A Variational Principle for Transport Phenomena I—Formulation of Principle

By

Masakatsu HIRAOKA* and Kanya TANAKA**

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A new variational principle was formulated to solve the first and second kind of boundary value problems in mometum transprot, heat and mass transfer process. The variational principle formulated in this paper is based on the concept of variational invariant and is applicable to the unsteady state. By deriving Euler's equation from the functional, the equivalence between the variational principle and the boundary value problem was verified. Then, the reciprocal principle was established by verifing that the functional should be a minimum and the reciprocal functional should be a maximum at the stationary point. Finally, the parameters included in the argument functions as the generalized coordinates were defined and Lagrange-Biot's equation which determined these paramtes was derived. Some applications of this principle to the problems of the transport phenomena will be discussed in the following paper.

1. Introduction

Analysis in transport phenomena which terats the transport of momentum, heat and mass are usualy conducted by solving the basic equations derived from the balance of these properties with the proper initial and boundary conditions. It was well known that many problems described as the boundary value problems in physics could be expressed by the equivalent variational principle such as Hamilton's Principle[1] in the realm of classiclal mechanics. This Variational principle was developed in the field of mechanics of continua[2], and also in the field of electric circuit by Thomson's principle.[3]

The difficulties encountered in the formulation of variational principle for transport phenomena depend on the fact that the irreversible process accompanied with dissipation of energy always occurs in these systems. Application of the variational principle in the field of transport phenomena has recently aroused the interest of the engineers of various fields. There exist many possibilities of the formulation of variational principle.

* Department of Sanitary Engineering,
** Department of Chemical Engineering, Doshisha University

Since the principle of minimum entropy production which characterizes the steady state in the irreversible process was proposed[4], it became obvious that the energy dissipation corresponding to the entropy production could be selected as the functional. This principle was worthy of notice for the researchers of non-Newtonian fluid mechanics and some attempts of developing this principle have been done.

Brid[5] proposed the generalized variational principle for non-Newtonian fluid and Johnson[6] researches the stationary properties of the functional with the generalized boundary conditions and derived the reciprocal theorem, and the maximum and minimum principle. Stewart[7] applied this principle to the laminar flow in the uniform duct. Glansdorff, Prigogine and Hays[8] have studied the case in which the principle of minimum entropy production is not valid even in the steady state and introduced the concept of local potential. Schechter[9] formulated the variational principle for Reiner-Rivlin fluid which includes the inertia term neglected by Bird and Johnson by use of this concept. Sparrowl and Siegel[10] applied the variational principle to solve the steady state heat transfer problem in duct.

For the unsteady state process, Rosen[11] suggested that the principle of minimum entropy production has a applicability for the transport phenomena. Chambers[12] proposed the variational priciple for the unsteady state heat transfer problems slightly different from Rosen. The most interesting method of formulation of the variational method was developed by Biot[13][17] and this method has been studied by Nigam and Agrawl[18], and Gupta[19]. These principles proposed by them were based on the concept of variational invariant[14].

There exists other methods formulating the variational principle to the unsteady state process. This is the one which uses the adjoint form suggested by Morse and Feshbach[20]. Slattery[21][22] has derived the generalized adjoint variational principle and applied to solve the flow of Newtonian fluid. Nichols and Bankoff[23] proposed the adjoint variationlal principle applicable to the convective diffusion. Finally, the approximate solution of the equations of change by use of Galarkin's direct method has been studied by Synder, Spriggs and Stewart[24].

The purpose of this paper is to propose a new variational principle, which is based on the concept of variational invariant and is applicable to the unsteady state. The variational principle was formulated in the form which could be applicable to the first and second kind boundary value problem, and the equivalence between the variational principle and the boundary value problem was verified. Then, the reciprocal principle was derived, and the maximum and minimum principle was established. Finally the parameters including in the argument functions as the generalized coordinates were defined and Lagrange-Biot's equation which determines these parameters was derived.

2. Variational Principle for Transport Phenomena

2.1 Variational Principle for Motin of Fluid

The equation of motion of incompressible fluid can be written as the next form.

$$\rho \frac{D\boldsymbol{v}}{Dt} \equiv \rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = -\nabla p-\nabla\cdot\boldsymbol{\tau}+\rho\boldsymbol{g}\,. \tag{1}$$

The equation of continuity

$$(\nabla\cdot\boldsymbol{v}) = 0\,. \tag{2}$$

The equation of stress tensor

$$\begin{aligned}\boldsymbol{\tau} &= -2\mu\boldsymbol{d}\\ \boldsymbol{d} &= \frac{1}{2}[(\nabla\boldsymbol{v})+{}^{t}(\nabla\boldsymbol{v})]\,,\end{aligned} \tag{3}$$

where, the density and the viscosity are constant and are a function of position $\boldsymbol{r}$.

The variational principle could be formulated to solve the boundary vlaue problems which are considered with the boundary conditions given on a surface S of domain V.

$$\boldsymbol{v} = \boldsymbol{v}^{\circ} \qquad \text{on} \quad S_V \tag{5}$$

$$\boldsymbol{\tau}+p\boldsymbol{\delta} = \boldsymbol{\pi}^{\circ}\,, \qquad \text{on} \quad S_\tau \tag{6}$$

where, S_V and S_τ are the non-overlapping regions, and $S=S_V+S_\tau$.

The augument functions $\boldsymbol{v}$, p by which the functional

$$J[\boldsymbol{v}, p] = \int_{t_0}^{t_1} I[\boldsymbol{v}, p]\,dt\,, \tag{7}$$

where,

$$\begin{aligned}I[\boldsymbol{v}, p] = \int_V [\rho\bar{\boldsymbol{v}}_t\cdot\boldsymbol{v}+(\rho\bar{\boldsymbol{v}}\cdot\nabla\bar{\boldsymbol{v}})\cdot\boldsymbol{v}+\Phi-p(\nabla\cdot\boldsymbol{v})\\ -\rho\boldsymbol{g}\cdot\boldsymbol{v}]dV+\int_{S_\tau}[\boldsymbol{\pi}^{\circ}\cdot\boldsymbol{n}]\cdot\boldsymbol{v}\,dS\,,\end{aligned} \tag{8}$$

is stationary with the admissibility conditions are the solutions of boundary value problem given above.

$$\Phi = \frac{1}{2}(-\boldsymbol{\tau}:\nabla\boldsymbol{v})\,, \tag{9}$$

in the functional Eq. 8 expresses the dissipation energy by viscocity and $\boldsymbol{n}$ is a unit vector normal outward to the boundary surface. It is important that the functional $I[\boldsymbol{v}, p]$ includes the two functions $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$; $\boldsymbol{v}$ is a ordinary argument function, but $\bar{\boldsymbol{v}}$ expresses $\boldsymbol{v}$ in the stationary state, that is, $\bar{\boldsymbol{v}}$ is a solution of given boundary value problem. $\bar{\boldsymbol{v}}_t$ is the partial derivative of $\bar{\boldsymbol{v}}$ about time. The integral variational formula including these stationary variables was called the variational invariant By Biot[14]. When we calculate the first variation

$$\delta J = \int_{t_0}^{t_1} \delta I \, dt \,, \tag{10}$$

the partial integral by t is not necessary because the differential term about time is only $\bar{\boldsymbol{v}}_t$ which is independent of the variational process. Therefore, it is obvious that the functional $I[\boldsymbol{v}, p]$ should be stationary in any time instead of the functional J.

Let us verify that the stationary points $\boldsymbol{v}$ and p of $I[\boldsymbol{v}, p]$ should be coincide with the solution of the boundary value problem initially given. The calculation of the first variation of $I[\boldsymbol{v}, p]$ leads to

$$\delta I = \int_V [\rho\bar{\boldsymbol{v}}_t \cdot \delta\boldsymbol{v} + (\rho\bar{\boldsymbol{v}} \cdot \nabla\bar{\boldsymbol{v}}) \cdot \delta\boldsymbol{v} + \delta\Phi - \delta p(\nabla \cdot \bar{\boldsymbol{v}}) - \bar{p}(\nabla \cdot \delta\boldsymbol{v})$$
$$-\rho\boldsymbol{g} \cdot \delta\boldsymbol{v}] \, dV + \int_{S_T} [\boldsymbol{\pi}^\circ \cdot \boldsymbol{n}] \cdot \delta\boldsymbol{v} \, dS \,. \tag{11}$$

From the symmetrical property of $\boldsymbol{\tau}$,

$$\delta\Phi = (-\boldsymbol{\tau} : \nabla\delta\boldsymbol{v}) \,, \tag{12}$$

and use of Gauss-Ostrogradskii's divergence theorem gives that

$$\int_V (-\boldsymbol{\tau} : \nabla\delta\boldsymbol{v}) \, dV = \int_V (\nabla \cdot [-\boldsymbol{\tau} \cdot \delta\boldsymbol{v}]) \, dV + \int_V [\nabla \cdot \boldsymbol{\tau}] \cdot \delta\boldsymbol{v} \, dV$$
$$= \int_S [-\boldsymbol{\tau} \cdot \delta\boldsymbol{v}] \cdot \boldsymbol{n} \, dS + \int_V [\nabla \cdot \boldsymbol{\tau}] \cdot \delta\boldsymbol{v} \, dV$$
$$= \int_S [-\boldsymbol{\tau} \cdot \boldsymbol{n}] \cdot \delta\boldsymbol{v} \, dS + \int_V [\nabla \cdot \boldsymbol{\tau}] \cdot \delta\boldsymbol{v} \, dS \,, \tag{13}$$

and also

$$\int_V p(\nabla \cdot \delta\boldsymbol{v}) \, dV = \int_V (\nabla \cdot p\delta\boldsymbol{v}) \, dV - \int_V (\nabla p) \cdot \delta\boldsymbol{v} \, dV$$
$$= \int_S [p\boldsymbol{\delta} \cdot \boldsymbol{n}] \cdot \delta\boldsymbol{v} \, dS - \int_V (\nabla p) \cdot \delta\boldsymbol{v} \, dV \,. \tag{14}$$

By substitution of the above equations into Eq. (11), we get

$$\delta I = \int_V [\rho\bar{\boldsymbol{v}}_t + \rho\bar{\boldsymbol{v}} \cdot \nabla\bar{\boldsymbol{v}} + \nabla \cdot \bar{\boldsymbol{\tau}} + \nabla\bar{p} - \rho\boldsymbol{g}] \cdot \delta\boldsymbol{v} \, dV - \int_V (\nabla \cdot \bar{\boldsymbol{v}}) \, \delta p \, dV$$
$$+ \int_{S_T} [(\boldsymbol{\pi}^\circ - \bar{\boldsymbol{\tau}} - \bar{p}\boldsymbol{\delta}) \cdot \boldsymbol{n}] \cdot \delta\boldsymbol{v} \, dS + \int_{S_V} [(-\bar{\boldsymbol{\tau}} - \bar{p}\boldsymbol{\delta}) \cdot \boldsymbol{n}] \cdot \delta\boldsymbol{v} \, dS \,. \tag{15}$$

The first variation of $I[\boldsymbol{v}, p]$ must be zero at a stationary state, so

$$\rho\bar{\boldsymbol{v}}_t+\rho\bar{\boldsymbol{v}}\cdot\nabla\bar{\boldsymbol{v}} = -\nabla\cdot\bar{\boldsymbol{\tau}}-\nabla\bar{p}+\rho\boldsymbol{g} \qquad \text{in} \quad V \tag{16}$$

$$(\nabla\cdot\bar{\boldsymbol{v}}) = 0 \tag{17}$$

$$\bar{\boldsymbol{\tau}}+\bar{p}\boldsymbol{\delta} = \boldsymbol{\pi}^\circ , \tag{18}$$

are reduced and from the admissiblity conditions, $\delta\boldsymbol{v}=0$ on the surface S_V and $\boldsymbol{v}=v^\circ$ are reduced.

