A Stochastic Theory of the Diffusion of Traffic Flow

By

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This paper presents a study of the diffusion of traffic flow and an observation by a moving observer, that is, a Doppler's effect. First we introduce a time process and a space process, and we show they are composed Poisson processes under the suitable assumptions. Secondly, we derive transformation formulae between these processes, interpreting velocity as the measure preserving transformation. Moreover, we analyze a Doppler's effect occurring in an observation by a moving observer, and finally we demonstrate, in a simple case that the time process is a homogeneous Poisson process.

Introduction

There seems to be two main approaches to the road traffic flow theory. One is the car-following theory, that is, a deterministic approach analogeous to hydrodynamics, thermodynamics, etc., in case the traffic density is high. The other is a probabilistic approach in case the traffic density is low, that is, it allows free travel of cars. In this paper we deal with the latter case.

Now, analyzing the traffic flow, we must take its fundamental characteristics that are the existence of time and space and the finiteness of cars' velocities[1]. We, at first, introduce the time process which is, roughly speaking, the number of cars passing through an arbitrarily fixed point with some velocities, and the space process which is the number of cars existing at an arbitrarily fixed time with some velocities in a given space interval.

Under suitable assumptions, we show these processes are Poission processes or composed Poisson processes. After that, we derive the transformation formulae between these processes, interpreting velocity as the measure preserving transformation. Moreover, we analyze a Doppler's effect occurring in an observation by a moving observer, and finally we demonstrate, in the simple case that the time process is a homogeneous Poisson process.

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Time Processes and Space Processes

Consider an n-lane one-way-road with no intersection. Let assume the following conditions.

(Condition A) The interaction between cars is negligible. Therefore, each car drives independently of any others.

(Condition B) The cars drive with their own constant speeds, and for sufficiently small h and sufficiently large H the cars which have velocities less than h or more than H are completely neglected.

Here we define $\Xi=[0, \Xi']$ as a space interval and $T=[0, T']$ as a time interval. Moreover we define $V=(0, \infty)$ as a velocity interval for the sake of formality, though, under the condition (B), V is $[h, H]$ in practical sense.

Let consider, on the probability space $(\Omega_\xi, B_\xi, P_\xi)$, the following stochastic process $x_\xi(t, v)$ which is called the time process at point ξ.

$x_\xi(t, v)$: the number of cars which pass through point $\xi\in\Xi$ at a time interval $[0, t]$ and whose velocities belong to a velocity interval $[0, v]$.

From the condition (A), $x_\xi(t, v)$ is obviously an integral-valued differential process. Particularly, we denote $x_\xi(t, \infty)$ as $x_\xi(t)$. Then it follows from the definitions of $x_\xi(t, v)$ and $x_\xi(t)$ that for arbitrary $\xi\in\Xi$, $v\in V$ and $s, t\ (s\leq t)\in T$,

$$x_\xi(t)-x_\xi(s) \geq x_\xi(t, v)-x_\xi(s, v) \geq 0 \qquad \text{w.p.1.}$$

Further we suppose the following condition (C) under which the process is absolutely continuous in probability.

(Condition C) For an arbitrary small $\varepsilon<0$, a positive number δ can be found such that for arbitrary $r=1, 2, \cdots$ and $s_1<t_1\leq s_2<t_2\leq \cdots s_r<t_r\leq T'$ for which $\sum_{j=1}^{r}(t_j-s_j)<\delta$, we have $\prod_{j=1}^{r} \boldsymbol{P}_\xi[\omega_\xi;\ x_\xi(t_j)-x_\xi(s_j)=0]>1-\varepsilon$, where ω_ξ is an element of Ω_ξ.

A. Renyi[2] has proved that an integral-valued differential process satisfying the condition (C) is an inhomogeneous composed Poisson process. Therefore, under the conditions (A) and (C), so is $x_\xi(t, v)$. According to the condition (C), $x_\xi(t, v)$ is obviously continuous in probability with respect to t, and then it follows from the definition of $x_\xi(t, v)$ that there is the standard modification of $x_\xi(t, v)$ which is separable and measurable[3].

Let suppose that $x_\xi(t, v)$ is separable and measurable. Moreover, we consider the following condition:

(Condition D) For $\xi\in\Xi$, $v\in V$ and $0\leq s<t\leq T'$,

$$\lim_{s\to t} \{\boldsymbol{P}_\xi[\omega_\xi;\ x_\xi(t, v)-x_\xi(s, v)\geq 1]/\boldsymbol{P}_\xi[\omega_\xi;\ x_\xi(t, v)-x_\xi(s, v)=1]\} = 1\ .$$

Under the conditions (A), (C) and (D), $x_\xi(t, v)$ is an inhomogeneous Poisson process. We denote the characteristic functions of $x_\xi(t)$ and $x_\xi(t, v)$ as $\varphi_\xi(t; z)$ and $\varphi_\xi(t, v; z)$, respectively. It follows that

$$\log \varphi_\xi(t;\ z) = \int_0^t \rho_\xi(u)\, du(e^{iz}-1) \qquad (1)$$

$$\log \varphi_\xi(t,\ v;\ z) = \int_0^t \rho_\xi^*(u,\ v)\, du(e^{iz}-1)\ , \qquad (2)$$

where $\rho_\xi(t)$ and $\rho_\xi^*(t, v)$ are non-negative valued Lebesgue integrable functions defined for almost everywhere with respect to t. Moreover we suppose that $\rho_\xi(t)$ and $\rho_\xi^*(t, v)$ are Borel measurable. We define a time velocity distribution function $F_\xi(t, v)$ as follows:

$$F_\xi(t,\ v) = \lim_{s\to t} \{\mathrm{E}[x_\xi(t,\ v)-x_\xi(s,\ v)]/\mathrm{E}[x_\xi(t)-x_\xi(s)]\} \quad \text{for} \quad 0\leq s<t\leq T'. \quad (3)$$

Obviously $F_\xi(t, v)$ is a non-decreasing function, continuous from the left and bounded by 0 and 1. It follows that $F_\xi(t, v)$ is a distribution function. If $\mathrm{E}[x_\xi(t)-x_\xi(s)]=0$, $F_\xi(u, v)$ for $u\in (s, t)]$ is a suitably chosen function of v satisfying the above mentioned properties. Thus defined $F_\xi(t, v)$ is equivalent to the following definition under the condition (D)

$$F_\xi(t,\ v) = \lim_{s\to t} \{\boldsymbol{P}_\xi[\omega_\xi;\ x_\xi(t,\ v)-x_\xi(s,\ v) = 1]/\boldsymbol{P}_\xi[\omega_\xi;\ x_\xi(t)-x_\xi(s) = 1]\}\ .$$

According to equation (3),

$$\rho_\xi^*(t,\ v) = F_\xi(t,\ v)\, \rho_\xi(t) \qquad \text{for} \quad t,\ a.e. \qquad (4)$$

Consequently,

$$\int_0^t \rho_\xi^*(u,\ v)\, du = \int_0^t F_\xi(u,\ v)\, \rho_\xi(u)\, du$$

Moreover, from equation (2),

$$\log \varphi_\xi(t,\ v;\ z) = \int_0^t F_\xi(u,\ v)\, \rho_\xi(u)\, du(e^{iz}-1) \qquad (5)$$

Let assume the following condition:

(Condition E) A time velocity distribution function $F_\xi(t, v)$ has not a singular part[4].

Under the condition (E), $F_\xi(t, v)$ is decomposed as follows:

$$F_\xi(t,\ v) = F_\xi^d(t,\ v)+F_\xi^c(t,\ v)\ ,$$

where $F_\xi^d(t, v)$ is purely discontinuous and $F_\xi^c(t, v)$ is absolutely continuous. Further,

$$F_\xi^d(t, v) = \sum_{v_i \in (0, v]} f_\xi^d(t, v_i) \quad \text{and} \quad F_\xi^c(t, v) = \int_0^v f_\xi(t, v)\, dv ,$$

where v_i is a point of discontinuity of $F_\xi^d(t, v)$ and $f_\xi^d(t, v_i)$ is a jump of $F_\xi^d(t, v)$ at v_i, and $f_\xi(t, v)$ is a non-negative valued Lebesgue integrable function defined for almost every v.

(Condition F) A time velocity density functions $f_\xi(t, v_i)$ and $f_\xi(t, v)$ are continuous in t.

Under the condition (F), $f_\xi(t, v_i)$ and $f_\xi(t, v)$ are measurable functions on $T \times V$. Therfore, from equation (2),

$$\log \varphi_\xi(t, v; z) = \int_0^t \{ \sum_{v_i \in (0, v]} f_\xi^d(u, v_i) + \int_0^v f_\xi(u, v)\, du\} \rho_\xi(u)\, du(e^{iz}-1)$$

$$= \{ \sum_{v_i \in (0, v]} \int_0^t f_\xi^d(u, v_i)\, \rho_\xi(u)\, du + \int_0^v \int_0^t f_\xi(u, v)\, \rho_\xi(u)\, du\, dv\} (e^{iz}-1) \qquad (6)$$

Here, we define a random interval function $x_\xi(I, K)$ as follows:

$x_\xi(I, K)$, the number of cars which pass through a point $\xi \in \Xi$ in a time interval $I=(s, t]$ and whose velocities belong to a velocity interval $K=(v, w]$.

