisfied. It must be remembered however that Stokes' equations can not provide a proper solution for the two-dimensional flow of viscous fluid past a cylindrical obstacle (Stokes' paradox). This means in effect that the linearized kinetic equation (5) is improper to treat this kind of problem. To remove this difficulty, we must refine (5) one step further in $S$. Namely, we replace the right-hand side of (5) by the expression

$$(1+S\Sigma)\Big\{-\Phi+\Sigma-\frac{3}{2}\Theta+2\hat{\boldsymbol{Q}}\cdot\boldsymbol{v}+\Theta v^2+S\Big[\frac{15}{8}\Theta^2-\frac{3}{2}\Sigma\Theta-\hat{\boldsymbol{Q}}^2 \\ +(2\Sigma-5\Theta)\hat{\boldsymbol{Q}}\cdot\boldsymbol{v}+\Big(\Sigma\Theta-\frac{5}{2}\Theta^2\Big)v^2+\frac{1}{2}(2\hat{\boldsymbol{Q}}\cdot\boldsymbol{v}+\Theta v^2)^2\Big]\Big\}. \quad (21)$$

Carrying out similar calculations as before, we arrive at refined forms of (11), (19) and (16):

$$\left.\begin{aligned}
&\frac{\partial\hat{Q}_x}{\partial x}+\frac{\partial\hat{Q}_y}{\partial y}+S\Big(\hat{Q}_x\frac{\partial}{\partial x}+\hat{Q}_y\frac{\partial}{\partial y}\Big)\Sigma=0\,,\\
&\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}(\Sigma+\Theta+S\Sigma\Theta)-K\Delta\begin{pmatrix}Q_x\\ \hat{Q}_y\end{pmatrix}\\
&\qquad+2S\Big(\hat{Q}_x\frac{\partial}{\partial x}+\hat{Q}_y\frac{\partial}{\partial y}\Big)\begin{pmatrix}\hat{Q}_x\\ \hat{Q}_y\end{pmatrix}+t_1=0\,,\\
&K\Delta\Theta-2S\Big(\hat{Q}_x\frac{\partial}{\partial x}+\hat{Q}_y\frac{\partial}{\partial y}\Big)\Theta+t_2=0\,.
\end{aligned}\right\} \quad (22)$$

Here the terms $t_1$, $t_2$ are of the order $O(SK)$ and consist of the second and higher order derivatives in $x$, $y$ of mean values. In the present case, $S\ll 1$ and $S/K\ll 1$ (small Reynolds number). Therefore all the corrections proportional to $S$ can be safely neglected at finite distances outside the Knudsen layer. At large distances ($\gtrsim O(K/S)$), however, the convection (or inertia) terms in the second and third equations may become larger than the diffusion terms since the former consist of the lower order derivatives. This fact is responsible for the Stokes' paradox, as first pointed out by Oseen. At large distances, $\hat{Q}_x\simeq 1$, $\hat{Q}_y\simeq 0$ and therefore we must retain the convection terms at least in simplified forms $2S(\partial/\partial x)$ $(\hat{Q}_x, \hat{Q}_y)$ and $-2S(\partial/\partial x)\Theta$ as suggested by Oseen. Again at these distances the terms $t_1$ and $t_2$ are negligible for $S\to 0$ compared with the respective preceding terms since they consist of higher order derivatives†. The term of $O(S)$ in the mass transport

† If $S\to 0$ but $S/K$ is not necessarily small, we must retain exact forms of the convection terms and we obtain the Navier-Stokes equation for incompressible fluid to $O(K^2)$. Here the terms $t_1$, $t_2$ may also be neglected since they are of $O(SK)$.

(or continuity) equation may be neglected everywhere. Thus, we are led to the conclusion that we may solve, as far as the approximation to $O(S)$ is concerned, the Stokes' equations at finite distances outside the Knudsen layer and the Oseen's equations at large distances from the cylinder. Further, we shall consider in the following the case in which the gas and the cylinder are in thermal equilibrium macroscopically. In that case, the energy equation may be approximated by the Stokes' equation

$$\varDelta\Theta = 0 \quad \text{or} \quad \varDelta T = 0\,, \tag{23}$$

throughout the asymptotic field. We may add here expressions for the stress $P_{ij}$ and heat flow $H_i$ in the asymptotic field:

$$\begin{aligned} P_{ij} &= -m\int (V_i-Q_i)(V_j-Q_j)F dV \\ &\simeq -P_\infty\delta_{ij}-2P_\infty S\int v_i v_j \Phi_{2n} E d\boldsymbol{v} \\ &= -P\delta_{ij}+P_\infty S\left\{K\left(\frac{\partial \hat{Q}_i}{\partial x_j}+\frac{\partial \hat{Q}_j}{\partial x_i}\right)-K^2\frac{\partial^2\Theta}{\partial x_i \partial x_j}\right. \\ &\qquad\left. +K^3\varDelta\left(\frac{\partial \hat{Q}_i}{\partial x_j}+\frac{\partial \hat{Q}_j}{\partial x_i}\right)+O(K^{2n+1})\right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} H_i &= \frac{1}{2}m\int (\boldsymbol{V}-\boldsymbol{Q})^2(V_i-Q_i)F d\boldsymbol{V} \\ &\simeq \frac{1}{2}mN_\infty C_m^3 S\left\{-\frac{5}{2}\hat{Q}_i+\int \boldsymbol{v}^2 v_i \Phi_{2n} E d\boldsymbol{v}\right\} \\ &= \frac{1}{2}mN_\infty C_m^3 S\left\{-\frac{5}{4}K\frac{\partial\Theta}{\partial x_i}+\frac{K^2}{2}\varDelta\hat{Q}_i+O(K^{2n+1})\right\}, \end{aligned} \tag{25}$$

where suffixes $i, j$ indicate, as usual, the rectangular components in the $x, y$ plane. It may be noted that these expressions involve additional terms other than those in Stokes' formulae.

