Negative Resistance in Semiconductors (InSb)

By

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We extend the theory of double injection in insulators, derived by Lampert, so as to adapt it to the case of extrinsic semiconductors. This new treatment is shown to agree reasonably well with our experimentally observed features for n^+p-InSb diodes at low temperature (77°K).

Three outstanding features are revealed by the present analysis:

(1) The relation of $V_M=(1/\beta)\cdot V_{TH}$ pointed out by Lampert is available only in the case that the recombination density N_R is much larger than the free carrier density P_0 (or N_0), i.e., $N_R>10^5\cdot P_0$, where V_M and V_{TH} are the minimum and maximum voltages respectively, β is the capture rate for electrons and holes.

(2) The greater the capture rate β is, the greater the region of the negative resistance becomes. In semiconductors, however, the magnitude of the region is vigorously depedent on a modified recombination density $\mathcal{R}=N_R/P_0$ (or N_R/N_0).

(3)The value of the mobility ratio $b=\mu_n/\mu_p$ is concerned with a rise in the semicoductor region, and the relation of $J\propto V^2$ is satisfactory, if $b<10$. In the material with high mobility as a case of InSb, however, the current rises steeply in proportion to the several powers of the voltage, e.g. $J\propto V^4$, when $b=50$.

As above mentioned, we can sufficiently explain the behavior of double injection in semiconductors or insulators by this treatment.

1. Introduction

The problem of double injection into semiconductors, that is, the simultaneous injection of electrons and holes, has drawn the greatest attention over the past decade. Extensive analytical works of double injection in insulators (semi-insulators) have been performed by Lampert[1,2]. His works are perhaps instructive to show what significant questions concerning its double injection behavior are suggested by systematic study of the problem on physical grounds. He has presented a physical model and an analysis of double injection in insulators from which he predicts the presence of a negative resistance under such conditions that carrier lifetime increases with current. The physical model of Lampert shows that the

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current can be carried to a great extent by recombination processes. He analyzed the negative resistance phenomenon in detail with respect to the double injection currents into trap-free insulators analogous to the vacuum diode and then into insulators with a single set of recombination centers. Then he showed how the presence of the recombination centers affected exhibiting the negative resistance. As a result, it has been theoretically pointed out that the presence of a marked negative resistance in the current-voltage chracteristics should be dependent upon the considerable difference of the capture cross sections for electrons and for holes in the recombination center.

The model of Lampert, however, was studied under the assumption that some effects of the ionized impurity carriers such as acceptors or donors could be neglected. Whereas, we find from the experimental confirmation that such carriers play an important role in the regime of the negative resistance. For example, we can obtain a result that the negative resistance leads to be reduced as the impurity density in a base crystal increases. The authors, therefore, have extended Lampert's theory to the case of a semiconductor in which the electron and hole distributions are non-degenerate. Furthermore, we have examined from both experimental and theoretical studies, the roles of the initial concentration, the mobility ratio and the capture rate for the injected carriers, respectively. To date, there is no satisfactory information about the double injection with negative resistance in semiconductors.

In this paper, we show the results of some computer analyses to suggest what significant parameters are effectively concerned with its double injection behavior. In addition, it is a purpose of this paper to derive a more qualified theory for the double injection negative resistance in semiconductors, and to adapt the theory to the observed current-voltage characteristics with our $n^{+}p$-InSb diodes.

2. Physical Model and Assumptions

Now, we consider the mechanism of the negative resistance for a p-type semiconductor which has a kind of recombination center in the forbidden band. The recombination center is vacant in thermal equilibrium.

In such a semiconductor that the capture probability for the injected electrons is larger than that for the injected holes, the injected electrons are rapidly captured by the recombination center and do not contribute to the current when voltage is applied so as to inject electrons and holes into the semiconductor. At this low injection level, most of the current flows with only free holes in the crystal. But raising the voltage to increase the quantity of the injection, the velocity of the electrons is gradually accelerated and at last within the lifetime of the electrons at the low injection level, they can pass across the cyrstal. This is a critical voltage at which the negative resistance occurs. Beyond this state, since the resistivity in the part where the recombination center traps the injected electrons decreases, the voltage between both terminals is reduced by degrees. This is clearly a phenomnon of the negative resistance. When the recombination center is all filled with the electrons, the terminal voltage becomes minimum, and over this state the sample shows a behavior like a normal semiconductor in high injection level.