If $K=V$, we denote $x_\xi(I, K)$ as $x_\xi(I)$. Further, we denote the characteristic function of $x_\xi(I)$ and $x_\xi(I, K)$ as $\varphi_\xi(I; z)$ and $\varphi_\xi(I, K; z)$, respectively.

If $I_1 \cap I_2 = \phi$ and $K_1 \cap K_2 = \phi$, then $x_\xi(I_1, K)$ and $x_\xi(I_2, K)$ are mutually independent, and so are $x_\xi(I, K_1)$ and $x_\xi(I, K_2)$, according to the Condition (A). Therefore, we have for $m=0, 1, 2, \cdots$

$$\log \varphi_\xi(I; z) = \int_I \rho_\xi(u)\, du(e^{iz}-1) , \qquad (7)$$

$$\boldsymbol{P}_\xi[\omega_\xi; x_\xi(I) = m] = \exp\, (-\int_I \rho_\xi(u)\, du) (\int_I \rho_\xi(u)\, du)^m/m! , \qquad (8)$$

$$\log \varphi_\xi(I, K; z) = \int_I \{ \sum_{V_i \in K} f_\xi^d(u, v_i) + \int_K f_\xi(u, v)\, dv\} \rho_\xi(u)\, du(e^{iz}-1)$$

$$= \left\{ \sum_{V_i \in K} \int_I f_\xi^d(u, v_i)\, \rho_\xi(u)\, du + \int_K \int_I f_\xi(u, v)\, \rho_\xi(u)\, du\, dv\right\} (e^{iz}-1) , \qquad (9)$$

and

$$\boldsymbol{P}_\xi[\omega_\xi; x_\xi(I, K) = m] = \exp\left[-\int_I \left\{ \sum_{V_i \in K} f_\xi^d(u, v_i) + \int_K f_\xi(u, v)\, dv\right\} \rho_\xi(u)\, du\right]$$

$$\times \left[\int_I \left\{ \sum_{V_i \in K} f_\xi^d(u, v_i) + \int_K f_\xi(u, v)\, dv\right\} \rho_\xi(u)\, du\right]^m /m! . \qquad (10)$$

In what follows, we ignore the condition (D). Then, similarly to equations (1) and (2), we have

$$\log \varphi_\xi(t;\, z) = \sum_{k=1}^{n} \int_0^t \rho_k(u;\, \xi)\, du (e^{ikz}-1)\,, \tag{11}$$

and
$$\log \varphi_\xi(t, v;\, z) = \sum_{k=1}^{n} \int_0^t \rho_k{}^*(u, v;\, \xi)\, du (e^{ikz}-1)\,, \tag{12}$$

where $\rho_k(t;\, \xi)$ and $\rho_k{}^*(t, v;\, \xi)$ are non-negative valued Lebesgue integrable and Borel measurable functions defined for almost every t, and are given by

$$\rho_k(t;\, \xi) = \lim_{s \uparrow t} \{\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi(t)-x_\xi(s)=k]/(t-s)\}$$

and
$$\rho_k{}^*(t, v;\, \xi) = \lim_{s \uparrow t} \{\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi(t, v)-x_\xi(s, v)=k]/(t-s)\}\,.$$

Then we define a function $F_k(t, v;\, \xi)$ as follows:

$$F_k(t, v;\, \xi) = \lim_{s \uparrow t} \{\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi(t, v)-x_\xi(s, v)=k]/\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi(t)-x_\xi(s)=k]\}\,. \tag{13}$$

$F_k(t, v;\, \xi)$ is a non-decreasing function, continuous from the left, and bounded by 0 and 1. It follows that $F_k(t, v\colon \xi)$ is a conditional distribution function. If $\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi(t)-x_\xi(s)=k]=0$, $F_k(u, v;\, \xi)$ for $u \in (s, t]$ is a properly chosen function satisfying the properties as described above.

Therefore, we have

$$\rho_k{}^*(t, v;\, \xi) = F_k(t, v;\, \xi)\, \rho_k(t;\, \xi) \qquad \text{for} \quad a.e.\, t \tag{14}$$

and then,

$$\int_0^t \rho_k{}^*(u, v;\, \xi)\, du = \int_0^t F_k(u, v;\, \xi)\, \rho_k(u;\, \xi)\, du\,.$$

It follows from (3) and (14) that, for a time velocity distribution function $F_\xi(t, v)$, we have

$$F_\xi(t, v) = \sum_{k=1}^{n} k F_k(t, v;\, \xi)\, \rho_k(t;\, \xi) \Big/ \sum_{k=1}^{n} k \rho_k(t;\, \xi)\,.$$

If $F_1(t, v;\, \xi) = \cdots = F_n(t, v;\, \xi)$, then $F_\xi(t, v) = F_k(t, v;\, \xi)$, $k=1, 2, \cdots n$.

Let suppose that $F_k(t, v;\, \xi)$ satisfies the conditions (E) and (F). Similarly to the equation (7)$\sim$(10), we have, for $m=0, 1, 2, \cdots$

$$\log \varphi_\xi(I;\, z) = \sum_{k=1}^{n} \int_I \rho_k(u;\, \xi)\, du (e^{ikz}-1)\,, \tag{15}$$

$$\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi(I)=m] = \exp\left\{-\sum_{k=1}^{n} \int_I \rho_k(u;\, \xi)\, du\right\} \sum_{r_1+2r_2+\cdots+nr_n=m} \prod_{k=1}^{n} \left\{\left(\int_I \rho_k(u;\, \xi)\, du\right)^{r_k} \Big/ r_k!\right\}, \tag{16}$$

$$\log \varphi_\xi(I, K;\, z) = \sum_{k=1}^{n} \int_I \left\{\sum_{v_i \in K} f_k{}^d(u, v_i;\, \xi) + \int_K f_k(u, v;\, \xi)\, dv\right\} \rho_k(u;\, \xi)\, du (e^{ikz}-1) \tag{17}$$

and

$$P_\xi[\omega_\xi; x_\xi(I, K)=m] = \exp\left\{-\sum_{k=1}^{n}\int_I \left(\sum_{V_i\in k} f_k^d(u, v_i; \xi)+\right.\right.$$
$$\left.\left.\int_K f_k(u, v; \xi)\,dv\right)\rho_k(u, \xi)\,du\right\}\sum_{r_1+2r_2+\cdots+nr_n=m}\prod_{k=1}^{n}\left[\left\{\int_I\left(\sum_{v_i\in K} f_k^d(u, v_i; \xi)+\right.\right.\right.$$
$$\left.\left.\left.\int_K f_k(u, v; \xi)\,dv\right)\rho_k(u; \xi)\,du\right\}^{r_k}\Big/ r_k!\right], \tag{18}$$

where $r_1, r_2, \cdots, r_n$ are non-negative integers. We have considered the time process $x_\xi(I, K)$ above.

In what follows, we consider a space process. First, we define a space process $y_t(J, K)$ on a probability space (Ω_t, B_t, P_t) as follows:

$y_t(J, K)$, the number of cars which exist in space interval $J=(\zeta, \xi]$ at time t and whose velocities belong to a velocity interval $K=(v, w]$.

Similarly to $x_\xi(I, K)$, we denote $y_t(J, K)$ as $y_t(J)$ for $K=V$, $y_t(\xi, v)$ for $J=[0, \xi]$ and $K=(0, v]$ and $y_t(\xi)$ for $J=[0, \xi]$ and $K=V$ respectively. Moreover we denote the characteristic functions of $y_t(J)$ and $y_t(J, K)$ as $\psi_t(J; z)$ and $\psi_t(J, K; z)$, respectively. Under similar conditions to (C), (D), (E) and (F), we have the following equations similar to the case of $\varphi_\xi(I; z)$ and $\varphi_\xi(I, K; z)$ for $m=0, 1, 2, \cdots$

$$\log \psi_t(J; z) = \int_J \lambda_t(\eta)\,d\eta\,(e^{iz}-1)\,, \tag{19}$$

$$P_t[\omega_t; y_t(J)=m] = \exp\left(-\int_J \lambda_t(\eta)\,d\eta\right)\left(\int_J \lambda_t(\eta)\,d\eta\right)^m\Big/ m!\,, \tag{20}$$

$$\log \psi_t(J, K; z) = \int_J\left\{\sum_{v_i\in K} g_t^d(\eta, v_i)+\int_K g_t(\eta, v)\,dv\right\}\lambda_t(\eta)\,d\eta\,(e^{iz}-1) \tag{21}$$

and
$$P_t[\omega_t; y_t(J, K)=m] = \exp\left[-\int_J\left\{\sum_{v_i\in K} g_t^d(\eta, v_i)+\right.\right.$$
$$\left.\left.\int_K g_t(\eta, v)\,dv\right\}\lambda_t(\eta)\,d\eta\right]\left[\int_J\left\{\sum_{v_i\in K} g_t^d(\eta, v_i)+\int_K g_t(\eta, v)\,dv\right\}\lambda_t(\eta)\,d\eta\right]^m\Big/ m!\,, \tag{22}$$

where $g_t^d(\xi, v)$ is a purely discontinuous part of a space velocity distribution function $G_t(\xi, v)$ defined similarly by equation (3), and $g_t(\xi, v)$ is a Radon-Nykodym's derivative of an absolutely continuous part of $G_t(\xi, v)$.