## 4. Asymptotic Field

We now proceed to solve the asymptotic equations just obtained. We may deal with the Oseen's equation alone, for it contains the Stokes equation at low Reynolds numbers. However, we shall make use here of a method which considers both equations in respective proper domains and matching of two solutions[9]. This method may have closer connection with the kinetic theory consideration in the present study. We introduce complex variables:

$$W = Q_X - iQ_Y\,, \quad Z = X+iY\,, \quad \bar{Z} = X-iY\,, \tag{26}$$

and write the equation of continuity and the vorticity $\omega$ as

$$\mathfrak{R}\left(\frac{\partial W}{\partial \bar{Z}}\right) = 0\,, \quad \omega \equiv \frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y} = 2i\frac{\partial W}{\partial \bar{Z}}\,, \tag{27}$$

where $\mathfrak{R}$ means real part. Then Oseen's equation of motion may be written as

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} - 2\hat{k}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial \bar{Z}}\right)W = 0\,, \quad \hat{k} = \frac{mN_\infty Q_\infty}{2\mu}\,. \tag{28}$$

A fundamental solution of this equation representing far field from the obstacle is well known to be

$$\frac{W}{Q_\infty} = 1 + \frac{C}{\hat{k}Z} - e^{\hat{k}X}\{\bar{C}K_0(\hat{k}R) + CK_1(\hat{k}R)e^{-i\theta}\}\,, \tag{29}$$

where $Z = Re^{i\theta}$, $K_0$, $K_1$ are modified Bessel functions and $C$ is an arbitrary constant. For sufficiently small Reynolds number $R_e = 2\hat{k}a$, there may exist a region in which $\hat{k}R \ll 1$ but $R \gg a$. In this region (29) may be approximated as

$$\frac{W}{Q_\infty} \sim 1 + \frac{\bar{C}}{2}\log(Z\bar{Z}) - \frac{C}{2}\frac{\bar{Z}}{Z} - \bar{C}\left\{\log\left(\frac{2}{\hat{k}}\right) - \gamma\right\} - \frac{C}{2}\,, \tag{30}$$

$\gamma \simeq 0.5772$ being Euler's constant. The constant $C$ is to be determined by a matching process with the near field subject to the Stokes equation. The expression (30) itself satisfies the Stokes equation and hence (30) may serve as a universal boundary condition at infinity for the Stokes equation. It is valid even when there are slip and jump of velocity on the surface of the cylinder (cf. (36)). A momentum consideration shows that $C$ is related to the force $D+iL$ experienced by the cylinder as

$$D + iL = 4\pi\mu Q_\infty C\,. \tag{31}$$

Now the Stokes' equation pertinent to the near field may be written in the form:

$$\frac{\partial}{\partial Z}(P + i\mu\omega) = 0\,. \tag{32}$$

Integrating this, we have

$$P - P_\infty = 4\mu Q_\infty \mathfrak{R}\left(\frac{dF}{dZ}\right), \tag{33a}$$

$$\omega = 2iQ_\infty\left(\frac{dF}{dZ} - \frac{d\bar{F}}{d\bar{Z}}\right), \tag{33b}$$

where $P_\infty = kN_\infty T_0$ is the pressure at infinity and $F(Z)$ is an arbitrary analytic function of $Z$. Remembering (27), we can integrate (33b) to have

$$\frac{W}{Q_\infty}=\bar{Z}\frac{dF}{dZ}-\bar{F}(\bar{Z})+G(Z)\,, \tag{34}$$

where $G(Z)$ is another arbitrary function of $Z$. Considering the asymptotic form (30) and regularity of the flow field, we may assume† the following expressions for $F(Z)$ and $G(Z)$:

$$\left.\begin{aligned} F(Z) &= -\frac{C}{2}\log\left(\frac{Z}{a}\right)+\sum_{j=1}^{\infty}\lambda_j\left(\frac{Z}{a}\right)^{-j}, \\ G(Z) &= \frac{\bar{C}}{2}\log\left(\frac{Z}{a}\right)+\mu_0+\sum_{j=1}^{\infty}\mu_j\left(\frac{Z}{a}\right)^{-j}, \end{aligned}\right\} \tag{35}$$

where $C$, $\lambda_j$, $\mu_j$ are constants to be determined by boundary conditions. The constant $C$ is seen to be real when the flow is symmetric with respect to $X$-axis as in the present case. We take hereafter $C$ to be real. Let $Q_R$, $Q_\theta$ be the $R$-and $\theta$-components of the flow velocity. The boundary condition for the asymptotic (continuum) flow velocity at the surface of the cylinder is, as is seen later, of the next forms:

$$\left.\begin{aligned} &\frac{Q_R}{Q_\infty}=CA\cos\theta\,,\quad \frac{Q_\theta}{Q_\infty}=CB\sin\theta\,, \\ \text{or}\quad &\frac{W}{Q_\infty}=\frac{C}{2}\left\{A-B+(A+B)\frac{a^2}{Z^2}\right\}\quad (|Z|=a)\,, \end{aligned}\right\} \tag{36}$$

where $A$, $B$ are constants to be determined by kinetic theory analysis. Taking into acount that $\bar{Z}=a^2/Z$ at $|Z|=a$, we have from (34) and (36) the equation

$$\frac{a^2}{Z}\frac{dF}{dZ}-\bar{F}\left(\frac{a^2}{Z}\right)+G(Z)=\frac{C}{2}\left\{A-B+(A+B)\frac{a^2}{Z^2}\right\}.$$