The above Eqs. (16)$\sim$(18) exactly express the boundary value problem described above. It became obvious that the equation of motion of Newtonian fluid can be obtained as Euler's equation of the variational problem, and the boundary conditions can be reduced from the national conditions. A notice should be paid that the pressure term p in Eq. (8) is a Lagrange's multiplier in Eq. (2) which expresses the condition of imcompressibility[25].

Let us consider the special case in which the state is in steady ($\boldsymbol{v}_t=0$), no convective term ($\boldsymbol{v}\cdot\nabla\boldsymbol{v}=0$) and the external force $\boldsymbol{g}$ has a potential $\Omega(\boldsymbol{g}=-\nabla\Omega)$ and a velocity at the boundary surface was given ($S=S_V$).

$$\begin{aligned}\int_V \rho\boldsymbol{g}\cdot\boldsymbol{v}\,dV &= -\int_V \rho(\nabla\Omega)\cdot\boldsymbol{v}\,dV \\ &= -\int_V (\nabla\cdot\rho\Omega\boldsymbol{v})\,dV+\int_V \rho\Omega(\nabla\cdot\boldsymbol{v})\,dV \\ &= -\int_S \rho\Omega(\boldsymbol{v}\cdot\boldsymbol{n})\,dS+\int_V \rho\Omega(\nabla\cdot\boldsymbol{v})\,dV .\end{aligned} \tag{19}$$

By considering that the first term of Eq. (19) becomes zero in the first variation, Eq. (8) is finally reduced to the next form.

$$I[\boldsymbol{v}, p] = \int_V [-(p+\rho\Omega)(\nabla\cdot\boldsymbol{v})+\Phi]\,dV . \tag{20}$$

To bring this functional to a miniumu value indicates the Helmholtz's principle[26] for Newtonian fluid.

2-2. Variational Principle for Heat and Mass Transfer

The basic equation of heat transfer can be written as the next form

$$\rho C_p\frac{D\theta}{Dt} \equiv \rho C_p\left(\frac{\partial\theta}{\partial t}+\boldsymbol{v}\cdot\nabla\theta\right) = \nabla\cdot(\lambda\nabla\theta)+Q_h(\boldsymbol{r}, t) , \tag{21}$$

where, ρ, C_p and λ are constant or a function of position.

The boundary surface S of domain V is divided into the two parts S_θ and S_q which do not duplicate, and then, the first kind and the second kind of boundary conditions are given

$$\theta = \theta^\circ \qquad \text{on} \quad S_\theta \tag{22}$$

$$-\lambda\nabla\theta = \boldsymbol{q}^\circ\,, \qquad \text{on} \quad S_q \tag{23}$$

where, $\theta = T - T_0$; T_0 is a temperature of reference state.
The basic equations of mass transfer can be written as the next from

$$\frac{DC}{Dt} \equiv \frac{\partial C}{\partial t} + \boldsymbol{v}\cdot\nabla C = \nabla\cdot(\mathscr{D}\nabla C) + R(\boldsymbol{r},\, t) \tag{24}$$

$$C = C^\circ \qquad \text{on} \quad S_V \tag{25}$$

$$-\mathscr{D}\nabla C = \boldsymbol{N}^\circ\,, \qquad \text{on} \quad S_N \tag{26}$$

where, $\mathscr{D}$ is a constant or a function of position and $S = S_C + S_N$, and $C = C_A - C_{A0}$ in which C_{A0} is a concentration of reference state. Q_h in Eq. (21) and R in Eq. (24) are the soruce of heat and mass, and when it accompanies with chemical reaction, these terms become the functions of both θ and C, but the analysis in this work will be limited to the case in which Q_h and R are only the function of independent variables $\boldsymbol{r}$ and t. Therefore, the heat and mass transfer are respectively independent and have the similar form.

The agrument functions which make the functional

$$J = \int_{t_0}^{t_1} I\,dt\,, \tag{27}$$

stationary with the admissibility conditions are the solutions of boundary vlaue problems given above. The functional for heat transfer is written as

$$I[\theta] = \int_V [\rho C_p \bar{\theta}_t \theta + \rho C_p(\boldsymbol{v}\cdot\nabla\bar{\theta})\theta + \frac{\lambda}{2}(\nabla\theta)^2 - Q_h\theta]\,dV + \int_{S_q} (\boldsymbol{q}^\circ\cdot\boldsymbol{n})\theta\,dS\,. \tag{28}$$

The functional for mass transfer is written as

$$I[C] = \int_V [\bar{C}_t C + (\boldsymbol{v}\cdot\nabla\bar{C})C + \frac{\mathscr{D}}{2}(\nabla C)^2 - RC]\,dV + \int_{S_N} (\boldsymbol{N}^\circ\cdot\boldsymbol{n})C\,dS\,. \tag{29}$$

Calaulation of the first variation gives

$$\delta I = \int_V [\rho C_p \bar{\theta}_t \delta\theta + \rho C_p(\boldsymbol{v}\cdot\nabla\bar{\theta})\delta\theta + (\lambda\nabla\theta)\cdot(\nabla\delta\theta) - Q_h\delta\theta]\,dV + \int_{S_q} (\boldsymbol{q}^\circ\cdot\boldsymbol{n})\delta\theta\,dS\,. \tag{30}$$

By use of Gaus-Ostraogradskii's divergence theorem, we get

$$\int_V (\lambda\nabla\theta)\cdot(\nabla\delta\theta)\,dV = \int_V \nabla\cdot(\delta\theta\cdot\lambda\nabla\theta)\,dV - \int_V (\nabla\cdot\lambda\nabla\theta)\delta\theta\,dV$$
$$= \int_S (\lambda\nabla\theta)\cdot\boldsymbol{n}\delta\theta\,dS - \int_V (\nabla\cdot\lambda\nabla\theta)\delta\theta\,dV\,, \tag{31}$$

and then, Eq. (30) reduced to the next form

$$\delta I = \int_V [\rho C_p\bar{\theta}_t + \rho C_p(\boldsymbol{v}\cdot\nabla\bar{\theta}) - \nabla\cdot(\lambda\nabla\bar{\theta}) - Q_h]\delta\theta\,dV$$
$$+ \int_{S_q} (\lambda\nabla\bar{\theta} + \boldsymbol{q}^\circ)\cdot\boldsymbol{n}\delta\theta\,dS + \int_{S_\theta} (\lambda\nabla\bar{\theta})\cdot\boldsymbol{n}\delta\theta\,dS\,. \tag{32}$$

As $\delta I=0$ in the stationary state and by considering the admissibility condition Eq. (22), the next expressions are derived as Euler's equation.

$$\rho C_p\bar{\theta}_t + \rho C_p(\boldsymbol{v}\cdot\nabla\bar{\theta}) = \nabla\cdot(\lambda\nabla\bar{\theta}) + Q_h \quad \text{in} \quad V \tag{33}$$
$$\bar{\theta} = \theta^\circ\,, \quad \delta\theta = 0 \quad \text{on} \quad S_\theta \tag{34}$$
$$-\lambda\nabla\bar{\theta} = \boldsymbol{q}^\circ\,, \quad \text{on} \quad S_q \tag{35}$$

These equations are equivalent with the boundary value problem given above. The verification for the mass transfer are the similar procedure described above. The special case could be reduced from Eq. (28) and Eq. (29), in which the state is in steady ($\theta_t=0$, $C_t=0$), no flow ($\boldsymbol{v}=0$), no sources ($Q_h=0$, $R=0$), and θ° and C° are given on all the points to boundary surface $S(S_q=0,\ S_N=0)$.

$$I[\theta] = \int_V \frac{\lambda}{2}(\nabla\theta)^2 dV \tag{36}$$

$$I[C] = \int_V \frac{\mathcal{D}}{2}(\nabla C)^2 dV\,. \tag{37}$$

These functionals express the energy dissipation caused by the irreversible process in the system and should have the minimum value according to the principle of the minimum entropy production by Prigogine.

3. Reciprocal Variational Pirnciple

We have formulated the new variational principle for the momentum, heat and mass transport in the previous sections. In this chapter, we will derive the reciprocal variational principles by use of Legendre transform or Friedrich transform. In this formulation, the momentum flux (shearing stress) $\boldsymbol{\tau}$, the heat flux $\boldsymbol{q}$ and the mass flux $\boldsymbol{N}$ are taken as the argument functions.

3.1. Reciprocal Variational Principle for Motion of Fluid

The free variational problems having all the variables $\boldsymbol{v}$, p, $\boldsymbol{d}$, $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ can be formulated as follows.

$$I[\boldsymbol{v}, p, \boldsymbol{d}, \boldsymbol{\tau}, \boldsymbol{\pi}] = \int_V \left[\rho \frac{D\bar{\boldsymbol{v}}}{Dt} \cdot \boldsymbol{v} - \left\{ \frac{1}{2} [(\nabla \boldsymbol{v}) + {}^t(\nabla \boldsymbol{v})] - \boldsymbol{d} \right\} : \boldsymbol{\tau} + \Phi(\boldsymbol{d}) \right.$$

$$\left. - p(\nabla \cdot \boldsymbol{v}) - \rho \boldsymbol{g} \cdot \boldsymbol{v} \right] dV + \int_{S_\tau} [\boldsymbol{\pi}^\circ \cdot \boldsymbol{n}] \cdot \boldsymbol{v} \, dS + \int_{S_V} [\boldsymbol{\pi} \cdot \boldsymbol{n}] \cdot (\boldsymbol{v} - \boldsymbol{v}^\circ) dS, \quad (38)$$

where,

$$\Phi(\boldsymbol{d}) = \mu(\boldsymbol{d} : \boldsymbol{d}) . \quad (39)$$

Calculating the first variation of Eq. (38)

$$\delta I = \int_V \left[\rho \frac{D\bar{\boldsymbol{v}}}{Dt} \cdot \rho \boldsymbol{v} - \left\{ \frac{1}{2} [(\nabla \boldsymbol{v}) + {}^t(\nabla \boldsymbol{v})] - \boldsymbol{d} \right\} : \delta \boldsymbol{\tau} \right.$$

$$- \left\{ \frac{1}{2} [(\nabla \delta \boldsymbol{v}) + {}^t(\nabla \delta \boldsymbol{v})] - \delta \boldsymbol{d} \right\} : \boldsymbol{\tau} + \delta \Phi - \delta p (\nabla \cdot \boldsymbol{v})$$

$$- p(\nabla \cdot \delta \boldsymbol{v}) - \rho \boldsymbol{g} \cdot \delta \boldsymbol{v}] dV + \int_{S_\tau} [\boldsymbol{\pi} \cdot \boldsymbol{n}] \cdot \delta \boldsymbol{v} \, dS$$

$$+ \int_{S_V} [\delta \boldsymbol{\pi} \cdot \boldsymbol{n}] \cdot (\boldsymbol{v} - \boldsymbol{v}^\circ) dS + \int_{S_V} [\boldsymbol{\pi} \cdot \boldsymbol{n}] \cdot \delta \boldsymbol{v} \, dS . \quad (40)$$