Under similar conditions to (C), (E) and (F), we have the following equations:

$$\log \psi_t(J; z) = \sum_{k=1}^{n}\int_J \lambda_k(\eta; t)\,d\eta\,(e^{ikz}-1)\,, \tag{23}$$

$$P_t[\omega_t; y_t(J)=m] = \exp\left\{-\sum_{k=1}^{n}\int_J \lambda_k(\eta; t)\,d\eta\right\}\sum_{r_1+2r_2+\cdots+nr_n=m}\prod_{k=1}^{n}$$
$$\left(\int_J \lambda_k(\eta; t)\,d\eta\right)^{r_k}\Big/ r_k!\,, \tag{24}$$

$$\log \psi_t(J, K; z) = \sum_{k=1}^{n} \int_J \left\{ \sum_{v_i \in K} g_k^d(\eta, v_i; t) + \int_K g_k(\eta, v; t)\, dv \right\} \lambda_k(\eta; t)\, d\eta (e^{ikz} - 1) \tag{25}$$

and

$$P_t[\omega_t; y_t(J, K) = m] = \exp\left[-\sum_{k=1}^{n} \int_J \left\{ \sum_{V_i \in K} g_k^d(\eta, v_i; t) + \int_K g_k(\eta, v; t)\, dv \right\} \lambda_k(\eta; t)\, d\eta \right] \sum_{r_1 + \cdots + nr_n = m} \prod_{k=1}^{n} \left[\int_J \left\{ \sum_{v_i \in K} g_k^d(\eta, v_i; t) + \int_K g_k(\eta, v; t)\, dv \right\} \lambda_k(\eta; t)\, d\eta \right]^{r_k} / r_k! \,. \tag{26}$$

Various Transformations

We deal with the transformations from a time process to a space process, from a time process to a time process, from a space process to a time process and from a space process to a space process. At first, we define that $x_\xi^d(I, v_i)$ is the number of cars which pass through a space point ξ in a time interval I with a certain constant velocity v_i. Similarly we define $y_t^d(J, v_i)$.

Now we propose to interpret the notion of velocity as a one to one measure preserving transformation, in case the car traffic flow satisfies the condition (B). That is, for $\theta < \zeta < \xi$, $r < s < t$, $v_i \in V$, $I = (r, s]$, $J = (\theta, \zeta]$ and $m = 0, 1, 2, \cdots$,

$$P_\theta[\omega_\theta; x_\theta^d(I, v_i) = m] = P_\xi[\omega_\xi; x_\xi^d(I + (\xi - \theta)/v_i, v_i) = m] \,, \tag{27}$$

where $I + (\xi - \theta)/v_i = (r + (\xi - \theta)/v_i,\ s + (\xi - \theta)/v_i]$,

$$P_\theta[\omega_\theta; x_\theta^d(I, v_i) = m] = P_t[\omega_t; y_t^d(J', v_i) = m] \,, \tag{28}$$

where $J' = [\theta + (t - s) v_i,\ \theta + (t - r) v_i)$,

$$P_r[\omega_r; y_r^d(J, v_i) = m] = P_t[\omega_t; y_t^d(J + (t - r) v_i, v_i) = m] \,, \tag{29}$$

where $J + (t - r) v_i = (\theta + (t - r) v_i,\ \zeta + (t - r) v_i]$,

and

$$P_r[\omega_r; y_r^d(J, v_i) = m] = P_\xi[\omega_\xi; x_\xi^d(I', v_i) = m] \,, \tag{30}$$

where $I' = [r + (\xi - \zeta)/v_i,\ r + (\xi - \theta)/v_i)$.

Consequently, we can obtain the following theorems.

(Theorem 1) Under conditions (A)$\sim$(F), suppose that $F_\xi(t, v)$ and $G_t(\xi, v)$ are purely discontinuous. Then, for $\theta \leq \zeta < \xi$, $r \leq s < t$, $v < w$, $I = (s, t]$, $J = (\zeta, \xi]$ and $K = (v, w]$, we have

$$\log \varphi_\xi(I, K; z) = (e^{iz} - 1) \left[\sum_{v_i \in K_1} \int_I v_i g_r^d(\xi + v_i(r - u), v_i) \lambda_r(\xi + v_i(r - u))\, du + \sum_{v_i \in K_2} \left\{ \int_{I_1} v_i g_r^d(\xi + v_i(r - u), v_i)\, \lambda_r(\xi + v_i(r - u)) du + \right.\right.$$

$$\int_{I_2} f_\theta^d(u-(\xi-\theta)/v_i, v_i)\,\rho_\theta(u-(\xi-\theta)/v_i)\,du\Big\}+$$
$$\sum_{v_i\in K_3}\int_I f_\theta^d(u-(\xi-\theta)/v_i, v_i)\,\rho_\theta(u-(\xi-\theta)/v_i)\,du\Big],$$

where $K_1=(v, (\xi-\theta)/(t-r))$, $K_2=((\xi-\theta)/(t-r), (\xi-\theta)/(s-r)]$,
$K_3=((\xi-\theta)/(s-r), w]$, $I_1=(s, r+(\xi-\theta)/v_i]$ and $I_2=(r+(\xi-\theta)/v_i, t]$,

and $$\log\psi_t(J, K; z)=(e^{iz}-1)\Big[\sum_{v_i\in K_1}\int_J g_r^d(\eta-v_i(t-r), v_i)\,\lambda_r(\eta-v_i(t-r))\,d\eta+$$
$$\sum_{v_i\in K_2}\Big\{\int_{J_1} f_\theta^d(t+(\theta-\eta)/v_i)\,\rho_\theta(t+(\theta-\eta)/v_i)/v_i\,d\eta+\int_{J_2} g_r^d(\eta-v_i(t-r), v_i)$$
$$\lambda_r(\eta-v_i(t-r))\,d\eta\Big\}+\sum_{v_i\in K_3}\int_J f_\theta^d(t+(\theta-\eta)/v_i, v_i)\,\rho_\theta(t+(\theta-\eta)/v_i)/v_i\,d\eta\Big],$$

where $K_1=(v, (\zeta-\theta)/(t-r)]$, $K_2=((\zeta-\theta)/(t-r), (\xi-\theta)/(t-r)]$,
$K_3=((\xi-\theta)/(t-r), w]$, $J_1=(\zeta, \theta+v_i(t-r)]$ and $J_2=(\theta+v_i(t-r), \xi]$.

(Proof). At first, we prove the first expression. Since $F_\xi(t, v)$ is purely discontinuous, $F_\xi(t, v)=F_\xi^d(t, v)=\sum_{v_i\in(0,v)} f_\xi^d(t, v_i)$. Therefore, according to (9),

$$\log\varphi_\xi(I, K; z)=\sum_{v_i\in K}\int_I f_\xi^d(u, v_i)\,\rho_\xi(u)\,du(e^{iz}-1).$$

Now we divide K into K_i $(i=1, 2, 3)$, where $K_1=(v, (\xi-\theta)/(t-r)]$,
$K_2=((\xi-\theta)/(t-r), (\xi-\theta)/(s-r)]$ and $K_3=((\xi-\theta)/(s-r), w]$.
Then it follows from a one to one measure preserving transformation which we have shown that if $v_i\in K_1$, then according to (28),

$$\boldsymbol{P}_\xi[\omega_\xi; x_\xi^d(I, v_i)=m]=\boldsymbol{P}_r[\omega_r; y_r^d(J', v_i)=m],$$

where $J'=[\xi+(r-t)v_i, \xi+(r-s)v_i)$, and then for any $I\in B_\xi$,

$$\int_I f_\xi^d(u, v_i)\,\rho_\xi(u)\,du(e^{iz}-1)=\int_{J'} g_r^d(\eta, v_i)\,\lambda_r(\eta)\,d\eta(e^{iz}-1)=$$
$$\int_I v_i g_r^d(\xi+(r-u)v_i, v_i)\,\lambda_r(\xi+(r-u)v_i)\,du(e^{iz}-1),\quad\text{consequently,}$$

$$f_\xi^d(t, v_i)\,\rho_\xi(t)=v_i g_r^d(\xi+(r-t)v_i, v_i)\,\lambda_r(\xi+(r-t)v_i)\quad\text{for almost every } t,$$

and if $v_i\in K_3$, then according to (27),

$$\boldsymbol{P}_\xi[\omega_\xi; x_\xi^d(I, v_i)=m]=\boldsymbol{P}_\theta[\omega_\theta; x_\theta^d(I-(\xi-\theta)/v_i, v_i)=m]$$

and then,

$$\int_I f_\xi^d(u, v_i)\,\rho_\xi(u)\,du(e^{iz}-1)=\int_{I-(\xi-\theta)/v_i} f_\theta^d(u, v_i)\,\rho_\theta(u)\,du(e^{iz}-1)=$$
$$\int_I f_\theta^d(u-(\xi-\theta)/v_i, v_i)\,\rho_\theta(u-(\xi-\theta)/v_i)\,du(e^{iz}-1),\quad\text{consequently,}$$

$$f_\xi^d(t, v_i)\,\rho_\xi(t) = f_\theta^d(t-(\xi-\theta)/v_i,\, v_i)\,\rho_\theta(t-(\xi-\theta)/v_i) \quad \text{for almost every } t.$$