Inserting (35) and comparing both sides, we can determine $F(Z)$ and $G(Z)$ except for the constant $C$ in the forms:

$$\left.\begin{aligned} F(Z) &= -\frac{C}{2}\log\left(\frac{Z}{a}\right), \\ G(Z) &= \frac{C}{2}\left\{\log\left(\frac{Z}{a}\right)+A-B+(1+A+B)\frac{a^2}{Z^2}\right\}, \end{aligned}\right\} \tag{37}$$

and we have

$$\frac{W}{Q_\infty}=\frac{C}{2}\left\{\log\left(\frac{\bar{Z}Z}{a^2}\right)-\frac{\bar{Z}}{Z}+A-B+(1+A+B)\frac{a^2}{Z^2}\right\}. \tag{38}$$

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† If the cross section of the cylinder is of general shape, we may transform it conformally onto a circle and then assume the expressions (35) in the new plane.

Then, we match this expression with the asymptotic form (30) and determine $C$ to be

$$C = \left\{\log\left(\frac{4}{R_e}\right) - \gamma + \frac{1}{2}(1+A-B)\right\}^{-1} . \tag{39}$$

We have also from (33a) and (37) the pressure field

$$P - P_\infty = -2\mu C Q_\infty R^{-1} \cos\theta . \tag{40}$$

If we put $A=B=0$ in the above result, we have obviously the conventional case of no slip velocity on the cylinder.

These results for the continuum flow are valid for the region $R-a \gg l$ ($l$ is the mean free path), but become incorrect for $R-a=O(l)$, i.e. in the Knudsen layer. Thus, kinetic theory consideration of the Knudsen layer is needed to clarify the effect of rarefaction on the flow of a gas past an obstacle. The solution from the kinetic theory is expected to tend asymptotically for $R-a \gg l$ to the continuum solution and hence (38)$\sim$(40) may provide boundary conditions at infinity for the Knudsen layer solution. One more condition regarding the temperature field is also needed. We shall consider below the simplest case in which the temperature of the gas outside the Knudsen layer is constant equal to that of the cylinder.

## 5. Kinetic Theory Analysis, First Approximate Solution

We next proceed to the kinetic theory analysis of the Knudsen layer. The governing equation is the B-G-K model of the Boltzmann equation (1) or its linearized version (5). It is convenient to change to cylindrical coordinates ($R$, $\theta$, $Z$) as shown in figure 1. Let $V_R$, $V_\theta$, $V_Z$ be the $R$-, $\theta$-, $Z$-components of $\boldsymbol{V}$ respectively. Then, the left-hand side of (1) in the new coordinates is seen to take on the form

$$V_R\frac{\partial F}{\partial R} + \frac{V_\theta}{R}\frac{\partial F}{\partial \theta} + \frac{V_\theta^2}{R}\frac{\partial F}{\partial V_R} - \frac{V_R V_\theta}{R}\frac{\partial F}{\partial V_\theta} . \tag{41}$$

We introduce new variables :

$$\left.\begin{aligned} &R/a-1 = Kn , \quad V/C_m = \boldsymbol{v} = (v_n, v_\theta, w) , \\ &\boldsymbol{Q}/Q_\infty = 2CK(q_n, q_\theta, 0) , \\ &(\Phi, \Sigma, \Theta, \Xi) = 2CK(\phi, \sigma, \vartheta, \xi) , \end{aligned}\right\} \tag{42}$$

where $C$ is the scale factor of the continuum solution given by (39). The linearized equations (5), (6) are then rewritten as

$$v_n\frac{\partial \phi}{\partial n} + \frac{K}{1+Kn}\left(v_\theta\frac{\partial \phi}{\partial \theta} + v_\theta^2\frac{\partial \phi}{\partial v_n} - v_n v_\theta\frac{\partial \phi}{\partial v_\theta}\right) = -\phi + \sigma + 2\boldsymbol{q}\cdot\boldsymbol{v} + \vartheta\left(\boldsymbol{v}^2 - \frac{3}{2}\right), \tag{43}$$

$$\left.\begin{aligned} &\sigma = \int E\phi d\boldsymbol{v}\,, \quad \boldsymbol{q} = \int \boldsymbol{v}E\phi d\boldsymbol{v}\,, \\ &\xi = \sigma + \vartheta = \frac{2}{3}\int \boldsymbol{v}^2 E\phi d\boldsymbol{v}\,, \quad E = \pi^{-3/2}e^{-v^2}\,. \end{aligned}\right\} \tag{44}$$

It is now assumed that molecules are reflected from the wall diffusely. Namely, the distribution function for $V_R>0$ at the wall is given by (la) with

$$N = N_\infty\{1+S\Sigma_W(\theta)\}\,, \quad T = T_0\,, \quad \boldsymbol{Q} = 0\,, \tag{45}$$

where the vairation $\Sigma_W(\theta)$ of the number density on the wall is yet to be determined. From this and (4), (42), we have

$$\phi_{v_n>0} = \sigma_W(\theta) \qquad (n=0)\,. \tag{46}$$

We have, in addition, the condition of no net mass flow at the wall, i. e.

$$q_n = 0 \qquad\qquad (n=0)\,. \tag{47}$$

Further, the solution of linearized B-G-K equation (43) must tend to the continuum solution (38)$\sim$(40) subject to the Stokes equation in hydrodynamics for $n\to\infty$ as discussed earlier.