Rearrangement by use of

$$\delta \Phi = 2\mu (\boldsymbol{d} : \delta \boldsymbol{d}) \quad (41)$$

$$\boldsymbol{\tau} : \nabla \delta \boldsymbol{v} = \boldsymbol{\tau} : {}^t(\nabla \delta \boldsymbol{v}) , \quad (42)$$

and Eqs. (13) and (14) gives

$$\delta I = \int_V \left[\left\{ \rho \frac{D\boldsymbol{v}}{Dt} + [\nabla \cdot \bar{\boldsymbol{\tau}}] + \nabla \bar{p} - \rho \boldsymbol{g} \right\} \cdot \delta \boldsymbol{v} - \left\{ \frac{1}{2} [(\nabla \bar{\boldsymbol{v}}) + {}^t(\nabla \bar{\boldsymbol{v}})] \right. \right.$$

$$\left. - \bar{\boldsymbol{d}} \right\} : \delta \boldsymbol{\tau} + (\bar{\boldsymbol{\tau}} + 2\mu \bar{\boldsymbol{d}}) : \delta \boldsymbol{d} - (\nabla \cdot \bar{\boldsymbol{v}}) \delta p] dV$$

$$+ \int_{S_\tau} [\boldsymbol{\pi}^\circ - \bar{\boldsymbol{\tau}} - \bar{p} \delta) \cdot \boldsymbol{n}] \cdot \delta \boldsymbol{v} \, dS + \int_{S_V} [(\bar{\boldsymbol{\pi}} - \bar{\boldsymbol{\tau}} - \bar{p} \boldsymbol{\delta}) \cdot \boldsymbol{n}] \cdot$$

$$\delta \boldsymbol{v} \, dS + \int_{S_V} [\delta \boldsymbol{\pi} \cdot \boldsymbol{n}] \cdot (\bar{\boldsymbol{v}} - \boldsymbol{v}^\circ) \, dS . \quad (43)$$

Euler's equation are derived from $\delta I = 0$ and by abbreviation of the symbol of stationary state "—",

$$\rho \frac{D\boldsymbol{v}}{Dt} = -[\nabla \cdot \boldsymbol{\tau}] - \nabla p + \rho \boldsymbol{g} \qquad \text{in } V \quad (44)$$

$$\boldsymbol{d} = \frac{1}{2} [(\nabla \boldsymbol{v}) + {}^t(\nabla \boldsymbol{v})] \qquad \text{in } V \quad (45)$$

$$\boldsymbol{\tau} = -2\mu \boldsymbol{d} \qquad \text{in } V \quad (46)$$

$$(\nabla \cdot \boldsymbol{v}) = 0 . \qquad \text{in } V \quad (47)$$

and the natural boundary conditions are

$$\boldsymbol{\tau}+p\boldsymbol{\delta} = \boldsymbol{\pi}^0 \qquad \text{on} \quad S_\tau \tag{48}$$

$$\boldsymbol{\tau}+p\boldsymbol{\delta} = \boldsymbol{\pi} \qquad \text{on} \quad S_V \tag{49}$$

$$\boldsymbol{v} = \boldsymbol{v}^0 . \qquad \text{on} \quad S_V \tag{50}$$

If Eqs. (45), (46), (48) and (50) as the admissibility conditions are added to the functional Eq. (38) of free variational problems, these subjects can be reduced to the variational problems about $\boldsymbol{v}$ which were described in the section 2–1.

Now, let us conduct Legendre transform on $\boldsymbol{\tau}$ and $\boldsymbol{d}$

$$-\Psi(\boldsymbol{\tau}) = \Phi(\boldsymbol{d})+(\boldsymbol{d}:\boldsymbol{\tau}) , \tag{51}$$

then, Eq. (38) is written in the next form

$$\begin{aligned} H[\boldsymbol{v}, p, \boldsymbol{\tau}, \boldsymbol{\pi}] = \int_V \Big[& \rho\frac{D\bar{\boldsymbol{v}}}{Dt}\cdot\boldsymbol{v} - \frac{1}{2}[(\nabla\boldsymbol{v})+{}^t(\nabla\boldsymbol{v})]:\boldsymbol{\tau}-\Psi(\boldsymbol{\tau}) \\ & -p(\nabla\cdot\boldsymbol{v})-\rho\boldsymbol{g}\cdot\boldsymbol{v}]dV \\ & +\int_{S_\tau}[\boldsymbol{\pi}^0\cdot\boldsymbol{n}]\cdot\boldsymbol{v}\,dS+\int_{S_V}[\boldsymbol{\pi}\cdot\boldsymbol{n}]\cdot(\boldsymbol{v}-\boldsymbol{v}^0)dS . \end{aligned} \tag{52}$$

By substitution of Eqs. (39) and (46) into Eq. (51), we get the expression of $\Psi(\boldsymbol{\tau})$

$$\begin{aligned} -\Psi(\boldsymbol{\tau}) &= \mu(\boldsymbol{d}:\boldsymbol{d})+(\boldsymbol{d}:\boldsymbol{\tau}) \\ &= \frac{1}{4\mu}(\boldsymbol{\tau}:\boldsymbol{\tau})-\frac{1}{2\mu}(\boldsymbol{\tau}:\boldsymbol{\tau}) \\ &= -\frac{1}{4\mu}(\boldsymbol{\tau}:\boldsymbol{\tau}) \end{aligned}$$

$$\therefore \quad \Psi(\boldsymbol{\tau}) = \frac{1}{4\mu}(\boldsymbol{\tau}:\boldsymbol{\tau}) . \tag{53}$$

Then, the substitution of the relations of Eq. (13) and (14) into the functional of Eq. (51) leads to

$$\begin{aligned} H[\boldsymbol{v}_0\, p, \boldsymbol{\tau}, \boldsymbol{\pi}] = & \int_V\Big[\Big\{\rho\frac{D\bar{\boldsymbol{v}}}{Dt}+[\nabla\cdot\boldsymbol{v}]+\nabla p-\rho\boldsymbol{g}\Big\}\cdot\boldsymbol{v}-\Psi(\boldsymbol{\tau})\Big]dV \\ & +\int_{S_\tau}[\boldsymbol{\pi}^0-\boldsymbol{\tau}-p\boldsymbol{\delta})\cdot\boldsymbol{n}]\cdot\boldsymbol{v}\,dS-\int_{S_V}[\boldsymbol{\pi}\cdot\boldsymbol{n}]\cdot\boldsymbol{v}^0\,dS . \end{aligned} \tag{54}$$

Insted of Eq. (44), when

$$\rho\frac{D\boldsymbol{v}}{Dt} = -[\nabla\cdot\boldsymbol{\tau}]+\nabla p+\rho\boldsymbol{g} , \tag{55}$$

and Eqs. (48) and (49) are added to Eq. (54) as the admissiblity condition, we get

$$H[\boldsymbol{\tau}] = -\int_V \Psi(\boldsymbol{\tau})dV - \int_{S_V}[\boldsymbol{\pi}\cdot\boldsymbol{n}]\cdot\boldsymbol{v}\,dS\,. \tag{56}$$

and it can be indicated that Euler's equation for this functional are

$$\boldsymbol{\tau} = -\mu[(\nabla\boldsymbol{v})+{}^t(\nabla\boldsymbol{v})]\,, \tag{57}$$

and Eq. (47), and the natural condition is Eq. (50).

It can be seen that the two variational problems described above mutually exchange the admissibility conditions and the natural conditions. Therefore, it is said that these two problems are mutually reciprocal. We will indicate that $I[\boldsymbol{v}]$ has a minimum value and on the other hand, $H[\boldsymbol{\tau}]$ has a maximum value in the stationary state in the following section. The reciprocal characters of these two principle are illustrated in Fig. 1. The transform in these variational problems

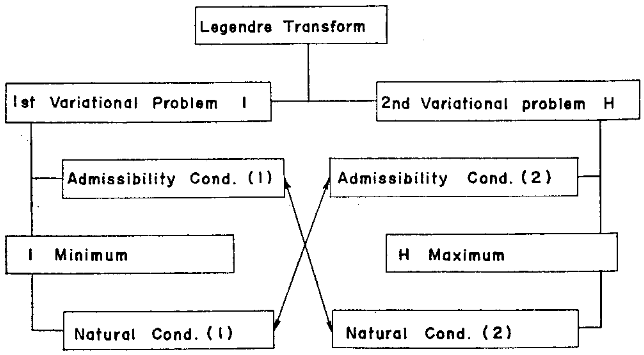

Fig. 1 Relation between the two variational principles.

does not affect the stationary function, therefore, the values of two functional in the stationary state are the same, that is $I_0=H_0$. By application of Eqs. (13) and (14) to Eq. (8), $I[\boldsymbol{v}, p]$ may be rearranged as follows.

$$I[\boldsymbol{v}, p] = \int_V [\rho\bar{\boldsymbol{v}}_t+(\rho\bar{\boldsymbol{v}}\cdot\nabla\boldsymbol{v})+\nabla\cdot\boldsymbol{\tau}+\nabla p-\rho\boldsymbol{g}]\cdot\boldsymbol{v}\,dV-\frac{1}{2}\int_V[\nabla\cdot\boldsymbol{\tau}]\cdot\boldsymbol{v}\,dV$$
$$+\int_{S_T}\left[\left(\boldsymbol{\pi}^0-\frac{\boldsymbol{\tau}}{2}-p\boldsymbol{\delta}\right)\cdot\boldsymbol{n}\right]\cdot\boldsymbol{v}\,dS-\int_{S_T}\left[\left(\frac{\boldsymbol{\tau}}{2}+p\boldsymbol{\delta}\right)\cdot\boldsymbol{n}\right]\cdot\boldsymbol{v}\,dS\,. \tag{58}$$

If we substitute Eqs. (16)$\sim$(18) into Eq. (58), the stationary value of the functional is obtained as follows.

$$I_0 = I[\bar{\boldsymbol{v}}, \bar{p}] = -\frac{1}{2}\int_V[\nabla\cdot\bar{\boldsymbol{\tau}}]\cdot\bar{\boldsymbol{v}}\,dV+\frac{1}{2}\int_{S_T}[(\boldsymbol{\pi}^0-\bar{p}\boldsymbol{\delta})\cdot\boldsymbol{n}]\cdot\boldsymbol{v}\,dS$$
$$-\int_{S_V}\left[\left(\frac{\bar{\boldsymbol{\tau}}}{2}+\bar{p}\boldsymbol{\delta}\right)\cdot\boldsymbol{n}\right]\cdot\boldsymbol{v}^0\,dS\,. \tag{59}$$