If $v_i \in K_2$, we divide I as $I=I_1+I_2$, where $I_1=(s,\, r+(\xi-\theta)/v_i]$ and $I_2=(r+(\xi-\theta)/v_i,\, t]$. Then,

$$\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi^d(I_1, v_i)=m] = \boldsymbol{P}_r[\omega_r;\, y_r^d(J_1', v_i)=m]\,,$$

where $J_1'=[\theta,\, \xi+(r-s)\,v_i)$, and $\boldsymbol{P}_\xi[\omega_\xi;\, x_\xi^d(I_2, v_i)=m] = \boldsymbol{P}_\theta[\omega_\theta;\, x_\theta^d(I_2', v_i)=m]$, where $I_2' = (r,\, t-(\xi-\theta)/v_i]$. Hence, if $v_i \in K_2$,

then $\int_I f_\xi^d(u, v_i)\,\rho_\xi(u)\,du(e^{iz}-1) = \Big\{\int_{J_1'} g_r^d(\eta, v_i)\,\lambda_r(\eta)\,d\eta +$

$\int_{I_2'} f_\theta^d(u, v_i)\rho_\theta(u)du\Big\}(e^{iz}-1) = \Big\{\int_{I_1} v_i g_r^d(\xi+(r-u)v_i, v_i)\lambda_r(\xi+(r-u)v_i)du +$

$\int_{I_2} f_\theta^d(u-(\xi-\theta)/v_i,\, v_i)\,\rho_\theta(u-(\xi-\theta)/v_i)\,du\Big\}(e^{iz}-1)\,.$

While, $\log \varphi_\xi(I, K;\, z) = \sum_{j=1}^{n}\Big\{\sum_{v_i \in K_j}\int_I f_\xi^d(u, v)\,\rho_\xi(u)\,du(e^{iz}-1)\Big\}\,.$

Consequently,

$$\log \varphi_\xi(I, K;\, z) = \Big[\sum_{v_i\in K_1}\int_I v_i g_r^d(\xi+(r-u)\,v_i,\, v_i)\,\lambda_r(\xi+(r-u)\,v_i)\,du +$$

$$\sum_{v_i\in K_2}\Big\{\int_{I_1} v_i g_r^d(\xi+(r-u)\,v_i,\, v_i)\,\lambda_r(\xi+(r-u)\,v_i)\,du + \int_{I_2} f_\theta^d(u-(\xi-\theta)/v_i,\, v_i)$$

$$\rho_\theta(u-(\xi-\theta)/v_i)\,du\Big\} + \sum_{v_i\in K_3}\int_I f_\theta^d(u-(\xi-\theta)/v_i, v_i)\rho_\theta(u-(\xi-\theta)/v_i)du\Big](e^{iz}-1)\,.$$

The second expression of the theorem is similarly derived as follows. Note that

$$\log \psi_t(J, K;\, z) = \sum_{j=1}^{3}\Big\{\sum_{v_i\in K_j}\int_J g_t^d(\eta, v_i)\lambda_t(\eta)d\eta(e^{iz}-1)\Big\}\,,$$

where $K_1=(v,\, (\zeta-\theta)/(t-r)]$, $K_2=((\zeta-\theta)/(t-r),\, (\xi-\theta)/(t-r)]$ and $K_3=((\xi-\theta)/(t-r),\, w]$. If $v_i \in K_1$, then according to (29),

$$\boldsymbol{P}_t[\omega_t;\, y_t^d(J, v_i) = m] = \boldsymbol{P}_r[\omega_r;\, y_r^d(J-v_i(t-r),\, v_i) = m]$$

Therefore for any $J \in B_t$,

$$\int_J g_t^d(\eta, v_i)\lambda_t(\eta)d\eta = \int_J g_r^d(\eta-(t-r)v_i,\, v_i)\lambda_r(\eta-(t-r)v_i)d\eta\,,$$

consequently, $g_t^d(\xi, v_i)\lambda_t(\xi) = g_r^d(\xi-(t-r)v_i,\, v_i)\lambda_r(\xi-(t-r)v_i)$ for almost every ξ. If $v_i \in K_3$, then according to (30),

$\boldsymbol{P}_t[\omega_t;\, y_t^d(J, v_i) = m] = \boldsymbol{P}_\theta[\omega_\theta;\, x_\theta^d(I', v_i) = m]$, where $I' = [t-(\xi-\theta)/v_i,\, t-(\zeta-\theta)/v_i)$, Therefore for any $J \in B_t$,

$$\int_J g_t^d(\eta, v_i)\lambda_t(\eta)d\eta = \int_J f_\theta^d(t-(\eta-\theta)/v_i, v_i)\rho_\theta(t-(\eta-\theta)/v_i)/v_i d\eta ,$$

consequently, $g_t^d(\xi, v_i)\lambda_t(\xi) = f_\theta^d(t-(\xi-\theta)/v_i, v_i)\rho_\theta(t-(\xi-\theta)/v_i)/v_i$ for almost every ξ. And if $v_i \in K_2$, then

$$\int_J g_t^d(\eta, v_i)\lambda_t(\eta)d\eta = \int_{J_1} f_\theta^d(t-(\eta-\theta)/v_i, v_i)\rho_\theta(t-(\eta-\theta)/v_i)/v_i d\eta + \int_{J_2} g_r^d(\eta-(t-r)v_i, v_i)\lambda_r(\eta-(t-r)v_i)d\eta ,$$

where $J_1 = (\zeta, \theta+(t-r)v_i]$ and $J_2 = (\theta+(t-r)v_i, \xi]$. Hence,

$$\log \psi_t(J, K; z) = (e^{iz}-1)\Big[\sum_{v_i\in K_1}\int_J g_r^d(\eta-v_i(t-r), v_i)\lambda_r(\eta-v_i(t-r))d\eta + \sum_{v_i\in K_2}\Big\{\int_{J_1} f_\theta^d(t+(\theta-\eta)/v_i)\rho_\theta(t+(\theta-\eta)/v_i)/v_i d\eta + \int_{J_2} g_r^d(\eta-v_i(t-r), v_i)\lambda_r(\eta-v_i(t-r))d\eta\Big\} + \sum_{v_i\in K_3}\int_J f_\theta^d(t+(\theta-\eta)/v_i, v_i)\rho_\theta(t+(\theta-\eta)/v_i)/v_i d\eta\Big],$$

as was to be proved.

This theorem shows that any time process or space process is constructed by the preceding time process and space process.

Suppose that $F_\xi(t, v)$ and $G_t(\xi, v)$ are absolutely continuous with respect to Lebesgue measure on V. Then we can obtain the following theorem.

(Theorem 2) Under the conditions (A)$\sim$(F), suppose that $F_\xi(t, v)$ and $G_t(\xi, v)$ are absolutely continuous with respect to Lebesgue measure on V. Then, for $\theta \le \zeta < \xi$, $r < s < t$, $v < w$, $I = (s, t]$, $J = (\zeta, \xi]$ and $K = (v, w]$, we have

$$\log \varphi_\xi(I, K; z) = (e^{iz}-1)\Big[\int_{K_1}\int_I v g_r(\xi+v(r-u), v)\lambda_r(\xi+v(r-u))du\,dv + \int_{K_2}\Big\{\int_{I_1} v g_r(\xi+v(r-u), v)\lambda_r(\xi+v(r-u))du + \int_{I_2} f_\theta(u-(\xi-\theta)/v, v)\rho_\theta(u-(\xi-\theta)/v)du\Big\}dv + \int_{K_3}\int_I f(u-(\xi-\theta)/v, v)\rho_\theta(u-(\xi-\theta)/v)du\,dv\Big],$$

where $K_1 = (v, (\xi-\theta)/(t-r)]$, $K_2 = ((\xi-\theta)/(t-r), (\xi-\theta)/(s-r)]$,

$K_3 = ((\xi-\theta)/(s-r), w]$, $I_1 = (s, r+(\xi-\theta)/v]$

and $I_2 = (r+(\xi-\theta)/v, t)]$,

and

$$\log \psi_t(J, K; z) = (e^{iz}-1)\Big[\int_{K_1}\int_J g_r(\eta-v(t-r), v)\lambda_r(\eta-v(t-r))\eta d dv + \int_{K_2}\Big\{\int_{J_1} f_\theta(t+(\theta-\eta)/v, v)\rho_\theta(t+(\theta-\eta)/v)/v d\eta + \int_{J_2} g_r(\eta-v(t-r), v)\lambda_r$$

$$(\eta-v(t-r))d\eta\Big\}dv+\int_{K_3}\int_J f_\theta(t+(\theta-\eta)/v,\,v)\rho_\theta(t+(\theta-\eta)/v)/vd\eta\Big],$$

where $K_1=(v,\,(\zeta-\theta)/(t-r)]$, $K_2=((\zeta-\theta)/(t-r),\,(\xi-\theta)/(t-r)]$, $K_3=((\xi-\theta)/(t-r),\,w]$, $J_1=(\zeta,\,\theta+v(t-r)]$
and $J_2=(\theta+v(t-r),\,\xi]$.