The assumptions on which the theory is based are as follows:

(1) We shall now deal with the simple model of p-type semiconductors with a single set of recombination center occupied by holes in thermal equilibrium.

(2) The double injection currents studied here are purely recombination-limited. A generalization of this model is well adjusted to the extrinsic semiconductors in which the density of free carriers is sufficiently high to extinguish a space charge.

(3) The sample thickness L is sufficiently large in comparison with the diffusion length L_d of the injected carriers in a small injection level. For a sample in this case $L/L_d > 5$ is used, and hence the assumption is justified.

(4) A condition of field-independent mobility is held within the voltage range in this study.

In the case of the negative resistance observed in the forward direction, the maximum voltage intensity is less than 50 V/cm, so that it is not necessary to consider the state of hot electrons.

3. The Basic Formulation for This Problem

With the preceding physical models in mind, we now formulate a quantitative mathematical description of the double injection in semiconductors. Using the equations of current flow and the continuity of electrons and holes as well as the experssion of neutrality, we may be able to characterize the behavior of the carriers in the applied electric fields.

$$\boldsymbol{J}_n = q\mu_n(N_0+\delta n)\boldsymbol{\varepsilon}+qD_n\nabla(\delta n)\,, \qquad \cdots\cdots\cdots(1)$$

$$\boldsymbol{J}_p = q\mu_p(P_0+\delta p)\boldsymbol{\varepsilon}-qD_p\nabla(\delta p), \qquad \cdots\cdots\cdots(2)$$

$$\partial(\delta n)/\partial t = (1/q)\nabla\cdot\boldsymbol{J}_n-\delta n/\tau_n\,, \qquad \cdots\cdots\cdots(3)$$

$$\partial(\delta p)/\partial t = -(1/q)\nabla\cdot\boldsymbol{J}_p-\delta p/\tau_p\,, \qquad \cdots\cdots\cdots(4)$$

$$\delta n = \delta p - n_r\,. \qquad \cdots\cdots\cdots(5)$$

The subscripts n and p refer to electrons and holes, respectively; $\boldsymbol{J}$ is the current density and $\boldsymbol{\varepsilon}$ is the electric field intensity; q is the magnitude of the electronic charge; μ and D are the mobility and the diffusion constant, respectively; τ is the lifetime for infected carriers; N_0 and P_0 are the equilibrium densities of electrons and holes; δ_n and δ_p are densities of injected carriers, respectively; n_r is the net capture density for electrons in the recombination center under steady-state conditions.

In order to simplify the mathematics, we shall first consider eliminating the term involving $\nabla\cdot\boldsymbol{\varepsilon}$, which derived from substitution of Eqs. (1) and (2) for $\nabla\cdot\boldsymbol{J}$ in Eqs. (3) and (4). From our previous considerations, we have seen that charge neutarlity holds throughout the bulk semiconductors at low injection levels. In treating the general case, however, it is a poor approximation to put $\nabla\cdot\boldsymbol{\varepsilon}=0$, since the effect of the space charge comes to dominate as the injection level increases. In our model, therefore, it may be considered to eliminate the space charge terms. Let us multiply the result of Eqs. (3), (4) which are combined with (1), (2) by σ_{n0} and σ_{p0} respectively and add, then we obtain the equation

$$\sigma_{p0}\frac{\partial(\delta n)}{\partial t}+\sigma_{n0}\frac{\partial(\delta p)}{\partial t}=\sigma_{p0}\mu_n\nabla\cdot(\delta n\boldsymbol{\varepsilon})-\sigma_{n0}\mu_p\nabla\cdot(\delta p\boldsymbol{\varepsilon})$$
$$+\{\sigma_{p0}D_n\nabla^2(\delta n)+\sigma_{n0}D_p\nabla^2(\delta p)\}-(\sigma_{p0}+\sigma_{n0})\gamma\ , \qquad \cdots\cdots\cdots(6)$$