3-2. Reciprocal Variational Principle for Heat and Mass Transfer

The free variational problems by use of all the variables θ and $\boldsymbol{q}$, and

$$\boldsymbol{e} = \nabla\theta \tag{60}$$

can be set as follows

$$I[\theta, \boldsymbol{q}, \boldsymbol{e}] = \int_V \left[\rho C_p \frac{D\bar{\theta}}{Dt}\theta + \Phi(\boldsymbol{e}) - Q_h\theta + \boldsymbol{q}\cdot(\boldsymbol{e}-\nabla\theta)\right] dV + \int_{S_q} (\boldsymbol{q}^0\cdot\boldsymbol{n})\theta\, dS + \int_{S_q} (\boldsymbol{q}\cdot\boldsymbol{n})(\theta-\theta^0) dS\,, \tag{61}$$

where,

$$\Phi(\boldsymbol{e}) = \frac{\lambda}{2}(\boldsymbol{e}\cdot\boldsymbol{e}) = \frac{\lambda}{2}\boldsymbol{e}^2\,. \tag{62}$$

By use of the relation

$$\int_V \boldsymbol{q}\cdot(\nabla\delta\theta)\, dV = \int_V [\nabla\cdot\boldsymbol{q}\delta\theta]\, dV - \int_V (\nabla\cdot\boldsymbol{q})\delta\theta\, dV = \int_S (\boldsymbol{q}\cdot\boldsymbol{n})\delta\theta\, dS - \int_V (\nabla\cdot\boldsymbol{q})\delta\theta\, dV\,, \tag{63}$$

the first varation of Eq. (61) can be calculated as follows

$$\delta I = \int_V \left[\left\{\rho C_p \frac{D\bar{\theta}}{Dt} + \nabla\cdot\bar{\boldsymbol{q}} - Q_h\right\}\delta\theta + (\boldsymbol{e}-\nabla\theta)\cdot\delta\boldsymbol{q} + (\boldsymbol{q}+\lambda\boldsymbol{e})\cdot\delta\boldsymbol{e}\right] dV + \int_{S_q} (\boldsymbol{q}^0-\boldsymbol{q})\cdot\boldsymbol{n}\delta\theta\, dS + \int_{S_\theta} (\delta\boldsymbol{q}\cdot\boldsymbol{n})(\theta-\theta^0)\, dS\,. \tag{64}$$

By abbreviation of the symbol of stationary value "$-$" and by putting $\delta I=0$, the next expressions are obtained as Euler's equation

$$\rho C_p \frac{D\theta}{Dt} = -(\nabla\cdot\boldsymbol{q}) + Q_h \qquad \text{in } V \tag{65}$$

$$\boldsymbol{e} = \nabla\theta \qquad \text{in } V \tag{66}$$

$$\boldsymbol{q} = -\lambda\boldsymbol{e}\,, \qquad \text{in } V \tag{67}$$

and the natural conditions are

$$\boldsymbol{q} = \boldsymbol{q}^0 \qquad \text{on } S_q \tag{68}$$

$$\theta = \theta^0\,. \qquad \text{on } S_\theta \tag{69}$$

If Eqs. (66), (67) and (69) are added to the functional Eq. (61) as the admissibility conditions, these subjects are reduced to the initial variational problem about θ.

Legendre transform can be conducted as in the previous section.

$$-\Psi(\boldsymbol{q}) = \Phi(\boldsymbol{e})+(\boldsymbol{e}\cdot\boldsymbol{q}) \,. \tag{70}$$

Then, Eq. (61) can be written as

$$H[\theta, \boldsymbol{q}] = \int_V\left[\rho C_p\frac{D\bar{\theta}}{Dt}\theta-\Psi(\boldsymbol{q})-Q_h\theta-(\nabla\cdot\theta)\cdot\boldsymbol{q}\right]dV$$
$$+\int_{S_q}(\boldsymbol{q}^0\cdot\boldsymbol{n})\theta\, dS+\int_{S_\theta}(\boldsymbol{q}\cdot\boldsymbol{n})(\theta-\theta^0)dS \,. \tag{71}$$

Substitution of Eq. (76) into Eq. (70) gives

$$\Psi(\boldsymbol{q}) = \frac{1}{2\lambda}(\boldsymbol{q}\cdot\boldsymbol{q}) = \frac{1}{2\lambda}\boldsymbol{q}^2 \,, \tag{72}$$

and from Eqs. (66) and (67), we get

$$\boldsymbol{q} = -\lambda(\nabla\theta) \,. \tag{73}$$

Now, by use of the same relation as Eq. (63), Eq. (71) can be rewritten as follows.

$$H[\theta, \boldsymbol{q}] = \int_V\left[\left\{\rho C_p\frac{D\bar{\theta}}{Dt}+(\nabla\cdot\boldsymbol{q})-Q_h\right\}\theta-\Psi(\boldsymbol{q})\right]dV$$
$$+\int_{S_q}(\boldsymbol{q}^0-\boldsymbol{q})\cdot\boldsymbol{n}\theta\, dS-\int_{S_N}(\boldsymbol{q}\cdot\boldsymbol{n})\theta^0\, dS \,, \tag{74}$$

and by adding the next equation and Eq. (68) as the admissibility conditions instead of Eq. (65)

$$\rho C_p\frac{D\bar{\theta}}{Dt} = -(\nabla\cdot\boldsymbol{q})+Q_h \,, \tag{75}$$

the next expression is obtained

$$H[\boldsymbol{q}] = -\int_V\Psi(\boldsymbol{q})dV-\int_{S_u}(\boldsymbol{q}\cdot\boldsymbol{n})\theta^0\, dS \,, \tag{76}$$

in which the reciprocal relation between $I[\theta]$ and $H[\boldsymbol{q}]$ is indicated.

By use of Gauss-Ostrogradskii divergence theorem to Eq. (28) and by subtitution of the natural conditions Eqs. (33)～(35), a expression of stationary value $H_0=I_0$ can be obtained.

$$I_0 = I[\bar{\theta}] = \frac{1}{2}\int_V(\nabla\cdot\lambda\nabla\bar{\theta})\bar{\theta}\, dV+\frac{1}{2}\int_{S_q}(\boldsymbol{q}^0\boldsymbol{n})\bar{\theta}\, dS+\frac{1}{2}\int_{S_\theta}(\lambda\nabla\bar{\theta})\cdot\boldsymbol{n}\theta^0\, \mathrm{d}S \,. \tag{77}$$

The same discussions are possible for mass transfer. By setting

$$\boldsymbol{M} = \nabla C \,, \tag{78}$$

the free variational problems related to Eq. (29) can be written as

$$I[C,\, \boldsymbol{N},\, \boldsymbol{M}] = \int_V \left[\frac{D\bar{C}}{Dt}\cdot C + \Phi(\boldsymbol{M}) - R\cdot C + \boldsymbol{N}\cdot(\boldsymbol{M}-\nabla C)\right] dV + \int_{S_N} (\boldsymbol{N}^0\cdot\boldsymbol{n}) C\, ds + \int_{S_C} (\boldsymbol{N}\cdot\boldsymbol{n})(C-C^0)\, dS\,. \tag{79}$$

$$\Phi(\boldsymbol{M}) = \frac{\mathcal{D}}{2}(\boldsymbol{M}\cdot\boldsymbol{M}) = \frac{\mathcal{D}}{2}\boldsymbol{M}^2\,. \tag{80}$$

The natural conditions are

$$\frac{DC}{Dt} = -(\nabla\cdot\boldsymbol{N}) + R \quad \text{in} \quad V \tag{81}$$

$$\boldsymbol{M} = \nabla C \quad \text{in} \quad V \tag{82}$$

$$\boldsymbol{N} = -\mathcal{D}\boldsymbol{M} \quad \text{in} \quad V \tag{83}$$

and

$$\boldsymbol{N} = \boldsymbol{N}^0 \quad \text{on} \quad S_N \tag{84}$$

$$C = C^0\,. \quad \text{on} \quad S_C \tag{85}$$

Legendre transform is

$$-\Psi(\boldsymbol{N}) = \Phi(\boldsymbol{M}) + (\boldsymbol{M}\cdot\boldsymbol{N})\,, \tag{86}$$

and substitution of Eq. (83) gives

$$\Psi(\boldsymbol{N}) = \frac{1}{2\mathcal{D}}(\boldsymbol{N}\cdot\boldsymbol{N}) = \frac{1}{2\mathcal{D}}\boldsymbol{N}^2 \tag{87}$$

$$H[C,\, \boldsymbol{N}] = \int_V \left[\frac{D\bar{C}}{Dt}\cdot C - \Psi(\boldsymbol{N}) - R\cdot C - (\nabla C)\cdot\boldsymbol{N}\right] dV + \int_{S_N} (\boldsymbol{N}^0\cdot\boldsymbol{n}) C\, dS + \int_{S_C} (\boldsymbol{N}\cdot\boldsymbol{n})(C-C^0)\, dS\,. \tag{88}$$

When we apply the same relation as Eq. (63) to this equation and add the next equation and Eq. (84) as the admissibility conditions instead of Eq. (81)

$$\frac{D\bar{C}}{Dt} = -(\nabla\cdot\boldsymbol{N}) + R\,, \tag{89}$$

$$H[\boldsymbol{N}] = -\int_V \Psi\, dV - \int_{S_C} (\boldsymbol{N}\cdot\boldsymbol{n}) C^0\, dS\,, \tag{90}$$

is derived. This is the reciprocal variational principle for the mass transfer. The expression for the stationary value $I_0 = H_0$ is

$$I_0 = I[\bar{C}] = \frac{1}{2}\int_V (\nabla\cdot\mathcal{D}\nabla\bar{C})\bar{C}\, dV + \frac{1}{2}\int_{S_N} (\boldsymbol{N}^0\cdot\boldsymbol{n})\bar{C}\, dS + \frac{1}{2}\int_{S_C} (\mathcal{D}\nabla\bar{C})\cdot\boldsymbol{n}\, C^0\, dS\,. \tag{91}$$

4. Maximum and Minimum Principle

We have discussed about the equivalence among the boundary value problem, the variational principle and its reciprocal variational principle in the previous sections. But the stationary condition alone is not enough in order to use the variational principle as the approximate method for solving the boundary value problem, because it is only a necessary condition and is not a sufficient condition. The concept of minimizing sequence which is used in the theory of approximate solution by the direct method is based on the premise that the functional could approach a limiting value.

It is the purpose of this chapter that the functional I should be in a minimum value in the stationary state, by calculation of the second variation and examination of the sufficient condition.