(Proof). Let prove the first formula of this theorem. Since $F_\xi(t,v)$ is absolutely continuous, according to (9), we have

$$\log\varphi_\xi(I,K;z)=\sum_{i=1}^{3}\int_{K_i}\int_I f_\xi(u,v)\rho_\xi(u)du\,dv(e^{iz}-1),$$

where $K_1=(v,\,(\zeta-\theta)/(t-r)]$, $K_2=((\zeta-\theta)/(t-r),\,(\xi-\theta)/(s-r)]$
and $K_3=((\xi-\theta)/(s-r),\,w]$.

We take properly chosen countable sets $\{v_{ij}\}\in K_i$ $(i=1,2,3)$ as follows:

$v=v_{10}<\cdots<v_{1p_1}=(\zeta-\theta)/(t-r)=v_{20}<\cdots<v_{2p_2}=(\xi-\theta)/(s-r)=v_{30}<\cdots<v_{3p_3}=w$. For $\{v_{ij}\}$ $(i=1,2,3)$, we define the process $x_\xi^d(I,K_i)$ as $x_\xi^d(I,K_i)=\sum_j x_\xi^d(I,v_{ij})$ and its characteristic function as $\varphi_\xi^d(I,K_i;z)$.

Then, we have

$$\log\varphi_\xi^d(I,K_i;z)=(e^{iz}-1)\sum_j\int_I f_\xi(u,v_{ij})(v_{ij+1}-v_{ij})\rho_\xi(u)du.$$

According to the proof of the theorem 1,

$$\log\varphi_\xi^d(I,K_1;z)=(e^{iz}-1)\sum_j\int_{J'}g_r(\eta,v_{1j})(v_{1j+1}-v_{1j})\lambda_r(\eta)d\eta,$$

where $J'=[\xi+v_{1j}(r-t),\,\xi+v_{1j}(r-s))$,

$$\log\varphi_\xi^d(I,K_2;z)=(e^{iz}-1)\sum_j\Big\{\int_{J_1'}g_r(\eta,v_{2j})(v_{2j+1}-v_{2j})\lambda_r(\eta)d\eta+\int_{I_2'}f_\theta(u,v_{2j})(v_{2j+1}-v_{2j})\rho_\theta(u)du\Big\},$$

where $J_1'=[\theta,\,\xi+v_{2j}(r-s))$ and $I_2'=(r,\,t-(\xi-\theta)/v_{2j}]$,

and $$\log\varphi_\xi(I,K_3;z)=(e^{iz}-1)\sum_j\int_{I'}f_\theta(u,v_{3j})(v_{3j+1}-v_{3j})\rho_\theta(u)du,$$

where $I'=(s-(\xi-\theta)/v_{3j},\,t-(\xi-\theta)/v_{3j}]$.

It is to be noted that for the characteristic function $\varphi_\xi^d(I,K;z)$ of $x_\xi^d(I,K)$ defined by $x_\xi^d(I,K)=\sum_{i=1}^{3}x_\xi^d(I,K_i)$,

$$\log\varphi_\xi^d(I,K;z)=\sum_{i=1}^{3}\log\varphi_\xi^d(I,K_i;z).$$

Let $\delta = \max_i \max_j (v_{ij+1} - v_{ij})$. It follows from the separability of $x_\xi(I, K)$ that for properly chosen countable sets $\{v_{ij}\}$, $x_\xi^d(I, K)$ converges to $x_\xi(I, K)$ with probability 1 as $\delta \to 0$. Therefore,

$$\log \varphi_\xi(I, K; z) = (e^{iz}-1)\Big[\int_{K_1}\int_{J'} g_r(\eta, v_{1j})(v_{1j+1}-v_{1j})\lambda_r(\eta)d\eta\, dv +$$
$$\int_{K_2}\Big\{\int_{J_1'} g_r(\eta, v_{2j})(v_{2j+1}-v_{2j})\lambda_r(\eta)d\eta + \int_{I_2'} f_\theta(u, v_{2j})(v_{2j+1}-v_{2j})$$
$$\rho_\theta(u)du\Big\}dv + \int_{K_3}\int_{I'} f_\theta(u, v_{3j})(v_{3j+1}-v_{3j})\rho_\theta(u)du\, dv\Big],$$

which is continuous at $z=0$. Consequently,

$$\log \varphi_\xi(I, K; z) = (e^{iz}-1)\Big[\int_{K_1}\int_I vg_r(\xi+v(r-u), v)\lambda_r(\xi+v(r-u))du\, dv +$$
$$\int_{K_2}\Big\{\int_{I_1} vg_r(\xi+v(r-u), v)\lambda_r(\xi+v(r-u))du + \int_{I_2} f_\theta(u-(\xi-\theta)/v, v)$$
$$\rho_\theta(u-(\xi-\theta)/v)du\Big\}dv + \int_{K_3}\int_I f_\theta(u-(\xi-\theta)/v, v)\rho_\theta(u-(\xi-\theta)/v)du\, dv\Big],$$

where $I_1 = (s, r+(\xi-\theta)/v]$ and $I_2 = (r+(\xi-\theta)/v, t]$.

Similarly, the second expression of the theorem can be proved. We shall omit the proof to avoid a conventional complication.

We easily obtain the following corollary with some modifications.

(Corollary 1)

Under the same conditions as theorem 2,

1) if $h \geq (\xi-\theta)/(s-r)$, then, $\log \varphi_\xi(I, K; z) = (e^{iz}-1)\int_K\int_I f_\theta(u-(\xi-\theta)/v, v)\rho_\theta(u-(\xi-\theta)/v)\, du\, dv$, and $f_\xi(t, v)\rho_\xi(t) = f_\theta(t-(\xi-\theta)/v, v)\rho_\theta(t-(\xi-\theta)/v)$ for almost every t and v,

2) if $H \leq (\xi-\theta)/(t-r)$, then, $\log \varphi_\xi(I, K; z) = (e^{iz}-1)\int_K\int_I vg_r(\xi+v(r-u), v)\lambda_r(\xi+v(r-u))du\, dv$ and $f_\xi(t, v)\rho_\xi(t) = vg_r(\xi+v(r-t), v)\lambda_r(\xi+v(r-t))$ for almost every t and v,

3) if $h \geq (\xi-\theta)/(t-r)$, then, $\log \psi_t(J, K; z) = (e^{iz}-1)\int_K\int_J f_\theta(t+(\theta-\eta)/v, v)\rho_\theta(t+(\theta-\eta)/v)/v\, d\eta\, dv$ and $g_t(\xi, v)\lambda_t(\xi) = f_\theta(t-(\xi-\theta)/v, v)\rho_\theta(t-(\xi-\theta)/v)$ for almost every ξ and v,

and

4) if $H \leq (\zeta-\theta)/(t-r)$, then, $\log \psi_t(J, K; z) = (e^{iz}-1)\int_K\int_J g_r(\eta+v(r-t), v)\lambda_r(\eta+v(r-t))d\eta\, dv$ and $g_t(\xi, v)\lambda_t(\xi) = g_r(\xi+v(r-t), v)\lambda_r(\xi+v(r-t))$ for almost every ξ and v.

We shall consider the more generalized case, that is, we suppose the conditions (A), (B), (C), (E) and (F), ignoring the condition (D). Then we can obtain the following theorem.