Now, we may neglect the second term on the left-hand side of (43) to the first approximation for $K\ll 1$. This is tantamount to neglecting the effect of the wall curvature owing to infinitesimal thickness of the Knudsen layer. Thus, the solution is expected to be locally that of the shear flow over an infinite plane wall. On the other hand, we may take $A=B=0$ (no slip on the cylinder) in the asymptotic form of flow velocity (38) to the present approximation. Putting $R/a=1+Kn$ there and expanding for small $K$, we have

$$q_\theta \sim -n\sin\theta\,, \quad q_n \sim 0\,. \tag{48a}$$

Also, from (40), (4), (42)

$$\xi \sim \sigma \sim -\cos\theta\,, \tag{48b}$$

since $T\sim T_0$ ($\vartheta\sim 0$). These results suggest the next form of $\phi$:

$$\phi = \varphi_1(n, \boldsymbol{v})\sin\theta - \cos\theta\,, \tag{49}$$

where $\varphi_1$ is supposed to be odd in $v_\theta$. This $\phi$ when substituted in (44) gives

$$q_\theta = q_{\theta,1}(n)\sin\theta\,, \qquad q_{\theta,1} = \int v_\theta E\varphi_1 d\boldsymbol{v}\,, \tag{50a}$$

$$\sigma = -\cos\theta\,, \qquad q_n = \vartheta \equiv 0\,. \tag{50b}$$

Also, inserting (49) in (43) (O($K$) being neglected), we have the equation for $\varphi_1$:

$$v_n \frac{\partial \varphi_1}{\partial n} = -\varphi_1 + 2v_\theta q_{\theta,1} \,. \tag{51}$$

The boundary condition (46) becomes

$$\varphi_{1,v_n>0} = 0 \qquad (n=0) \,, \tag{52}$$

since we may expect $\sigma_W = -\cos\theta$ to the present approximation. The condition (47) is automatically satisfied by (50b). We next solve (51) formally for $\varphi_1$ subject to (52) and to the condition that $\varphi_1$ should not be infinite exponentially for $n \to \infty$, $v_n < 0$. The result is

$$\varphi_1 = \frac{2v_\theta}{v_n} \int_c^n \exp\left(-\frac{n-\eta}{v_n}\right) q_{\theta,1}(\eta) d\eta \,, \tag{53}$$

where $c=0$ for $v_n > 0$, and $c = \infty$ for $v_n < 0$. Substitution of this in (50a) yields an integral equation for $q_{\theta,1}$ in the form:

$$\sqrt{\pi}\, q_{\theta,1}(n) = \int_0^\infty g_{-1}(n-\eta) q_{\theta,1}(\eta) d\eta \,, \tag{54}$$

where the function $g_j(x)$ ($j=0, \pm 1, \pm 2 \cdots$) is defined as†

$$g_j(x) = (\operatorname{sgn} x)^{j+1} \int_0^\infty \xi^j \exp\left\{-\xi^2 - \frac{|x|}{\xi}\right\} d\xi \,. \tag{54a}$$

Equation (54) and the boundary condition $q_{\theta,1} \sim -n$ (from (48a)) are seen to be the same as those for the steady shear flow along a plane wall as was expected earlier. Since this problem has already been solved[4),6),7)], $q_{\theta,1}$ may be taken as known. In ref. 7, there is given an approximate solution of the form:

$$\left.\begin{aligned} &q_{\theta,1}(n) = -\{n + \alpha_1 + q^*_{\theta,1}(n)\} \,, \\ &q^*_{\theta,1}(n) = \sum_{i=0}^{3} a_i g_i(n) \,, \qquad \alpha_1 = 1.0161 \,, \\ &a_0 = -0.1189, \quad a_1 = -1.0389, \quad a_2 = 1.7650, \quad a_3 = -0.8974 \,. \end{aligned}\right\} \tag{55}$$

A small correction is needed for $n \simeq 0$ as given in ref. 7. From (55) we have the result $q_\theta|_{n=0} = -0.707 \sin\theta$ and so there occurs an actual slip at the wall surface. Also, extending the asymptotic form $q_\theta \sim -(n+\alpha_1)\sin\theta$ to $n=0$ and considering (7), (42), we obtain the following relation

$$Q_\theta|_{R=a} = \tau [dQ_\theta/dR]_{R=a} \,, \quad \tau = 0.901 l \,, \tag{56}$$

where $\tau$ is the so-called slip coefficient. It may be noted that there appears no

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† Some relevant characteristics of $g_j(x)$ may be found in ref. 6. Also, a numerical table is given in ref. 10.

variation of $q_n$, $\vartheta$ and $\sigma$ across the Knudsen layer to the first approximation.

The foregoing analysis has revealed that the asymptotic solution (cotinuum flow) should accompany a slip of the form

$$Q_\theta/Q_\infty = -2CK\alpha_1 \sin\theta\,, \quad Q_R = 0\,, \tag{57}$$

if it is extrapolated to the wall disregarding the Knudsen layer variation. Comparing (57) and (36), the continuum solution which takes this result into account can be obtained simply by putting in (38) and (39)

$$A = 0\,, \quad B = -2K\alpha_1\,. \tag{58}$$

## 6. Second Approximate Solution

We next proceed to the second approximate solution. Equation (43) may now be approximated in the form:

$$\begin{aligned} v_n\frac{\partial\phi}{\partial n} + K\left(v_\theta\frac{\partial\phi}{\partial\theta} + v_\theta{}^2\frac{\partial\phi}{\partial v_n} - v_n v_\theta\frac{\partial\phi}{\partial v_\theta}\right) \\ = -\phi+\sigma+2\boldsymbol{q}\cdot\boldsymbol{v}+\vartheta\left(\boldsymbol{v}^2-\frac{3}{2}\right). \end{aligned} \tag{59}$$