4-1. Maximum and Minimum Principle for Motion of Fluid

Calculation of the second variation gives

$$\delta^2 I = \int_V \Big[-\Big\{ \frac{1}{2}[(\nabla \delta \boldsymbol{v}) + {}^t(\nabla \delta \boldsymbol{v})] - \delta \boldsymbol{d} \Big\} : \delta \boldsymbol{\tau} + \frac{\partial^2 \Phi}{\partial \boldsymbol{d} \partial \boldsymbol{d}} : \delta \boldsymbol{d} \delta \boldsymbol{d} - \delta p (\nabla \cdot \delta \boldsymbol{v}) \Big] dV + \int_{S_V} [\delta \boldsymbol{\pi} \cdot \boldsymbol{n}] \cdot \delta \boldsymbol{v} \, dS \,, \tag{94}$$

and then, the variation in the above integral must satisfy the admissibility conditions, so from Eqs. (45), (46) and (50)

$$\left. \begin{aligned} \delta \boldsymbol{d} &= \frac{1}{2}[(\nabla \delta \boldsymbol{v}) + {}^t(\nabla \delta \boldsymbol{v})] && \text{in } V \\ \delta \boldsymbol{\tau} &= -2\mu \delta \boldsymbol{d} && \text{in } V \\ \delta \boldsymbol{v} &= 0 && \text{on } S_V \end{aligned} \right\} \tag{95}$$

and the equation of continuity

$$(\nabla \cdot \delta \boldsymbol{v}) = 0 \,, \tag{96}$$

we can get

$$\delta^2 I = \int_V \frac{\partial^2 \Phi}{\partial \boldsymbol{d} \partial \boldsymbol{d}} : \delta \boldsymbol{d} \delta \boldsymbol{d} \, dV \,. \tag{97}$$

By use of the dummy index, the function in the above integral can be written as the next form from Eq. (39)

$$\frac{\partial^2 \Phi}{\partial \boldsymbol{d} \partial \boldsymbol{d}} : \delta \boldsymbol{d} \delta \boldsymbol{d} = \mu \frac{\partial^2 (d_{pq} d_{pq})}{\partial d_{ij} \partial d_{kl}} \delta d_{ij} \delta d_{kl} \,, \tag{98}$$

and

$$\frac{\partial(d_{pq}d_{pq})}{\partial d_{ij}} = 2d_{ij}\,, \quad \frac{\partial^2(d_{pq}d_{pq})}{\partial d_{ij}\partial d_{kl}} = 2\delta_{ik}\delta_{jl} \tag{99}$$

$$\therefore \quad \delta^2 I = \int_V 2\mu\delta_{ik}\delta_{jl}\delta d_{ij}\delta d_{kl}\,dV$$

$$= \int_V 2\mu\delta d_{ij}\delta d_{ij}\,dV\,. \tag{100}$$

It is obvius that always $\delta d_{ij}\delta d_{ij} \geqq 0$, and $\mu > 0$ for Newtonian fluid, therefore, we can indicate that

$$\delta^2 I \geqq 0\,, \tag{101}$$

from Eq. (100). This is the sufficient condition that $I[\boldsymbol{v}]$ has a minimum value. Then, let us consider the reciprocal problem. The second variation is calculated from Eq. (54)

$$\delta^2 H = \int_V \left[\{[\nabla\cdot\delta\boldsymbol{\tau}]+\nabla\delta p\} : \delta\boldsymbol{v} - \frac{\partial^2\Psi}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}} : \delta\boldsymbol{\tau}\delta\boldsymbol{\tau} \right] dV$$

$$-\int_{S_\tau} [(\delta\boldsymbol{\tau}+\delta p\boldsymbol{\delta})\cdot\boldsymbol{n}]\cdot\delta\boldsymbol{v}\,dS + \int_{S_V} [(\delta\boldsymbol{\pi}-\delta\boldsymbol{\tau}-\delta p\boldsymbol{\delta})\cdot\boldsymbol{n}]\cdot\delta\boldsymbol{v}\,dS\,. \tag{102}$$

From the admissibility conditions Eqs. (55), (48) and (49), we get

$$\left.\begin{aligned} -[\nabla\cdot\delta\boldsymbol{\tau}]-\nabla\delta p &= 0 && \text{in } V \\ \delta\boldsymbol{\tau}+\delta p\boldsymbol{\delta} &= 0 && \text{on } S_\tau \\ \delta\boldsymbol{\tau}+\delta p\boldsymbol{\delta} &= \delta\boldsymbol{\pi} && \text{on } S_V \end{aligned}\right\} \tag{103}$$

Substitution of these relations into Eq. (102) leads to

$$\delta^2 H = -\int_V \frac{\partial^2\Psi}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}} : \delta\boldsymbol{\tau}\delta\boldsymbol{\tau}\,dV\,, \tag{104}$$

and from Eq. (53)

$$\frac{\partial^2\Psi}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}} : \delta\boldsymbol{\tau}\delta\boldsymbol{\tau} = \frac{1}{4\mu}\cdot 2\delta_{ik}\delta_{jl}\delta\tau_{ij}\delta\tau_{kl}$$

$$= \frac{1}{2\mu}\delta\tau_{ij}\delta\tau_{ij} \tag{105}$$

$$\therefore \quad \delta^2 H = -\int_V \frac{1}{2\mu}\delta\tau_{ij}\delta\tau_{ij}\,dV\,. \tag{106}$$

$\delta\tau_{ij}\delta\tau_{ij} \geqq 0$ and $\mu > 0$ are valid for Newtonian fluid, therefore, we can indicate that

$$\delta^2 H \leqq 0\,. \tag{107}$$

This is the sufficient condition that $H[\boldsymbol{\tau}]$ has a maximum value. As $I_0 = H_0$ at the stationary state, the maximum and minimum principle can be expressed as

$$H[\boldsymbol{\tau}] \leqq H_0 = I_0 \leqq I[\boldsymbol{v}]\,. \tag{108}$$

4-2. Maximum and Minimum Principle for Heat and Mass Transfer

The same procedure described above could be done for heat and mass transfer. The second variation of Eq. (61) is calculated as

$$\delta^2 I = \int_V \left[\frac{\partial^2 \Phi}{\partial \boldsymbol{e} \partial \boldsymbol{e}} : \delta\boldsymbol{e}\delta\boldsymbol{e} + \delta\boldsymbol{q}\cdot(\delta\boldsymbol{e} - \nabla\delta\theta)\right] dV + \int_{S_\theta} (\delta\boldsymbol{q}\cdot\boldsymbol{n})\delta\theta\, dS\,. \tag{109}$$

From the admissibilty conditions Eqs. (66), (67) and (69), we get

$$\left.\begin{aligned} \delta\boldsymbol{e} - \nabla\delta\theta &= 0 \quad \text{in} \quad V \\ \delta\boldsymbol{q} + \lambda\delta\boldsymbol{e} &= 0 \quad \text{in} \quad V \\ \delta\theta &= 0 \quad \text{on} \quad S_\theta \end{aligned}\right\}. \tag{110}$$

Substitution of these relations into Eq. (109) gives

$$\delta^2 I = \int_V \frac{\partial^2 \Phi}{\partial \boldsymbol{e} \partial \boldsymbol{e}} : \delta\boldsymbol{e}\delta\boldsymbol{e}\, dV\,. \tag{111}$$

By use of Eq. (62), we get

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial \boldsymbol{e} \partial \boldsymbol{e}} : \delta\boldsymbol{e}\delta\boldsymbol{e} &= \frac{\lambda}{2}\cdot\frac{\partial^2 (e_m e_m)}{\partial e_i \partial e_j}\delta e_i \delta e_j \\ &= \frac{\lambda}{2}\cdot 2\delta_{ij}\delta e_i \delta e_j = \lambda\delta e_i \delta e_i \end{aligned} \tag{112}$$

$$\therefore \quad \delta^2 I = \int_V \lambda\delta e_i \delta e_i\, dV\,. \tag{113}$$

$\delta e_i \delta e_i \geqq 0$, and $\lambda > 0$, then we can indicate that

$$\delta^2 I \geqq 0\,. \tag{114}$$

This is the sufficient condition that $I[\theta]$ has a minimum value. The second variation of the reciprocal varaitional functional Eq. (74) is

$$\begin{aligned} \delta^2 H = &\int_V \left[(\nabla\cdot\delta\boldsymbol{q})\delta\theta - \frac{\partial^2 \Psi}{\partial \boldsymbol{q} \partial \boldsymbol{q}} : \delta\boldsymbol{q}\delta\boldsymbol{q}\right] dV \\ &- \int_{S_q} (\delta\boldsymbol{q}\cdot\boldsymbol{n})\delta\theta\, dS\,. \end{aligned} \tag{115}$$

From the admissibility conditions Eqs. (75) and (68), we get

$$\left.\begin{aligned} \nabla\cdot\delta\boldsymbol{q} &= 0 \quad \text{in} \quad V \\ \delta\boldsymbol{q} &= 0 \quad \text{on} \quad S_q \end{aligned}\right\}. \tag{116}$$

Substitution of these equation into Eq. (115) gives

$$\delta^2 H = -\int_V \frac{\partial^2 \Psi}{\partial \boldsymbol{q} \partial \boldsymbol{q}} : \delta \boldsymbol{q} \delta \boldsymbol{q} dV \,, \tag{117}$$

and from Eq. (72)

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial \boldsymbol{q} \partial \boldsymbol{q}} : \delta \boldsymbol{q} \delta \boldsymbol{q} &= \frac{1}{2\lambda} \cdot \frac{\partial^2 (q_m q_m)}{\partial q_i \partial q_j} \delta q_i \delta q_j \\ &= \frac{1}{2\lambda} \cdot 2 \delta_{ij} \delta q_i \delta q_j = \frac{1}{\lambda} \delta q_i \delta q_i \end{aligned} \tag{118}$$

$$\therefore \quad \delta^2 H = -\int_V \frac{1}{\lambda} \delta q_i \delta q_i dV \,. \tag{119}$$

$\delta q_i \delta q_i \geqq 0$ and $\lambda > 0$ therefore, we can indicate that

$$\delta^2 H \leqq 0 \,. \tag{120}$$

This is the sufficient condition that $H[\boldsymbol{q}]$ has a minimum value. The maximum and minimum principle for heat transfer can be written as

$$H[\boldsymbol{q}] \leqq H_0 = I_0 \leqq I[\theta] \,. \tag{121}$$

The same discussions could be given for mass transfer and the results are as follows. The second variation of Eq. (29)

$$\begin{aligned} \delta^2 I &= \int_V \frac{\partial^2 \Phi}{\partial \boldsymbol{M} \partial \boldsymbol{M}} : \delta \boldsymbol{M} \delta \boldsymbol{M} dV \\ &= \int_V \mathcal{D} \delta M_i \delta M_i dV \geqq 0 \,. \end{aligned} \tag{122}$$

For the reciprocal functional Eq. (90)

$$\begin{aligned} \delta^2 H &= -\int_V \frac{\partial^2 \Psi}{\partial \boldsymbol{N} \partial \boldsymbol{N}} : \delta \boldsymbol{N} \delta \boldsymbol{N} dV \\ &= -\int_V \frac{1}{\mathcal{D}} \delta N_i \delta N_i dV \leqq 0 \,. \end{aligned} \tag{123}$$

The maximum and minimum principle for mass transfer is

$$H[\boldsymbol{N}] \leqq H_0 = I_0 \leqq I[C] \,. \tag{124}$$

5. Lagrange-Biot's Equation

Advantage of the formulation of the variational principle for the boundary value problems is to offer the powerful tool for the approximate solution of boundary value problem by use of the direct method. In this chapter, we will establish the method in which the partial differential equation could be reduced to the ordinary differential equation by use of Ritz's or Galerkin's direct method.