(Theorem 3)

Under conditions (A), (B), (C), (E) and (F), for $\theta \leq \zeta < \xi$, $r < s < t$, $v < w$, $I=(s, t]$, $J=(\zeta, \xi]$ and $K=(v, w]$, we have

$$\log \varphi_\xi(I, K; z) = \sum_{k=1}^{n} (e^{ikz}-1)\Big[\int_I \Big\{ \sum_{v_i \in K_1} v_i g_k{}^d(\xi+v_i(r-u), v_i; r)$$
$$\lambda_k(\xi+v_i(r-u); r) + \int_{K_1} v g_k(\xi+v(r-u), v; r)\lambda_k(\xi+v(r-u); r)dv\Big\}du +$$
$$\sum_{v_i \in K_2} \Big\{\int_{I_1'} v_i g_k{}^d(\xi+v_i(r-tu), v_i; r)\lambda_k(\xi+v_i(r-u); r)du +$$
$$\int_{I_2'} f_k{}^d(u-(\xi-\theta)/v_i, v_i; \theta)\rho_k(u-(\xi-\theta)/v_i; \theta)\, du\Big\} +$$
$$\int_{K_2} \Big\{\int_{I_1} v g_k{}^d(\xi+v(r-u), v; r)\lambda_k(\xi+v(r-u); r)du +$$
$$\int_{I_2} f_k(u-(\xi-\theta)/v, v; \theta)\rho_k(u-(\xi-\theta)/v; \theta)du\Big\}dv +$$
$$\int_I \Big\{\sum_{v_i \in K_3} f_k{}^d(u-(\xi-\theta)/v_i, v_i; \theta)\rho_k(u-(\xi-\theta)/v_i; \theta) +$$
$$\int_{K_3} f_k(u-(\xi-\theta)/v, v; \theta)\rho_k(u-(\xi-\theta)/v; \theta)dv\Big\}du\Big],$$

where $K_1 = (v, (\xi-\theta)/(t-r)]$, $K_2 = ((\xi-\theta)/(t-r), (\xi-\theta)/(s-r)]$,
$K_3 = ((\xi-\theta)/(s-r), w]$, $I_1' = (s, r+(\xi-\theta)/v_i]$,
$I_2' = (r+(\xi-\theta)/v_i, t]$, $I_1 = (s, r+(\xi-\theta)/v]$ and
$I_2 = (r+(\xi-\theta)/v, t]$,

and

$$\log \psi_t(J, K; z) = \sum_{k=1}^{n} (e^{ikz}-1)\Big[\int_J \Big\{ \sum_{v_i \in K_1} g_k{}^d(\eta-v_i(t-r), v_i; r)$$
$$\lambda_k(\eta-v_i(t-r); r) + \int_{K_1} g_k(\eta-v(t-r), v; r)\lambda_k(\eta-v(t-r); r)dv\Big\}d\eta +$$
$$\sum_{v_i \in K_2} \Big\{\int_{J_1'} f_k{}^d(t+(\theta-\eta)/v_i, v_i; \theta)\rho_k(t+(\theta-\eta)/v_i; \theta)/v_i)d\eta +$$
$$\int_{J_2'} g_k{}^d(\eta-v_i(t-r), v_i; r)\lambda_k(\eta-v_i(t-r); r)d\eta\Big\} +$$
$$\int_{K_2} \Big\{\int_{J_1} f_k(t+(\theta-\eta)/v, v; \theta)\rho_k(t+(\theta-\eta)/v; \theta)/v)d\eta +$$
$$\int_{J_2} g_k(\eta-v(t-r), v; r)\lambda_k(\eta-v(t-r); r)d\eta\Big\}dv +$$
$$\int_J \Big\{\sum_{v_i \in K_3} f_k{}^d(t+(\theta-\eta)/v_i, v_i; \theta)\rho_k(t+(\theta-\eta)/v_i; \theta)/v_i +$$

$$\int_{K_3} f_k(t+(\theta-\eta)/v,\ v;\ \theta)\rho_k(t+(\theta-\eta)/v;\ \theta)/v\,du\Big\}d\eta\Big],$$

where $K_1=(v,\ (\zeta-\theta)/(t-r)]$, $K_2=((\zeta-\theta)/(t-r),\ (\xi-\theta)/(t-r)]$, $K_3=((\xi-\theta)/(t-r),\ w]$, $J_1'=(\zeta,\ \theta+v_i(t-r)]$, $J_2'=(\theta+v_i(t-r),\ \xi]$, $J_1=(\zeta,\ \theta+v(t-r)]$ and $J_2=(\theta+v(t-r),\ \xi]$.

If $\zeta<\xi<\theta$ and $s<t<r$, then the first expression is valid for $K_1=(v,\ (\theta-\xi)/(r-s)]$, $K_2=((\theta-\xi)/(r-s),\ (\theta-\xi)/(r-t)]$ and $K_3=((\theta-\xi)/(r-t),\ w]$ and the second expression is valid for $K_1=(v,\ (\theta-\xi)/(r-t)]$, $K_2=((\theta-\xi)/(r-t),\ (\theta-\zeta)/(r-t)]$ and $K_3=((\theta-\zeta)/(r-t),\ w]$.

(Proof). Let prove the first expression of the theorem. It follows from equation (17) that

$$\log\varphi_\xi(I,\ K;\ z)=\sum_{k=1}^{n}(e^{ikz}-1)\int_I\Big\{\sum_{v_i\in K}f_k^d(u,\ v_i;\ \xi)+\int_K f_k(u,\ v;\ \xi)dv\Big\}\rho_k(u;\ \xi)du\,.$$

We divide K into $K_r(r=1,2,3)$, where $K_1=(v,\ (\xi-\theta)/(t-r)]$, $K_2=((\xi-\theta)/(t-r),\ (\xi-\theta)/(s-r)]$ and $K_3=((\xi-\theta)/(s-r),\ w]$.

Next, for properly chosen countable sets $\{v_{rj}\}$, in similar manner to the proof of theorem 2, we define $x_\xi^d(I,\ K_r)$ and $\varphi_\xi^d(I,\ K_r;\ z)$. Hence,

$$\log\varphi_\xi^d(I,\ K_r;\ z)=\sum_{k=1}^{n}(e^{ikz}-1)\int_I\Big\{\sum_{v_i\in K_r}f_k^d(u,\ v_i;\ \xi)+\sum_j f_k(u,\ v_{rj};\ \xi)(v_{rj+1}-v_{rj})\Big\}\rho_k(u;\ \xi)du\,.$$

Therefore

$$\log\varphi_\xi^d(I,\ K;\ z)=\sum_{r=1}^{3}\log\varphi_\xi^d(I,\ K_r;\ z)=\sum_{k=1}^{n}(e^{ikz}-1)\Big[\sum_{v_i\in K_1}\int_{J(v_i)}g_k^d(\eta,\ v_i;\ r)\lambda_k(\eta;\ r)d\eta+\sum_j\int_{J(v_{1j})}g_k(\eta,\ v_{1j};\ r)\lambda_k(\eta;\ r)(v_{1j+1}-v_{1j})d\eta+$$
$$\sum_{v_i\in K_2}\Big\{\int_{J_1(v_i)}g_k^d(\eta;\ v_i;\ r)\lambda_k(\eta;\ r)d\eta+\int_{I_2(v_i)}f_k^d(u,\ v_i;\ \theta)\rho_k(u;\ \theta)du+$$
$$\sum_j(v_{2j+1}-v_{2j})\Big\{\int_{J_1(v_{2j})}g_k(\eta,\ v_{2j};\ r)\lambda_k(\eta;\ r)d\eta+$$
$$\int_{I_2(v_{2j})}f_k(u,\ v_{2j};\ \theta)\rho_k(u;\ \theta)du\Big\}+\sum_{v_i\in K_3}\int_{I(v_i)}f_k^d(u,\ v_i;\ \theta)\rho_k(u;\ \theta)du+$$
$$\sum_j\int_{I(v_{3j})}f_k(u,\ v_{3j};\ \theta)\rho_k(u;\ \theta)(v_{3j+1}-v_{3j})du\Big],$$

where $J(v)=(\xi+v(r-t),\ \xi+v(r-s)]$, $J_1(v)=(\theta,\ \xi+v(r-s)]$, $I_2(v)=(r,\ t-(\xi-\theta)/v]$ and $I(v)=(s-(\xi-\theta)/v,\ t-(\xi-\theta)/v]$. It follows that, similarly to the proof of theorem 2,

$$\log \varphi_\xi(I, K;\ z) = \sum_{k=1}^{n}(e^{ikz}-1)\Big[\sum_{v_i\in K_1}\int_I v_i g_k{}^d(\xi(v_i),\ v_i;\ r)\lambda_k(\xi(v_i);\ r)du+$$
$$\int_{K_1}\int_I v g_k(\xi(v),\ v;\ r)\lambda_k(\xi(v);\ r)du\,dv+\sum_{v_i\in K_2}\Big\{\int_{I_1'} v_i g_k{}^d(\xi(v_i),\ v_i;\ r)\lambda_k(\xi(v_i);\ r)du+$$
$$\int_{I_2'} f_k{}^d(t(v_i),\ v_i;\ \theta)\rho_k(t(v_i);\ \theta)\,du\Big\}+\int_{K_2}\Big\{\int_{I_1} v g_k(\xi(v),\ v;\ r)\lambda_k(\xi(v);\ r)du+$$
$$\int_{I_2} f_k(t(v),\ v;\ \theta)\rho_k(t(v);\ \theta)du\Big\}dv+\sum_{v_i\in K_3}\int_I f_k{}^d(t(v_i),\ v_i;\ \theta)\rho_k(t(v_i);\ \theta)du+$$
$$\int_{K_3}\int_I f_k(t(v),\ v;\ \theta)\rho_k(t(v);\ \theta)du\,dv\Big],$$

where $\xi(v) = \xi+v(r-u)$ and $t(v) = u-(\xi-\theta)/v$.