Writing $R/a=1+Kn$ in the continuum solution (38), (39), (40) with $A$, $B$ given by (58) and expanding the result for small $K$ we have the asymptotic conditions for $n\to\infty$ (cf. (48)):

$$\left.\begin{aligned} &q_\theta \sim \{-n-\alpha_1+K(n^2+\alpha_1 n)\}\sin\theta\,, \\ &q_n \sim K\left(\frac{1}{2}n^2+\alpha_1 n\right)\cos\theta\,, \\ &\xi = \sigma+\vartheta\sim(-1+Kn)\cos\theta\,, \quad \vartheta\sim 0\,. \end{aligned}\right\} \tag{60}$$

From these results we may infer the proper form† of $\phi$ such that (cf. (49))

$$\phi = \{\varphi_1(n, \boldsymbol{v})+K\varphi_2(n, \boldsymbol{v})\}\sin\theta + \{-1+K\psi_2(n, \boldsymbol{v})\}\cos\theta\,, \tag{61}$$

where $\varphi_1$ and $\varphi_2$ are supposed to be odd in $v_\theta$. This $\phi$ when inserted in (44) gives

$$\left.\begin{aligned} q_\theta &= \{q_{\theta,1}(n)+Kq_{\theta,2}(n)\}\sin\theta\,, \quad q_n = Kq_{n,2}(n)\cos\theta\,, \\ \sigma &= \{-1+K\sigma_2(n)\}\cos\theta\,, \quad \vartheta = K\vartheta_2(n)\cos\theta\,, \\ \xi &= \{-1+K\xi_2(n)\}\cos\theta\,, \end{aligned}\right\} \tag{62}$$

with

† Since $\phi$ for the free molecular flow is seen to have the same form in $\theta$, it is likely that the exact solution takes also on the form

$$\phi = \varphi_s(n, \boldsymbol{v})\sin\theta + \psi_c(n, \boldsymbol{v})\cos\theta\,.$$

$$\left.\begin{aligned} q_{\theta,2} &= \int v_\theta E\varphi_2 d\boldsymbol{v}\,, \quad q_{n,2} = \int v_n E\psi_2 d\boldsymbol{v}\,, \\ \sigma_2 &= \int E\psi_2 d\boldsymbol{v}\,, \quad \xi_2 = \sigma_2 + \vartheta_2 = \frac{2}{3}\int \boldsymbol{v}^2 E\psi_2 d\boldsymbol{v}\,. \end{aligned}\right\} \tag{62a}$$

Since the first order terms have already been obtained in the preceding paragraph, we only calculate the second order terms indicated by suffix 2. Substituting (61) in (59), we have the equations for $\varphi_2$ and $\psi_2$:

$$v_n\frac{\partial\varphi_2}{\partial n}+v_\theta+v_\theta{}^2\frac{\partial\varphi_1}{\partial v_n}-v_nv_\theta\frac{\partial\varphi_1}{\partial v_\theta} = -\varphi_2+2v_\theta q_{\theta,2}\,, \tag{63}$$

$$v_n\frac{\partial\psi_2}{\partial n}+v_\theta\varphi_1 = \sigma_2-\psi_2+2v_nq_{n,2}+\vartheta_2\left(\boldsymbol{v}^2-\frac{3}{2}\right). \tag{64}$$

Writing $\sigma_W=(-1+K\tilde{\gamma}_2)\cos\theta$, the conditions (46) and (47) at the wall give

$$\left.\begin{aligned} &\varphi_{2,v_n>0} = 0\,, \quad \psi_{2,v_n>0} = \tilde{\gamma}_2\,, \\ &q_{n,2} = 0\,, \end{aligned}\right\} \quad (n=0)\,, \tag{65}$$

where $\tilde{\gamma}_2$ is a constant to be determined in matching process with the continuum solution. We substitute $\varphi_1$ from (53) into (63) and solve the resulting equation for $\varphi_2$ subject to (65) and to the condition that $\varphi_2$ should not diverge exponentially as $n\to\infty$. The result is

$$\begin{aligned} \varphi_2 = \frac{1}{v_n}\int_c^n \exp\left(-\frac{n-\eta}{v_n}\right)\Big\{&v_\theta(-1+2q_{\theta,2}(\eta)) \\ &+2\int_c^\eta \exp\left(-\frac{\eta-\zeta}{v_n}\right)\left[\frac{v_\theta{}^3}{v_n{}^2}+v_\theta-\frac{v_\theta{}^3}{v_n{}^3}(\eta-\zeta)\right]q_{\theta,1}(\zeta)d\zeta\Big\}d\eta\,, \end{aligned} \tag{66}$$

where $c$ is to be taken zero for $v_n>0$ and $\infty$ for $v_n<0$. Introducing (66) in the first equation of (62a) and making some reductions we obtain an integral equation for $q_{\theta,2}$. Considering there the asymptotic form of $q_\theta$ in (60), we put

$$q_{\theta,2}(n) = n^2+\alpha_1 n-\alpha_2+q^*_{\theta,2}(n)\,, \quad q^*_{\theta,2}(n)\to 0 \qquad (n\to\infty)\,, \tag{67}$$

and remembering (54), (55), we may write the equation for $q^*_{\theta,2}$ in the form:

$$\begin{aligned} \sqrt{\pi}\,q^*_{\theta,2}(n) &= \left(\alpha_2+\frac{1}{2}\right)g_0(n)+\alpha_1 g_1(n)-2g_2(n) \\ &\quad -\frac{\sqrt{\pi}}{2}\int_n^\infty q^*_{\theta,1}(\eta)d\eta+\int_0^\infty g_{-1}(n-\eta)q^*_{\theta,2}(\eta)d\eta\,. \end{aligned} \tag{68}$$

We may use the approximate form (55) for $q^*_{\theta,1}$. Since $g_j{}'(x)=-g_{j-1}(x)$, we have

$$\int_n^\infty q^*_{\theta,1}(\eta)d\eta = \sum_{i=1}^4 a_{i-1}g_i(n)\,. \tag{68a}$$