Now, let us consider the variational problem that the functional $I[y]$ becomes stationary about the argument function $y(\boldsymbol{r}, t)$. We assume the admissible function as the next form.

$$y(\boldsymbol{r}, t) = \sum_{k=1}^{n} q_k(t)\cdot y_k(\boldsymbol{r}) , \tag{125}$$

where, $\{y_k(\boldsymbol{r})\}$ are the linearly independent sequence of functions which satisfy the given admissibility conditions and $\{q_k(t)\}$ are the sequende of unknown parameters which are only the function of time t. As we would treat the nonlinear problems, it is better to consider the more generalized form as follows.

$$y(\boldsymbol{r}, t) = y(q_1, q_2, \cdots, q_n : \boldsymbol{r}) . \tag{125$'$}$$

When we substitute Eq. (125)$'$ into the functional $I[y]$, the next form could be obtained.

$$I[y] = I[q_1, q_2, \cdots, q_n] . \tag{126}$$

Then, if we calculate the variation about the parameters $q_1, q_2, \cdots, q_n$, the simultaneous ordinary equations which should determine $\{q_k(t)\}$ could be obtained as Euler-Lagrange's equation. When we solve these equations with the initial condition, the optimal solution colud be obtained in the sense that the functional $I[y]$ has the stationary value or the extremum in the given form of $y(\boldsymbol{r}, t)$. If the initially assumed function may include the from of function of our purpose as the special case, we could obtain the rigorous solution.

Let us consider a space of n th order in which $q_1, q_2, \cdots, q_n$ compose the main coordinates. The state of the system at any time can be expressed as a trajectory drawing from a starting point in this space. In this sense, we designate the parameters $q_1, q_2, \cdots, q_n$ as the generalized coordinates.

5-4. Lagrange-Biot's Equation for Motion of Fluid

We assume that

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}(q_1, q_2, \cdots, q_n; \boldsymbol{r}) \tag{127}$$

$$p(\boldsymbol{r}, t) = p(r_1, r_2, \cdots, r_m; \boldsymbol{r}) , \tag{128}$$

where, $\{q_i\}$ and $\{r_j\}$ are the generalized coordinates and are only the functions of t. The variations of t and p are

$$\delta \boldsymbol{v} = \sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right) \delta q_i \tag{129}$$

$$\delta p = \sum_{j=1}^{m} \left(\frac{\partial p}{\partial r_j}\right) \delta r_j \,, \tag{130}$$

where, the stationary symbols "—" are abbreviated.

$$\frac{\partial \boldsymbol{v}}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right)\left(\frac{\partial q_i}{\partial t}\right) = \sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right) \dot{q}_i \tag{131}$$

$$\therefore \quad \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial \boldsymbol{v}}{\partial t}\right) = \frac{\partial \boldsymbol{v}}{\partial q_i} \tag{132}$$

then,

$$\begin{aligned} \bar{\boldsymbol{v}}_t \cdot \delta \boldsymbol{v} &= \left(\frac{\partial \boldsymbol{v}}{\partial t}\right) \cdot \sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right) \delta q_i = \left(\frac{\partial \boldsymbol{v}}{\partial t}\right) \cdot \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial \boldsymbol{v}}{\partial t}\right) \delta q_i \\ &= \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)^2 \delta q_i \,. \end{aligned} \tag{133}$$

By use of this relation, the first term of Eq. (8) can be expressed as

$$\begin{aligned} \delta \int_V \rho \bar{\boldsymbol{v}}_t \cdot \boldsymbol{v} \, dV &= \int_V \rho \bar{\boldsymbol{v}}_t \cdot \delta \boldsymbol{v} \, dV \\ &= \sum_{i=1}^{n} \delta q_i \left[\frac{\partial}{\partial \dot{q}_i} \int_V \frac{\rho}{2} \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)^2 dV \right] \\ &= \sum_{i=1}^{n} \delta q_i \left(\frac{\partial D}{\partial \dot{q}_i}\right), \end{aligned} \tag{134}$$

where,
$$D = \int_V \frac{\rho}{2} \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)^2 dV \,. \tag{135}$$

The second term of Eq. (8) becomes

$$\begin{aligned} \delta \int_V (\rho \bar{\boldsymbol{v}} \cdot \nabla \boldsymbol{v}) \cdot \boldsymbol{v} \, dV &= \int_V (\rho \bar{\boldsymbol{v}} \cdot \nabla \bar{\boldsymbol{v}}) \cdot \delta \boldsymbol{v} \, dV \\ &= \sum_{i=1}^{n} \delta q_i \int_V (\rho \boldsymbol{v} \cdot \nabla \boldsymbol{v}) \cdot \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right) dV \,. \end{aligned} \tag{136}$$

For the third term of Eq. (8), setting

$$U = \int_V \Phi \, dV = \int_V \mu(\boldsymbol{d} : \boldsymbol{d}) \, dV \,, \tag{137}$$

then,

$$\delta \int_V \Phi \, dV = \delta U = \sum_{i=1}^{n} \delta q_i \left(\frac{\partial U}{\partial q_i}\right). \tag{138}$$

For the fourth term of Eq. (8)

$$\delta\int_V p(\nabla\cdot\boldsymbol{v})\,dV = \int_V \delta p(\nabla\cdot\boldsymbol{v})\,dV + \int_V p(\nabla\cdot\delta\boldsymbol{v})\,dV$$
$$= \sum_{j=1}^{m}\delta r_j \int_V \left(\frac{\partial p}{\partial r_j}\right)(\nabla\cdot\boldsymbol{v})dV$$
$$+\sum_{i=1}^{n}\delta q_i \int_V p\frac{\partial}{\partial q_i}(\nabla\cdot\boldsymbol{v})\,dV\,. \tag{139}$$

For the fifth term of Eq. (8)

$$\delta\int_V \rho\boldsymbol{g}\cdot\boldsymbol{v}\,dV = \int_V \rho\boldsymbol{g}\cdot\delta\boldsymbol{v}\,dV$$
$$= \sum_{i=1}^{n}\delta q_i \int_V \rho\boldsymbol{g}\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q_i}\right)dV\,. \tag{140}$$

Finally, for the term of surface integral

$$\delta\int_{S_C}[\boldsymbol{\pi}^0\cdot\boldsymbol{n}]\cdot\boldsymbol{v}\,dS = \int_{S_T}[\boldsymbol{\pi}^0\cdot\boldsymbol{n}]\cdot\delta\boldsymbol{v}\,dS$$
$$= \sum_{i=1}^{n}\delta q_i \int_{S_T}[\boldsymbol{\pi}^0\cdot\boldsymbol{n}]\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q_i}\right)dS\,. \tag{141}$$

By use of Eqs. (131)$\sim$(141), the first variation of Eq. (8) is consequently expressed as follows.

$$\delta I = \sum_{i=1}^{n}\delta q_i\left[\frac{\partial D}{\partial\dot{q}_i} + \int_V(\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q_i}\right)dV + \frac{\partial U}{\partial q_i} - \int_V p\frac{\partial}{\partial q_i}(\nabla\cdot\boldsymbol{v})\,dV\right.$$
$$\left.-\int_V\rho\boldsymbol{g}\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q_i}\right)dV + \int_{S_T}[\boldsymbol{\pi}^0\cdot\boldsymbol{n}]\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q_i}\right)dS\right]$$
$$-\sum_{j=1}^{m}\delta r_j\left[\int_V\left(\frac{\partial p}{\partial r_j}\right)(\nabla\cdot\boldsymbol{v})\,dV\right]. \tag{142}$$

We can derived Euler-Lagrange's equation by putting $\delta I=0$

$$\frac{\partial U}{\partial q_i} + \frac{\partial D}{\partial\dot{q}_i} = Q_i \qquad (i=1,2,\cdots,n) \tag{143}$$

$$R_j = \int_V\left(\frac{\partial p}{\partial r_j}\right)(\nabla\cdot\boldsymbol{v})\,dV = 0\,, \qquad (j=1,2,\cdots,m) \tag{144}$$

where,

$$Q_i = Q_i^V + Q_i^p + Q_i^g - Q_i^{S_T} \tag{145}$$

$$Q_i^V = -\int_V(\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q_i}\right)dV \tag{146}$$

$$Q_i^p = \int_V p\frac{\partial}{\partial q_i}(\nabla\cdot\boldsymbol{v})\,dV \tag{147}$$

$$Q_i^g = \int_V \rho \boldsymbol{g} \cdot \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right) dV \tag{148}$$

$$Q_i^{S_T} = -\int_{S_T} [\boldsymbol{\pi}^0 \cdot \boldsymbol{n}] \cdot \left(\frac{\partial \boldsymbol{v}}{\partial q_i}\right) dV . \tag{149}$$

If we consider the equation of continuity as the admissibility condition, Eq. (144) is naturally satisfied and then $Q_i^p = 0$, therefore Eq. (147) is not necessary. Euler-Lagrange's equation Eq. (143) in this case is analogous to Lagrange's equation of the mechanical system which has the energy U, dissipation function D and generalized force Q. We wish to designate this equation as Lagrange-Biot's equation connecting with Biot who initially derived this relation. U in Eq. (143) has a physical meaning which expresses the dissipation rate of energy by viscosity in the given field V, therefore, analogy between the mechanical system and the system under consideration is only in the form of equation. For this reason, we call the analogical potential for U and the analogical dissipation function for D to avoid confusion among these of the mechanical system.

5-2. Lagrange-Biot's Equation for Heat and Mass Transfer

We assume that

$$\theta(\boldsymbol{r}, t) = \theta\ (q_1, q_2, \cdots, q_n; \boldsymbol{r}) , \tag{150}$$

where, $q_1, q_2, \cdots, q_n$ are the generalized corrdinates.
The following procedure is analogous to the one of the previous section.

$$\delta\theta = \sum_{i=1}^{m} \left(\frac{\partial \theta}{\partial q_i}\right) \delta q_i \tag{151}$$

$$\frac{\partial \theta}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial \theta}{\partial q_i}\right)\left(\frac{\partial q_i}{\partial t}\right) = \sum_{i=1}^{n} \left(\frac{\partial \theta}{\partial q_i}\right) \dot{q}_i \tag{152}$$

$$\therefore \quad \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial \theta}{\partial t}\right) = \frac{\partial \theta}{\partial q_i} \tag{153}$$

$$\bar{\theta}_t \delta\theta = \left(\frac{\partial \theta}{\partial t}\right) \sum_{i=1}^{n} \left(\frac{\partial \theta}{\partial q_i}\right) \delta q_i = \left(\frac{\partial \theta}{\partial t}\right) \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \theta}{\partial t}\right) \delta q_i$$
$$= \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial \theta}{\partial t}\right)^2 \delta q_i . \tag{154}$$

For the first term of Eq. (28)

$$\delta \int_V \rho C_p \bar{\theta}_t \theta\, dV = \int_V \rho C_p \bar{\theta}_t \delta\theta\, dV = \sum_{i=1}^{n} \delta q_i \left(\frac{\partial D}{\partial \dot{q}_i}\right) , \tag{155}$$

where, the analogical dissipation function can be written as

$$D = \int_V \frac{1}{2} \rho C_p \left(\frac{\partial \theta}{\partial t}\right)^2 dV \tag{156}$$

For the second term of Eq. (28)

$$\delta \int_V \rho C_p (\boldsymbol{v} \cdot \nabla \bar{\theta}) \theta \, dV = \int_V \rho C_p (\boldsymbol{v} \cdot \nabla \bar{\theta}) \delta \theta \, dV$$
$$= \sum_{i=1}^{n} \delta q_i \int_V \rho C_p (\boldsymbol{v} \cdot \nabla \theta) \left(\frac{\partial \theta}{\partial q_i}\right) dV . \tag{157}$$