Similarly, we can obtain the second expression of the theorem. We shall omit the proof to avoid a conventional complication. The latter half of the theorem is trivial, as was to be proved.

A Doppler's Effect in the Road Traffic Flow

In this part, we analyze the Doppler's effect i.e., the difference between the observation of the traffic flow by an observer moving with varying velocity (for example, an observer on a car or a helicopter) and the observation by an observer at a fixed space point.

We define the following three stochastic processes.

$X(I, J, K)$, the number of cars which exist in a space interval J during a time interval I and whose velocities belong to a velocity interval K;

$x_\xi(I, K\,|\,x_\zeta)$, the number of cars which pass through a space point ξ in a time interval I passing through another space point ζ during the same time interval I and whose velocities belong to a velocity interval K,

and

$y_t(J, K\,|\,y_s)$, the number of cars which existed in a space interval J at time s and exist in the same space interval J at time t and whose velocities belong to a velocity interval K,

where $I=(s,\ t]$, $J=(\zeta,\ \xi]$ and $K=(v,\ w]$.

Then, evidently,

$$X(I, J, K) = x_\xi(I, K)+y_t(J, K) = x_\zeta(I, K)+y_s(J, K) \tag{31}$$

As for $x_\xi(I, K\,|\,x_\zeta)$ and $y_t(J, K\,|\,y_s)$, if the velocity distributions are pruely discontinuous, then

$$x_\xi(I, K\,|\,x_\zeta) = \sum_{v_i \in K_\xi} x_\xi^d(I', v_i), \text{ where } I' = (s+|J|/v_i, t],$$
$$K_\xi = K \cap [|J|/|I|, \infty), \ |J| = |\xi-\zeta| \text{ and } |I| = |t-s|. \tag{32}$$

and

$$y_t(J, K\,|\,y_s) = \sum_{i \in K_t} y_t^d(J', v_i), \text{ where } J' = (\zeta+v_i|I|, \xi]$$
$$\text{and } K_t = K \cap (0, |J|/|I|]. \tag{33}$$

In the above we denote Lebesgue measures of I and J as $|I|$ and $|J|$, respectively.

Moreover we denote the characteristic functions of $X(I, J, K)$, $x_\xi(I, K\,|\,x_\zeta)$ and $y_t(J, K\,|\,y_s)$ as $\varphi_X(I, J, K;\, z)$, $\varphi_\xi(I, K\,|\,\zeta;\, z)$ and $\psi_t(J, K\,|\,s;\, z)$, respectively.

Here, in order to avoid conventional complications, we assume that under conditions A), B), C), E), and F), the velocity distributions are absolutely continuous.

Consequently, from (17), (25) and (31),

$$\log \varphi_X(I, J, K;\, z) = \sum_{k=1}^{n} (e^{ikz}-1)\Big[\int_K \Big\{\int_I f_k(u, v;\, \xi)\, \rho_k(u;\, \xi)\, du +$$
$$\int_J g_k(\eta, v;\, t)\, \lambda_k(\eta;\, t)\, d\eta\Big\} dv\Big] \tag{34}$$
$$= \sum_{k=1}^{n} (e^{ikz}-1)\Big[\int_K \Big\{\int_I f_k(u, v;\, \zeta)\, \rho_k(u;\, \zeta)\, du + \int_J g_k(\eta, v;\, s)\, \lambda_k(\eta;\, s)\, d\eta\Big\} dv\Big]. \tag{35}$$

and the equality of equation (34) and (35) is guaranteed by theorem (3). Moreover from (32) and (33),

$$\log \varphi_\xi(I, K\,|\,\zeta;\, z) = \sum_{k=1}^{n} (e^{ikz}-1) \int_K \int_{I'} f_k(u, v;\, \xi)\, \rho_k(u;\, \xi)\, du\, dv, \tag{36}$$

where $I' = (s+|J|/v, t]$

$$\log \psi_t(J, K\,|\,s;\, z) = \sum_{k=1}^{n} (e^{ikz}-1) \int_{K_t} \int_{J'} g_k(\eta, v;\, t)\, \rho_k(\eta;\, t)\, d\eta\, dv, \tag{37}$$

where $J' = (\zeta+v|I|, \xi]$.

Applying theorem (3) to the above, equations (34)$\sim$(37) are converted into the following equations:

$$\log \varphi_X(I, J, K;\, z) = \sum_{k=1}^{n} (e^{ikz}-1)\Big[\int_K \int_{I_1} f_k(u, v;\, \xi)\, \rho_k(u;\, \xi)\, du\, dv\Big]$$
$$= \sum_{k=1}^{n} (e^{ikz}-1)\Big[\int_K \int_{J_1} g_k(\eta, v;\, t)\, \lambda_k(\eta;\, t)\, d\eta\, dv\Big], \tag{38}$$

where $J_1=(\zeta, \xi+v|I|]$ and $I_1=(s, t+|J|/v]$,

for $K_2=(|J|/|I|, \infty)$,

$$\log \varphi_\xi(I, K|\zeta; z) = \sum_{k=1}^{n} (e^{ikz}-1)\left[\int_{K\cap K_2}\int_{I_2} f_k(u, v; \xi)\, \rho_k(u; \xi)\, du\, dv\right]$$
$$= \sum_{k=1}^{n} (e^{ikz}-1)\left[\int_{K\cap K_2}\int_{I_2} g_k(\eta, v; t)\, \lambda_k(\eta; t)\, d\eta\, dv\right], \tag{39}$$

where $J_2=(\xi, \zeta+v|I|]$ and $I_2=(s+|J|/v, t]$,

and for $K_1=(0, |J|/|I|]$,

$$\log \psi_t(J, K|s; z) = \sum_{k=1}^{n} (e^{ikz}-1)\left[\int_{K\cap K_1}\int_{I_3} f_k(u, v; \xi)\, \rho_k(u; \xi)\, du\, dv\right]$$
$$= \sum_{k=1}^{n} (e^{ikz}-1)\left[\int_{K\cap K_1}\int_{J_3} g_k(\eta, v; t)\, \lambda_k(\eta; t)\, d\eta\, dv\right], \tag{40}$$

where $J_3=(\zeta+v|I|, \xi]$ and $I_3=(t, s+|J|/v]$

Next we define the following observation processes:

$p(I, K; t_0, \xi_0)$, the number of cars whose velocities belong to a velocity interval K and which are observed in a time interval I by an observer who, starting from a point ξ_0 at time t_0, is moving with a velocity $V(t)$ in the opposite direction of the car traffic flow, that is, in the negative direction of the ξ-axis, where $I=(s, t]$;

$q(I, K; t_0, \xi_0)$, the number of cars whose velocities belong to a velocity interval K and which are observed by an observer who, starting from a point ξ_0 at time t_0, is moving with a velocity $V(t)$ in the same direction of the car traffic flow, that is, in the positive direction of the ξ-axis, where we assume that cars which are once passed through by an observer and after that pass ahead of the observer are not counted and conversely.

Then

$$p(I, K; t_0, \xi_0) = X(I, J_p, K), \quad \text{where} \quad J_p = \left(\xi_0-\int_{t_0}^{t} V(t)\, dt,\ \xi_0-\int_{t_0}^{s} V(t)\, dt\right] \tag{41}$$

and $$q(I, K; t_0, \xi_0) = x_\xi(I, K|x_\zeta)+y_t(J, K|y_s)\,, \tag{42}$$

where $\zeta=\xi_0+\int_{t_0}^{s} V(t)\, dt$, $\xi=\xi_0+\int_{t_0}^{t} V(t)\, dt$ and $J=(\zeta, \xi]$.

We denote the characteristic function of $p(I, K; t_0, \xi_0)$ and $q(I, K; t_0, \xi_0)$ as $\chi_p(I, K; z)$ and $\chi_q(I, K; z)$ respectively and put $v^*|I|=\int_s^t V(t)\, dt$. Then from equations (38)~(42), we have the following equations:

$$\log \chi_p(I, K; z) = \sum_{k=1}^{n} (e^{ikz}-1)\left[\int_K \int_{I_p} f_k(u, v; \xi)\, \rho_k(u; \xi)\, du\, dv\right], \tag{43}$$

where $\xi=\xi_0-\int_{t_0}^{t} V(t)\,dt$ and $I_p=(s-v^*|I|/v, t]$,

and $\log \chi_q(I, K; z) = \sum_{k=1}^{n} (e^{ikz}-1)\Big[\int_{K\cap K_1}\int_{I_q} f_k(u, v; \xi)\, \rho_k(u; \xi)\, du\, dv$

$$+\int_{K\cap K_2}\int_{I_q'} f_k(u, v; \xi)\, \rho_k(u; \xi)\, du\, dv\Big], \tag{44}$$

where $\xi=\xi_0+\int_{t_0}^{t} V(t)\,dt$, $K_1=(0, v^*]$, $K_2=(v^*, \infty)$, $I_q=(t, s+v^*|I|/v]$

and $I_q' = (s+v^*|I|/v, t]$.