An effective method to solve (68) may be the one used in ref. 6. That is, taking (67) into account, we assume

$$q^*_{\theta,2}(n) = \sum_{i=1}^{N} b_i g_i(n) , \tag{69}$$

and introduce this in (68). Multiplying both sides of the resulting equation by $n^m$ ($m=0, 1, \cdots, N$) and integrating from $n=0$ to $n=\infty$, we get simultaneous linear algebraic equations for $N+1$ unknowns $b_i$'s and $\alpha_2$. The solution of these equations for $N=4$ is as follows:

$$\begin{aligned} b_1 &= \phantom{1}4.924 , \quad b_2 = -15.852 , \\ b_3 &= 16.667 , \quad b_4 = \phantom{1}-5.064 ; \quad \alpha_2 = 0.2502 . \end{aligned} \tag{69a}$$

The constant $\alpha_2$ determines the second order slip. On the other hand, if we integrate (64) throughout the $\boldsymbol{v}$-space after multiplying both sides by $E$, we have the continuity equation

$$\frac{dq_{n,2}}{dn} + q_{\theta,1} = 0 . \tag{70}$$

Integrating this with (55) and (65), we get the result

$$\left.\begin{aligned} q_{n,2}(n) &= \frac{1}{2}n^2 + \alpha_1 n - \beta_2 + q^*_{n,2}(n) , \\ q^*_{n,2}(n) &= -\sum_{i=1}^{4} a_{i-1} g_i(n) , \qquad \beta_2 = 0.2336 . \end{aligned}\right\} \tag{71}$$

The constant $\beta_2$ represents a jump at the wall of the normal component of continuum (asymptotic) flow velocity.

We next solve (64) formally for $\psi_2$. Boundary conditions are given in (65) and regularity at infinity, and $\varphi_1$ may be substituted from (53). We thus obtain

$$\begin{aligned} \psi_2 = \varepsilon \tilde{\gamma}_2 e^{-(n/v_n)} + \int_c^n \exp\left(-\frac{n-\eta}{v_n}\right) \Big\{ 2q_{n,2} + \left(\sigma_2 - \frac{3}{2}\vartheta_2\right)\frac{1}{v_n} \\ + \vartheta_2 \frac{\boldsymbol{v}^2}{v_n} - 2\frac{v_\theta^2}{v_n^2} \int_c^\eta \exp\left(-\frac{\eta-\zeta}{v_n}\right) q_{\theta,1}(\zeta)\, d\zeta \Big\} d\eta , \end{aligned} \tag{72}$$

where $\varepsilon=1$, $c=0$ for $v_n>0$, and $\varepsilon=0$, $c=\infty$ for $v_n<0$. Inserting this into the last three equations in (62a), we have three integral equations involving $q_{\theta,1}$, $q_{n,2}$, $\sigma_2$ and $\vartheta_2$. We can eliminate there $q_{\theta,1}$ and $q_{n,2}$ by the aid of (70). Thus, after lengthy calculation we are led to the following result:

$$\begin{aligned} \sqrt{\pi}\, \sigma_2^*(n) = (\tilde{\gamma}_2 - \gamma_2) g_0(n) - \delta_2 \left\{ g_2(n) - \frac{1}{2} g_0(n) \right\} + g_1(n) \\ + \int_0^\infty \left\{ g_{-1}\sigma_2^*(\eta) + \left(g_1 - \frac{1}{2} g_{-1}\right) \vartheta_2^*(\eta) \right\} d\eta , \end{aligned} \tag{73}$$

$$\frac{3}{2}\sqrt{\pi}\,\vartheta_2^*(n) = (\tilde{r}_2-r_2)\left\{g_2(n)-\frac{1}{2}g_0(n)\right\}$$
$$-\delta_2\left\{g_4(n)-g_2(n)+\frac{5}{4}g_0(n)\right\}+g_3(n)-\frac{1}{2}g_1(n)$$
$$+\int_0^\infty\left\{\left(g_1-\frac{1}{2}g_{-1}\right)\sigma_2^*(\eta)+\left(g_3-g_1+\frac{5}{4}g_{-1}\right)\vartheta_2^*(\eta)\right\}d\eta\,, \tag{74}$$

where we have written in view of (60)

$$\left.\begin{aligned}&\sigma_2 = n+r_2+\sigma_2^*(n)\,, \quad \vartheta_2 = \delta_2+\vartheta_2^*(n)\,,\\ &\sigma_2^*(n)\to 0\,, \quad \vartheta_2^*(n)\to 0 \qquad (n\to\infty)\,,\end{aligned}\right\} \tag{75}$$

and the argument $n-\eta$ of $g_j$ in the integrals is abbreviated. We have also an additional equation

$$\frac{\sqrt{\pi}}{2} = \frac{1}{2}\delta_2-(\tilde{r}_2-r_2)+\int_0^\infty\{2g_0(\eta)\sigma_2^*(\eta)+[2g_2(\eta)-g_0(\eta)]\vartheta_2^*(\eta)\}d\eta\,. \tag{76}$$