For the third term, the analogical potential is defined as

$$U = \int_V \Phi \, dV = \int_V \frac{\lambda}{2} (\nabla \theta)^2 dV , \tag{158}$$

then, we get

$$\delta U = \sum_{i=1}^{n} \left(\frac{\partial U}{\partial q_i}\right) \delta q_i . \tag{159}$$

For the fourth term of Eq. (28)

$$\delta \int_V Q_h \theta \, dV = \sum_{i=1}^{n} \delta q_i \int_V Q_h \left(\frac{\partial \theta}{\partial q_i}\right) dV . \tag{160}$$

Finally, for the surface integral

$$\delta \int_{S_q} (\boldsymbol{q}^0 \cdot \boldsymbol{n}) \theta \, dS = \sum_{i=1}^{n} \delta q_i \int_{S_q} (\boldsymbol{q}^0 \cdot \boldsymbol{n}) \left(\frac{\partial \theta}{\partial q_i}\right) dS . \tag{161}$$

By use of Fqs. (155)~(161), the first variationiof Eq. (28) can be written as follws.

$$\delta I = \sum_{i=1}^{n} \delta q_i \left[\frac{\partial D}{\partial \dot{q}_i} + \int_V \rho C_p (\boldsymbol{v} \cdot \nabla \theta) \left(\frac{\partial \theta}{\partial q_i}\right) dV + \frac{\partial U}{\partial q_i} \right.$$
$$\left. - \int_V Q_h \left(\frac{\partial \theta}{\partial q_i}\right) dV + \int_{S_q} (\boldsymbol{q} : \boldsymbol{n}) \left(\frac{\partial \theta}{\partial q_i}\right) dS \right] . \tag{162}$$

From $\delta I = 0$, Lagrange-Biot's equation for heat transfer is obtained.

$$\frac{\partial U}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i , \tag{163}$$

where,

$$Q_i = Q_i^V + Q_i^H + Q_i^{S_q} \tag{164}$$

$$Q_i^V = - \int_V \rho C_p (\boldsymbol{v} \cdot \nabla \theta) \left(\frac{\partial \theta}{\partial q}\right) dV \tag{165}$$

$$Q_i^H = \int_V Q_h \left(\frac{\partial \theta}{\partial q_i}\right) dV \tag{166}$$

$$Q_i^{S_q} = - \int_{S_q} (\boldsymbol{q}^0 \cdot \boldsymbol{n}) \left(\frac{\partial \theta}{\partial q_i}\right) dS . \tag{176}$$

For mass transfer, the same discussion is possible, so the results obtained are written as follows.

$$C(\boldsymbol{r}, t) = C(q_1, q_2, \cdots, q_n; \boldsymbol{r}) . \tag{168}$$

The analogical dissipation function is

$$D = \int_V \frac{1}{2}\left(\frac{\partial C}{\partial t}\right)^2 dV . \tag{169}$$

The analogical potential is

$$U = \int_V \Phi \, dv = \int_V \frac{\mathscr{D}}{2} (\nabla C)^2 dV . \tag{170}$$

The Lagrange-Bio's equation is the same as Eq. (163)

$$Q_i = Q_i^V + Q_i^R + Q_i^{S_N} \tag{171}$$

$$Q_i^V = -\int_V (\boldsymbol{v}\cdot\nabla C)\left(\frac{\partial C}{\partial q_i}\right) dV \tag{172}$$

$$Q_i^R = \int_V R\left(\frac{\partial C}{\partial q_i}\right) dV \tag{173}$$

$$Q_i^{S_N} = -\int_{S_N} (\boldsymbol{N}^0\cdot\boldsymbol{n})\left(\frac{\partial C}{\partial q_i}\right) dS . \tag{174}$$

Example

Let us consider the one-dimensional heat conduction problem.

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2} \qquad (\alpha = \lambda/\rho C_p) \tag{175}$$

$$\left.\begin{array}{llll} I.C. & ; \text{ at } & t = 0, & \theta = 0 \\ B.C.\,1 & ; \text{ at } & x = 0, & \theta = 1 \\ B.C.\,2 & ; \text{ at } & x = L, & \theta = 0 \end{array}\right\} \tag{176}$$

As the steady solution is

$$\Theta = 1 - \frac{x}{L} , \tag{177}$$

therefore, we assume the admissible function that

$$\Theta(q_1, q_2, \cdots, q_m, \cdots) = 1 - \frac{x}{L} + \sum_{m=1}^{\infty} q_m(t)\cdot\sin\frac{m\pi x}{L} . \tag{178}$$

By use of the property of orthogonality

$$D = \frac{1}{2}\rho C_p \int_0^L \left(\sum_{m=1}^{m} \dot{q}_m \sin \frac{m\pi x}{L} \right)^2 dx$$

$$= \frac{1}{2}\rho C_p \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \dot{q}_m \dot{q}_k \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx$$

$$= \frac{1}{2}\rho C_p \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \dot{q}_m \cdot \dot{q}_k \cdot \frac{L}{2}\delta_{mk} = \frac{L}{4}\rho C_p \sum_{m=1}^{\infty} \dot{q}_m^2 \tag{179}$$

$$U = \frac{\lambda}{2} \int_0^L \left(\sum_{m=1}^{\infty} q_m \frac{m\pi}{L} \cos \frac{m\pi x}{L} - \frac{1}{L} \right)^2 dx$$

$$= \frac{\lambda \pi^2}{4L} \sum_{m=1}^{\infty} m^2 q_m^2 + \frac{\lambda}{2L} \tag{180}$$

$$\therefore \quad \frac{\partial U}{\partial q_m} = \frac{\lambda}{2} \cdot \frac{\pi^2 m^2}{L} \cdot q_m, \quad \frac{\partial D}{\partial \dot{q}_m} = \frac{L}{2}\rho C_p \dot{q}_m . \tag{181}$$

As $Q_i = 0$, The Lagrange-Biot's equation can be written as

$$\frac{\partial U}{\partial q_m} + \frac{\partial D}{\partial \dot{q}_m} = 0 . \tag{182}$$

Substitution of Eq. (181) gives

$$\dot{q}_m + \left(\frac{m\pi}{L} \right)^2 \alpha q_m = 0 . \tag{183}$$

Solving this equation, we get

$$q_m(t) = C_m e^{-(m\pi/L)^2 \alpha t} . \tag{184}$$

The coefficients C_m can be determined by subatitution of Eq. (184) into Eq. (178) and by use of the initial condition.

$$1 - \frac{x}{L} = - \sum_{m=1}^{\infty} C_m \cdot \sin \frac{m\pi x}{L} . \tag{185}$$

We can write $1 - x/L$ in the form of Fouriee's series

$$1 - \frac{x}{L} = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \cdot \sin \frac{m\pi x}{L} . \tag{186}$$

Comparison with these equations leads to

$$C_m = - \frac{2}{\pi} \cdot \frac{1}{m} . \tag{187}$$

$$\therefore \quad \Theta(x, t) = 1 - \frac{x}{L} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} e^{-(m\pi/L)^2 \alpha t} \cdot \sin \left(\frac{m\pi x}{L} \right) . \tag{188}$$

5-3. Discussion and Conclusions

The Lagrange's equation of the classical mechanics was initially derived from the base of Newtonian's equation and has been developed to the irreversible process for the lumped systems. In generally, we can write the Lagrange's equation as the next from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}+\frac{\partial D}{\partial \dot{q}_i}=Q_i\ , \tag{189}$$

where, L is Lagrangean, D is Rayleigh's dissipation function and Q_i is the generalized force. If we assume that the kinetic energy T is the function $\dot{q}_i$; $T=T(\dot{q}_i, \dot{q}_2, \cdots, \dot{q}_n)$ and U is the function of q; $U=U(q_1, q_2, \cdots, q_n)$, then Eq. (189) can be written in the next form.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial U}{\partial q_i}+\frac{\partial D}{\partial \dot{q}_i}=Q_i\ . \tag{190}$$

This equation can be applied to the mechanical system and also to the electrical circuit, hydrodynamic system and thermodynamic system. On the other hand, in the systems of continua which were treated in this report, the elements composed the systems are continuously distributed in the space (the distributed system), therefore, consistent treatment seems to be more difficult than the lumped systems.

It becomes possible to treat the distributed system as a analogous lumped system by expressing in the form of Lagrange-Biot's equations which were derived in the present work. The unsteady state behaviour of the fluid dynamical systems, the heat transfer systems and the mass transfer systems can be described by the one equation

$$\frac{\partial U}{\partial q_i}+\frac{\partial D}{\partial \dot{q}_i}=Q_i\ , \tag{191}$$

in which the elements characterized the systems are the the analogical potential U and the analogical dissipation energy D. The generalized forces Q_i are consisted of the following four terms.

(1) convective term Q_i^V

(2) internal source Q_i^H, Q_i^R

(3) external force Q_i^g

(4) boundary conditions $Q_i^{S_T}$, $Q_i^{S_q}$, $Q_i^{S_N}$

The comparison of Eqs. (190) and (191) is reduced to the fact that the systems we trated in this work have not the elements corresponding to the kinetic energy T. This means the fact that the thermodynamical system has the one element of energy and the one element of dissipation term. The reciprocal variational principles which are formulated in this work are not suitable to solve the boundary value problems by use of the direct method because of the strictness of the admissibility conditions, but it is usefull to estimate the error in the calculation by the direct method on the bases of maximum and minimum principle. In the reciprocal variational principles it should be noticed that the argument functions corresponding to the argument functions $\boldsymbol{v}$, θ and C in the variational functional become $\boldsymbol{\tau}$, $\boldsymbol{q}$ and $\boldsymbol{N}$ which express the flux of each variables. The obtained results in this work are summmarized in Table 1∼4.

Table 1. Boundary Value Problems for Transport Phonmena.

	Basic Equation	Boundary Condition
Momentum	$\rho\frac{Dv}{Dt}\equiv\rho\left(\frac{\partial v}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla\cdot\tau-\nabla p+\rho\boldsymbol{g}$ (1) $(\nabla\cdot\boldsymbol{v})=0$ (2) $\tau=-2\mu\boldsymbol{d}$ (3) $\boldsymbol{d}=\frac{1}{2}[(\nabla\boldsymbol{v})+{}^t(\nabla\boldsymbol{v})]$ (4)	$\boldsymbol{v}=\boldsymbol{v}^0$ on S_v (5) $\tau+p\delta=\pi^0$ on S_τ (6)
Heat	$\rho\frac{D\theta}{Dt}\equiv\rho\left(\frac{\partial\theta}{\partial t}+\boldsymbol{v}\cdot\nabla\theta\right)=\nabla\cdot(\lambda\nabla\theta)+Q_h(\boldsymbol{r},t)$ (21) $(\theta=T-T_0)$	$\theta=\theta^0$ on S_θ (22) $-\lambda\nabla\theta=\boldsymbol{q}^0$ on S_q (23)
Mass	$\rho\frac{Dc}{Dt}\equiv\rho\left(\frac{\partial c}{\partial t}+\boldsymbol{v}\cdot\nabla c\right)\nabla\cdot(\mathscr{D}\nabla c)=R(\boldsymbol{r},t)$ (24) $(c=C_A-C_{A0})$	$C=C^0$ on S_c (25) $-\mathscr{D}\nabla c=\boldsymbol{N}^0$ on S_N (26)

Table 2. Variational Principle for Transport Phenomena.