For example, in order to simplify, if we suppose condition (D), put $I=(t-1, t)$ and denote the position of the observer at time t as ξ, then we get

$$\log \chi_p(I, K; z) = (e^{iz}-1)\int_K \int_{t-(v+v^*)/v}^{t} f_\xi(u, v)\, \rho_\xi(u)\, du\, dv$$

and $\log \chi_q(I, K; z) = (e^{iz}-1)\Big[\int_{K\cap K_1}\int_{\xi}^{t+(v^*-v)/v} f_\xi(u, v)\, \rho_\xi(u)\, du\, dv$

$$+\int_{K\cap K_2}\int_{t-(v-v^*)/v}^{t} f_\xi(u, v)\, \rho_\xi(u)\, du\, dv\Big].$$

The differences from the time process observed at the fixed point ξ are revealed as follows:

$$\mathbf{E}[p(I, K; t_0, \xi_0)] = \mathbf{E}[x_\xi(I, K)]+\int_K \int_{t-(v+v^*)/v}^{t-1} f_\xi(u, v)\, \rho_\xi(u)\, du\, dv$$

and $\mathbf{E}[q(I, K; t_0,\xi_0)] = \mathbf{E}[x_\xi(I, K)]+\int_{K\cap K_1}\Big\{\int_{t}^{t+(v^*-v)/v} f_\xi(u, v)\, \rho_\xi(u)\, du$

$$-\int_{t-1}^{t} f_\xi(u, v)\, \rho_\xi(u)\, du\Big\}\, dv-\int_{K\cap K_2}\int_{t-1}^{t-(v-v^*)/v} f_\xi(u, v)\, \rho_\xi(u)\, du\, dv\,.$$

Applications and Conclusion

In this part, we apply the results of the above sections to a simple case, that is, a car traffic flow which is a homogeneous Poisson process with an absolutely continuous velocity distribution.

Before we do it, we must point out the following facts; generally speaking, if either a car traffic flow i.e. $x_\xi(I)$ or $y_t(J)$, or a velocity ditsribution is inhomogeneous in time or space, the other must be inhomogeneous, too, and homogeneity is stable if and only if both a car traffic flow and a velocity distribution are homogeneous. These facts are easily obtained by the equation (3) and the discussion of "Various Transformations".

Therefore, it is relevant to assume the characteristic function of $x_\xi(I, K)$, that is, $\varphi_\xi(I, K; z)$ as the following equation:

$$\log \varphi_\xi(I, K; z) = (e^{iz}-1)\,\rho_\xi\,|I|\int_K f_\xi(v)\,dv\,, \tag{45}$$

where $I=(s, t]$ and $K=(v, w]$.

We denote ρ_0 and $f_0(v)$ by ρ and $f(v)$, respectively.

In the following, we show the results deduced from the above assumptions. At first, from corollary (1), for $\xi<hs$,

$$\log \varphi_\xi(I, K; z) = (e^{iz}-1)\,\rho\,|I|\int_K f(v)\,dv\,.$$

Consequently, for $\xi<hs$, $\rho_\xi|I|\int_K f_\xi(v)\,dv=\rho\,|I|\int_K f(v)\,dv$. If we put $K=V$, then $\rho_\xi=\rho$, therefore, $f_\xi(v)=f(v)$ for *a.e.v.*

Therefore, from (45), for arbitrary $\xi\in\Xi$, $I\subset T$ and $K\subset V$

$$\log \varphi_\xi(I, K; z) = (e^{iz}-1)\,\rho\,|I|\int_K f(v)\,dv\,. \tag{46}$$

It follows from the latter half of theorem (3) that for $J=(\zeta, \xi]$ and t such that $t+|J|/m\leq T'$,

$$\log \psi_t(J, K; z) = (e^{iz}-1)\,\rho\,|J|\int_K f(v)/v\,dv\,. \tag{47}$$

Therefore, in accordance with corollary (1), the equation (47) holds for arbitrary $t\in T$, $J\subset\Xi$ and $K\subset V$. That is, the space process $y_t(J, K)$ is also a homogeneous Poisson process with an absolutely continuous velocity distribution.

Let us put a characteristic function $\psi_t(J, K; z)$ of a space process as follows:

$$\log \psi_t(J, K; z) = \lambda\,|J|\int_K g(v)\,dv(e^{iz}-1)\,. \tag{48}$$

It follows from equations (47) and (48) that

$$\lambda\int_K g(v)\,dv = \rho\int_K f(v)/v\,dv\,,$$

therefore

$$\lambda g(v) = \rho f(v)/v \qquad \text{for } a.e.v.$$

Consequently,

$$\rho = \lambda\int_0^\infty vg(v)\,dv = \lambda\mathbf{E}_g(v) \quad \text{and} \quad \lambda = \rho\int_0^\infty f(v)/v\,dv = \rho\mathbf{E}_f(1/v)\,.$$

We can obtain the well-known relationship between a space velocity density function $g(v)$ and a time velocity density function $f(v)$,

$$\mathbf{E}_g(v) f(v) = v g(v) \qquad \text{for } a.e.v.$$

It follows from "A Doppler's effect in the road traffic flow" that for a characteristic function $\chi_p(I, K; z)$ of an observation process $p(I, K; t_0, \xi_0)$,

$$\log \chi_p(I, K; z) = (e^{iz}-1)\,\rho\,|I| \left\{ \int_K f(v)\,dv + v^* \int_K f(v)/v\,dv \right\}$$

$$= (e^{iz}-1)\,\lambda\,|I| \left\{ \int_K v g(v)\,dv + v^* \int_K g(v)\,dv \right\}$$

and $\quad \mathbf{E}[p(I, V; t_0, \xi_0)] = \rho\,|I|\,\{1+v^*\mathbf{E}_f(1/v)\} = \lambda\,|I|\,\{v^*+\mathbf{E}_g(v)\}$.

Similarly, we can obtain for a characteristic function $\chi_q(I, K; z)$ of an observation process $q(I, K; t_0, \xi_0)$.

$$\log \chi_q(I, K; z) = (e^{iz}-1)\,\rho\,|I| \left\{ v^* \left(\int_v^{v^*} f(v)/v\,dv - \int_{v^*}^{w} f(v)/v\,dv \right) \right.$$

$$\left. + \int_{v^*}^{w} f(v)\,dv - \int_v^{v^*} f(v)\,dv \right\} = (e^{iz}-1)\,\lambda\,|I| \left\{ v^* \left(\int_v^{v^*} g(v)\,dv - \int_{v^*}^{w} g(v)\,dv \right) \right.$$

$$\left. + \int_{v^*}^{w} v g(v)\,dv - \int_v^{v^*} v g(v)\,dv \right\}$$

and $\quad \mathbf{E}[q(I, V; t_0, \xi_0)] = \rho\,|I| \left\{ v^* (\mathbf{E}_f(1/v) - 2\int_{v^*}^{\infty} f(v)/v\,dv) + 2\int_{v^*}^{\infty} f(v)\,dv - 1 \right\}$

$$= \lambda\,|I| \left\{ v^* (1 - 2\int_{v^*}^{\infty} g(v)\,dv) + 2\int_{v^*}^{\infty} v g(v)\,dv - \mathbf{E}_g(v) \right\}.$$

Now, let consider the opposite lane traffic flow together. It is relevant to assume that a traffic flow and the opposite traffic flow are independent of each other. We shall make use of the affix 1 for a traffic flow with the same direction as an observer and the affix 2 for the opposite traffic flow. Then we have the following equation for a characteristic function $\chi(I, K; z)$ of the number of cars which the observer observes in a time interval I and whose velocities belong to K,

$$\log \chi(I, K; z) = (e^{iz}-1) \Big[\rho_1\,|I| \left\{ v^* \left(\int_v^{v^*} f_1(v)/v\,dv - \int_{v^*}^{w} f_1(v)/v\,dv \right) + \int_{v^*}^{w} f_1(v)\,dv \right.$$

$$\left. - \int_v^{v^*} f_1(v)\,dv \right\} + \rho_2\,|I| \left\{ \int_K f_2(v)\,dv + v^* \int_K f_2(v)/v\,dv \right\} \Big].$$

It is hoped to investigate the problem of queues for a traffic light, the problem of equilibrium analysis, and so on.

The next correction to the present theory would involve some form of interaction between cars. However, such correction seems to be more difficult than the present theory.

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