Equations (73), (74) and (76) are seen to be the same as those for the stagnation-point flow studied by Tamada[8)†]. We may quote the approximate solution given there. Namely,

$$\left.\begin{aligned}&\sigma_2^* = \sum_{i=0}^{3} c_i g_i(n)\,, \quad \vartheta_2^* = \sum_{i=0}^{3} d_i g_i(n)\,,\\ &c_0 = -0.2704\,, \quad c_1 = 0.3164\,, \quad c_2 = -0.3431\,, \quad c_3 = 0.1077\,,\\ &d_0 = -0.002763\,, \quad d_1 = 0.002786\,, \quad d_2 = -0.01703\,, \quad d_3 = -0.02327\,,\end{aligned}\right\} \tag{77}$$

$$\tilde{r}_2-r_2 = -0.8427\,, \quad \delta_2 = 0.22338\,. \tag{77a}$$

The results (55), (67), (71) with (42), (62) imply that there are slip and jump of the continuum (asymptotic) flow velocity at the surface of the cylinder in the forms (cf. (57)):

$$\left.\begin{aligned}&\frac{Q_\theta}{Q_\infty} = -2CK(\alpha_1+K\alpha_2)\sin\theta\,,\\ &\frac{Q_R}{Q_\infty} = -2CK^2\beta_2\cos\theta\,.\end{aligned}\right\} \tag{78}$$

The continuum flow solution subject to this boundary condition (cf. (36)) can be obtained immediately by putting in (38) and (39)

$$A = -2K^2\beta_2\,, \quad B = -2K(\alpha_1+K\alpha_2)\,. \tag{79}$$

† Correspondence between the two systems is as follows:

$$\sigma_2^* = -\frac{1}{2}\sigma^*, \quad \vartheta_2^* = -\frac{1}{2}\theta^*\,; \ \tilde{r}_2-r_2 = -\frac{1}{2}\sigma_W, \quad \delta_2 = \frac{1}{2}\theta_W\,; \ g_i \equiv J_i\,.$$

Thus, we have

$$\left.\begin{aligned}\frac{Q_\theta}{Q_\infty} &= -C\left\{\log\left(\frac{R}{a}\right)+\frac{1}{2}\left(1-\frac{a^2}{R^2}\right)+K\alpha_1\left(1+\frac{a^2}{R^2}\right)\right.\\ &\quad \left.+K^2\left[\alpha_2\left(1+\frac{a^2}{R^2}\right)-\beta_2\left(1-\frac{a^2}{R^2}\right)\right]\right\}\sin\theta\,,\\ \frac{Q_R}{Q_\infty} &= C\left\{\log\left(\frac{R}{a}\right)-\frac{1}{2}\left(1-\frac{a^2}{R^2}\right)+K\alpha_1\left(1-\frac{a^2}{R^2}\right)\right.\\ &\quad \left.+K^2\left[\alpha_2\left(1-\frac{a^2}{R^2}\right)-\beta_2\left(1+\frac{a^2}{R^2}\right)\right]\right\}\cos\theta\,,\end{aligned}\right\} \tag{80}$$

$$C=\left\{\log\left(\frac{4}{R_e}\right)-\gamma+\frac{1}{2}+K\alpha_1+K^2(\alpha_2-\beta_2)\right\}^{-1}. \tag{81}$$

We have also from (62), (75) the perturbation pressure

$$\xi\sim\{-1+K(n+\gamma_2+\delta_2)\}\cos\theta\,. \tag{82}$$

This should be match with the result

$$\xi=\{-1+Kn+O(K^2)\}\cos\theta\,,$$

from the continuum solution (40). Therefore,

$$\gamma_2+\delta_2=0\,, \tag{83}$$

which together with the results (77a) determine the constants $\gamma_2$, $\tilde{\gamma}_2$ as

$$\gamma_2=-0.22338\,,\quad \tilde{\gamma}_2=-1.0661\,. \tag{84}$$

Further, we have to consider the temperature field. From (62), (75) we get

$$\vartheta\sim K\delta_2\cos\theta\,, \tag{85}$$

which gives, with the aid of (4), (42), the boundary condition for the continuum flow temperature:

$$\frac{T-T_0}{T_0}=2CSK^2\delta_2\cos\theta\qquad (R=a)\,. \tag{85a}$$

The solution of (23) subject to (85a) is easily obtained as

$$\frac{T-T_0}{T_0}=2CSK^2\delta_2\left(\frac{a}{R}\right)\cos\theta\,. \tag{86}$$

Also, from (40), (86) and the relation $P=kNT$, we may obtain the field of density.

Finally, the drag $D$ acting on the cylinder can be calculated to the second approximation by means of the formula (31) combined with (81). The result may be written as

$$C_D = \frac{D}{(1/2)\rho Q_\infty^2 (2a)}$$
$$= \frac{8\pi}{R'_e}\left\{\log\left(\frac{8}{R'_e}\right) - 0.0772 + 1.0161K + 0.0166K^2\right\}^{-1}. \tag{87}$$

where $R'_e = 2R_e$ is the Reynolds number referred to the diameter $2a$ of the cylinder. Parameters here may be changed by the relations

$$K = \frac{S}{(R'_e/4)}, \quad S = (\kappa/2)^{1/2} M. \tag{87a}$$

The formula (87) can be confirmed by the direct integration of the stress acting on the wall of the cylinder. Details will however be omitted for brevity.

## 7. Discussions

In figure 2 are shown plotted $q^*_{\theta,1}$, $q^*_{\theta,2}$, $\sigma_2{}^*$ and $\vartheta_2{}^*$ against $n$, the distance from the wall normalized by the mean free path, as calculated by (55), (69) and (77) respectively. It will be seen that these Knudsen layer variations fade out within several mean free paths. Figure 3 represents the distribution of the tangential flow velocity $q_\theta$ (divided by $\sin\theta$) across the Knudsen layer at $K=0$, 0.1,

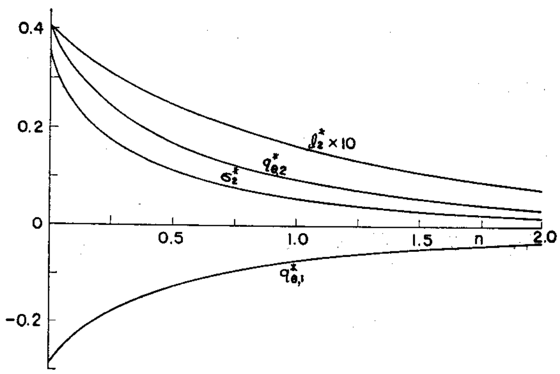


Fig. 2. Knudsen layer variations of $q_\theta$, $\sigma$ and $\vartheta$.