	Functional	Dissipation Function	Admissibility Condition
Momentum	$I[\boldsymbol{v}]=\int_V[\rho\bar{\boldsymbol{v}}_t\cdot\boldsymbol{v}+(\rho\bar{\boldsymbol{v}}\cdot\nabla\bar{\boldsymbol{v}})\cdot\boldsymbol{v}$ $+\Phi-\rho\boldsymbol{g}\cdot\boldsymbol{v}]dV$ $+\int_{S_\tau}[\pi^0\cdot\boldsymbol{n}]\cdot\boldsymbol{v}dS$ (8′)	$\Phi=\frac{1}{2}(-\tau:\nabla\boldsymbol{v})$ $=\mu(\boldsymbol{d}:\boldsymbol{d})$ (9) (39)	$\boldsymbol{v}=\boldsymbol{v}^0$ on S_V (5) $(\nabla\cdot\boldsymbol{v})=0$ (2)
Heat	$I[\theta]=\int_V\rho C_p\bar{\theta}_t\theta+\rho C_p(\boldsymbol{v}\cdot\nabla\bar{\theta})\theta$ $+\Phi-Q_h\theta]dV$ $+\int_{S_q}(\boldsymbol{q}^0\cdot\boldsymbol{n})\theta dS$ (28)	$\Phi=\frac{\lambda}{2}(\nabla\theta)^2$	$\theta=\theta^0$ on S_θ (22)
Mass	$I[c]=\int_V[\bar{C}_t\cdot C+(\boldsymbol{v}\cdot\nabla\bar{C})\cdot C$ $+\Phi-R\cdot c]dV$ $+\int_{S_N}(\boldsymbol{N}^0\cdot\boldsymbol{n})cdS$ (29)	$\Phi=\frac{\mathscr{D}}{2}(\nabla C)^2$	$C=C^0$ on S_C (25)

Table 3. Reociprocal Variational Principle.

	Functional	Dual Dissipation Function	Admissibility Condition
Momentum	$H[\tau]=-\int_V \psi dV$ $-\int_{S_V}[\pi\cdot\boldsymbol{n}]\cdot\boldsymbol{v}^0 dS$ (56)	$\psi=\frac{1}{4\mu}(\tau:\tau)$ (53)	$\rho\frac{D\bar{v}}{Dt}=-[\Delta\cdot\tau]-\nabla p+\rho\boldsymbol{g}$ (55) $\tau+p\delta=\pi^0$ on S_τ (48) $\tau+p\delta=\pi$ on S_V (49)
Heat	$H[\boldsymbol{q}]=-\int_V \psi dV$ $-\int_{S_u}(\boldsymbol{q}\cdot\boldsymbol{n})\theta^0 dS$ (76)	$\psi=\frac{1}{2\lambda}\boldsymbol{q}^2$ (72)	$\rho C_p\frac{D\bar{\theta}}{Dt}=-(\nabla\cdot\boldsymbol{q})+Q_h$ (75) $\boldsymbol{q}=\boldsymbol{q}^0$ on S_q (68)
Mass	$H[\boldsymbol{N}]=-\int_V \psi dV$ $-\int_{S_c}(\boldsymbol{N}\cdot\boldsymbol{n}^0)C^0 dS$ (90)	$\psi=\frac{1}{2\mathcal{D}}\boldsymbol{N}^2$ (87)	$\frac{D\bar{c}}{Dt}=-(\nabla\cdot\boldsymbol{N})+R$ (89) $\boldsymbol{N}=\boldsymbol{N}^0$ on S_N (84)

Table 4. Lagrange-Biot's Eauation.

	Analogical Potential	Analogical Dissipation Function	Generalized Force
Momentum	$U=\int_V \Phi dV$ $=\int_V \mu(\boldsymbol{d}:\boldsymbol{d})dV$ (137)	$D=\int_V\frac{\rho}{2}\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)^2 dV$ (135)	$Q_i=Q_i^V+Q_i^p+Q_i^g+Q_i^{S_\tau}$ (145) $Q_i^V=-\int_V(\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q^i}\right)dV$ (146) $Q_i^p=\int_V p\frac{\partial}{\partial q_i}(\nabla\cdot\boldsymbol{v})dV$ (147) $Q_i^g=\int_V\rho\boldsymbol{g}\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q^i}\right)dV$ (148) $Q_i^{S_\tau}=-\int_{S_\tau}(\pi^0\cdot\boldsymbol{n})\cdot\left(\frac{\partial\boldsymbol{v}}{\partial q^i}\right)dS$ (149)
Heat	$U=\int_V \Phi dV$ $=\int_V\frac{\lambda}{2}(\nabla\theta)^2 dV$ (158)	$D=\int_V\frac{1}{2}\rho C_p\left(\frac{\partial\theta}{\partial t}\right)^2 dV$ (156)	$Q_i=Q_i^V+Q_i^H+Q_i^{S_q}$ (164) $Q_i^V=-\int_V\rho C_p(\boldsymbol{v}\cdot\nabla\theta)\left(\frac{\partial\theta}{\partial q^i}\right)dV$ (165) $Q_i^H=\int_V Q_h\left(\frac{\partial\theta}{\partial q^i}\right)dV$ (166) $Q_i^{S_q}=-\int_{S_q}(\boldsymbol{q}^0\cdot\boldsymbol{n})\left(\frac{\partial\theta}{\partial q^i}\right)dS$ (167)
Mass	$U=\int_V \Phi dV$ $=\int_V\frac{\mathcal{D}}{2}(\nabla C)^2 dV$ (170)	$D=\int_V\frac{1}{2}\left(\frac{\partial c}{\partial t}\right)^2 dV$ (169)	$Q_i=Q_i^V+Q_i^R+Q_i^{S_N}$ (171) $Q_i^V=-\int_V(\boldsymbol{v}\cdot\nabla C)\left(\frac{\partial c}{\partial q^i}\right)dV$ (172) $Q_i^R=\int_V R\left(\frac{\partial c}{\partial q^i}\right)dV$ (173) $Q_i^{S_N}=-\int_{S_N}(\boldsymbol{N}^0\cdot\boldsymbol{n})\left(\frac{\partial c}{\partial q^i}\right)dS$ (174)

Notation

C_p	heat capacity at constant pressure, per unit mass	[cal/g°C]
$c=c_A-c_{A0}$	concentration with reference to base condition	[g/cm³]
D	analogical dissipation function	
$\mathcal{D}$	diffusion coefficient	[cm²/sec]
$\boldsymbol{d}$	rate of deformation tensor	[1/sec]
$\boldsymbol{e}$	a vector defined in Eq. 60	[°C/cm]
$\boldsymbol{g}$	external force	[dyne/g]=[cm/sec²]
$\boldsymbol{M}=\nabla c$	a vector defined in Eq. 78	[g/cm⁴]
$\boldsymbol{N}$	mass flux	[g/cm² sec]
$\boldsymbol{n}$	unit normal vector with outward direction	
p	pressure	[dyne/cm²]
Q_h	rate of heat production	[cal/cm³ sec]
Q_i	generalized force ($i=1, 2, \cdots, n$)	
Q_V	flow rate	[cm³/ sec]
$\boldsymbol{q}$	heat flux	[cal/cm² sec]
q_i	generalized coordinates ($i=1, 2, \cdots, n$)	
R_M	rate of substance production	[g/cm³ sec]
$\boldsymbol{r}$	position vector	[cm]
S	surface of domain	
T	kinetic energy	[erg]
T	absolute temperature	[°K]
T_0	reference temperature	[°K]
t	time	[sec]
U	analogical potential	
V	domain	
$\boldsymbol{v}$	velocity vector	[cm/sec]
$\alpha=\lambda/\rho C_p$	thermal diffusivity	[cm²/sec]
$\boldsymbol{\delta}$	unit tensor	
δ_{ij}	Kronecker's delta	
Θ	non-dimensional temperature	[—]
θ	temperature with reference to base condition	[°K] [°C]
λ	thermal conductivity	[cal/cm sec°C]
μ	viscosity	[g/cm sec]
ν	kinematic viscosity	[cm²/sec]
$\boldsymbol{\pi}$	pressure tensor	[dyne/cm²]
ρ	density	[g/cm³]

$\boldsymbol{\tau}$	:	shear stress tensor	[dyne/cm^2]
Ω	:	potential of external force ($\boldsymbol{g}=-\nabla\Omega$)	[eag/g]

Literature Cited

1) H. Goldstein; Classical Mechanics, Addison-Wesly Pub. (1957)
2) A.E.H. Love; A treatise on the Mathematical Theory of Elasticity, Dover Pub. (1944)
3) H. Takahashi; Denziki-gauk (Electromagnetism), Shōkabō (1962)
4) I. Prigogine; Introduction to Thermodynamics of Irreversible Processes, 2nd Ed., Interscience Pub. (1961)
5) R.B. Bird; The Physics of Fluids, **3**, 539–541 (1960)
6) M.W. Johnson; The Pyhsics of Fluids, **3**, 871–878 (1960)
7) W.E. Stewart; A.I. Ch. E. Journal, **8**, 425–428 (1962)
8) P. Glansdorff, I. Prigogine & D.F. Hays; The Physics of Fluids, **5**, 144–149 (1962)
9) R.S. Schechter; Chem. Eng. Sci., **17**, 803–808 (1962)
10) E.M. Sparrow & R. Siegel; J. Heat Transfer (Trans. ASME, Series C), **81**, 157–167 (1959)
11) P. Rosen; J. Chem. Phys., **21**, 1220–1221 (1953)
11) L.L.G. Chambers; Quart. Journ. Mech. and Applied Math., **9**, Part 2, 234–235 (1956)
13) M.A. Biot; J. Appl. Physi., **27**, 240–253 (1956)
14) M.A. Biot; J. Aeronautical Sciences, **24**, 857–873 (1957)
15) M.A. Biot: J. Aero/Space Sciences, **26**, 367–381 (1959)
16) M.A. Biot; Ibid., **29**, 558–567 (1962)
17) M.A. Biot; ibid., **29**, 568–577 (1962)
18) S.D. Nigam & H.C. Agrawal; J. Math. Mech., **9**, 869–883 (1960)
19) S.G. Gupta; Appl. Sci. Res., Section A, **10**, 85–101, 229–234 (1961)
20) P.M. Mose & H. Feshbach; Methods of Theoretical Physics, Vol. 1, McGraw-Hill (1958)
21) J.C. Slattery; Chem. Eng. Sci., **19**, 801–806 (1964)
22) R.W. Flumerfelt & J.C. Slattery; Chem. Eng. Sci., **20**, 157–163 (1965)
23) R.A. Nichols & S.G. Bankoff; Int. J. Heat Mss Transfer, **8**, 329–335 (1965)
24) J. Snyder, T.W. Spriggs & W.E. Stewart: A.I. Ch. E. Journal, **10**, 535–540 (1964)
25) A. Sommerfeld; Mechanics of Deformable Bodies, p. 89, Academic Press (1946)
26) R.B. Bird, W.E. Stewart & E.N. Lightfoot; Transport Phemonomena, p. 12, John Wiley & Sons (1960)