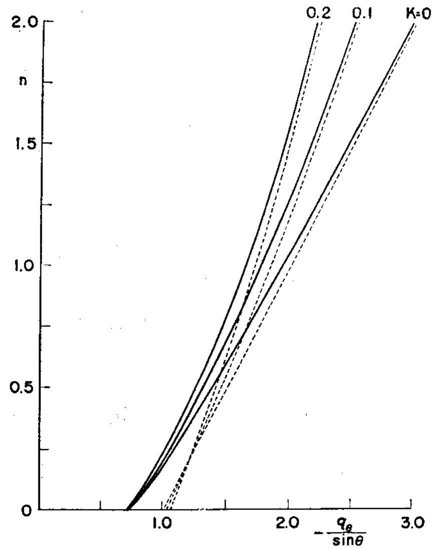


Fig. 3. Distributions of the tangential velocity across the Knudsen layer.

0.2. Dotted lines are the asymptotic curves obtained from (80). The difference between the full and dotted lines is equal to $q^*_{\theta,1}+Kq^*_{\theta,2}$. Actual and asymptotic (continuum) slips of the flow velocity on the wall may be noted (second order slip is included.). In figure 4 are drawn curves of $C_D$ versus $R'_e$ by (87) for several

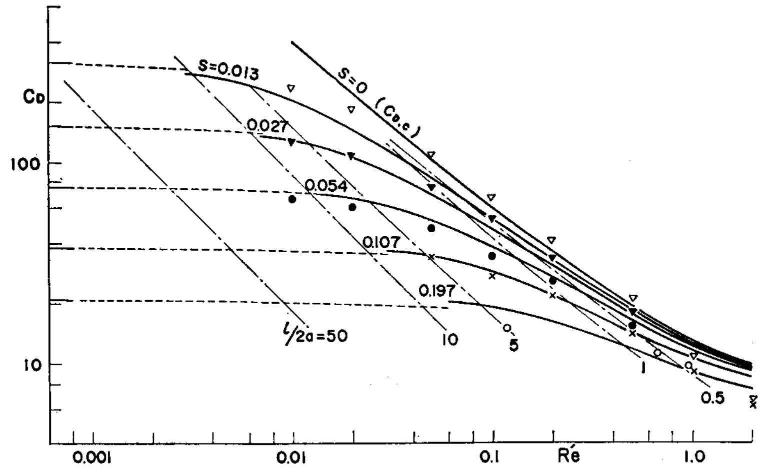


Fig. 4. Drag coefficient versus Reynolds number; full-line curves from (87) and dotted-line curves from (89).

values of $S$ and compared with the result of recent experiment by Coudeville, Trepaud & Brun[11]. It will be seen that agreement between the theory and the experiment is fairly satisfactory. The drag coefficient $C_{D.f}$ for the free molecular flow ($K\to\infty$) is easily calculated to be

$$C_{D.f}=\sqrt{\pi}\left(\frac{3}{2}+\frac{\pi}{4}\right)\frac{1}{S}. \tag{88}$$

Quite recently, the present authors have carried out a calculation of the first order correction to the free molecular flow and obtained the following result†:

$$C_D=C_{D.f}-\left\{\left(\frac{4}{\pi}-\frac{7}{12}+\frac{\pi}{4}-\frac{\pi^2}{32}+\frac{\pi^3}{64}\right)\log K+\mathrm{O}(1)\right\}\frac{1}{KS}. \tag{89}$$

In figure 4 is also shown $C_D$ from (89) by dotted lines. The transition between the two theoreis for small and large $K$ is seen to be smooth. Sherman[12] proposed a simple interpolation formula

$$C_D/C_{D.f}=\{1+(C_{D.f}/C_{D.c})\}^{-1}, \tag{90}$$

† Details of the analysis are intended to be published soon.

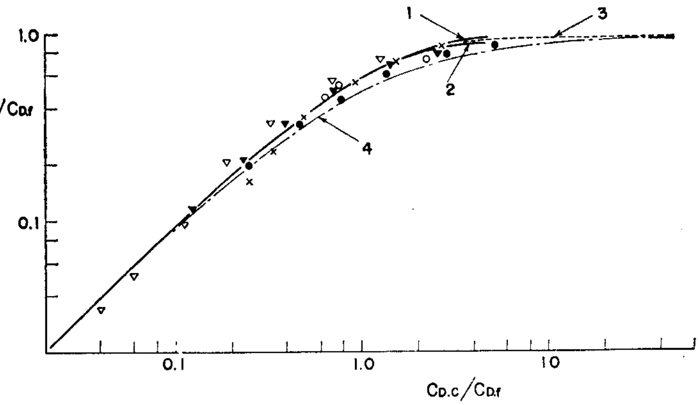


Fig. 5. Sherman plot of drag coefficient; curve 1 ($S$=0.197) and curve 2 ($S$=0.013) from (87), curve 3 from (89) and curve 4 from (90).

$C_{D\cdot c}$ being the continuum limit ($K\rightarrow 0$) of $C_D$. In figure 5 are shown plotted this curve together with the result of the present study and experimental data for comparison. The dependence of $C_D/C_{D\cdot f}$ on $S$ is seen to be almost eliminated in this manner of plot.

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