where $\sigma_{p0}=qP_0\mu_p$ and $\sigma_{n0}=qN_0\mu_n$ are the conductivites on the ionized free holes and electrons, respectively, γ is the recombination rate density which can be defined by $\gamma=\delta n/\tau_n=\delta_p/\tau_p$. It is true, at low injection levels where the injected excess densities δn and δp are less than the defect density N_R and accordingly the corresponding lifetimes τ_n and τ_p are unequal, this condition will be realized. In a steady-state and equilbrium condition, the lifetimes τ_n and τ_p are represented by the equations $\tau_n=1/\langle v_ns_n\rangle p_r$ and $\tau_p=1/\langle v_ps_p\rangle n_r$, respectevely. When the quantity $\langle vs\rangle$ represents the average probability that the injected electrons or holes are captured by the recombination center, it has the dimension of cm^3/sec as is probably seen from the definition of $\langle vs\rangle$. We have already noted that p_r and n_r are the densities of holes and electrons, respectively, in the recombination center, so that the total density N_R of this center is equal to (n_r+p_r). Since the densiteis have the diemension cm^{-3}, $\langle v_ns_n\rangle p_r$ has the dimension of sec^{-1}; it represents the fraction of space swept out per unit time. In this consideration, the recombination rate density γ may be written in the form

$$\gamma=\delta n\langle v_ns_n\rangle p_r=\delta p\langle v_ps_p\rangle n_r\ . \qquad \cdots\cdots\cdots(7)$$

Under a condition of one-dimensional geometry and also assuming that the diffusion currents are negligible, we have a steady-state equation which is very appropriate to our problem, that is,

$$\sigma_{p0}\mu_n \frac{\partial}{\partial x}(\delta n\varepsilon) - \sigma_{n0}\mu_p \frac{\partial}{\partial x}(\delta p\varepsilon) = (\sigma_{p0}+\sigma_{n0})r \,. \qquad \cdots\cdots\cdots(8)$$

3.1 Intrinsic Semiconductors ($P_0 = N_0$)

We first derive the current-voltage characteristics for intrinsic semiconductors under conditions that $P_0 = N_0$ and the both carrier densities are small as compared with the injected hole density δp and electron density δn, i.e. $P_0 = N_0 \ll \delta p$, δn. Now, dividing both sides in equation (8) by $\sigma_{p0}\mu_n$ and taking $P_0 = N_0$, we obtain the equation

$$\frac{\partial}{\partial x}[(\delta n - \delta p)\varepsilon] = (1+b)\frac{r}{\mu_n} \,, \qquad \cdots\cdots\cdots(9)$$

where $b = \mu_n/\mu_p$ is the mobility ratio for both carriers.

The above equation gets close to what is given by Lampert[2]. Now, using the quantity $u = \delta n/\delta p$ (the injected carrier density ratio), the net density of the captured electrons n_r is given by

$$n_r = \frac{\delta n\langle v_n s_n\rangle N_R}{\delta n\langle v_n s_n\rangle + \delta p\langle v_p s_p\rangle} = \frac{\beta u N_R}{1+\beta u} \,, \qquad \cdots\cdots\cdots(10)$$

where $\quad \beta = \langle v_n s_n\rangle / \langle v_p s_p\rangle \,.$

Thus, from the equations (5) and (10), we have

$$\left.\begin{aligned} \delta n &= \frac{u}{1-u} n_r = \frac{\beta u^2}{(1-u)(1+\beta u)} N_R \\ \delta p &= \frac{1}{1-u} n_r = \frac{\beta u}{(1-u)(1+\beta u)} N_R \,. \end{aligned}\right\} \qquad \cdots\cdots\cdots(11)$$

Also, taking $\delta n \gg N_0$ and $\delta p \gg P_0$ in Eqs. (1) and (2), the electric field intensity ε is reduced to the form

$$\varepsilon = \frac{J}{q\mu_p \delta p(1+bu)} = \frac{J}{q\mu_p N_R} \cdot \frac{1}{\beta} g(u) \,, \qquad \cdots\cdots\cdots(12)$$

where

$$g(u) = (1-u)(1+\beta u)/u(1+bu) \,. \qquad \cdots\cdots\cdots(13)$$

Substituting these expressions into (9) and carrying out the indicated differentiation, it may be re-written as

$$h(u) \cdot \frac{\partial u}{\partial x} = \left(\frac{1}{bJ}\right) q\langle v_p s_p\rangle \beta^2 N_R^2 \,, \qquad \cdots\cdots\cdots(14)$$

where

$$h(u) = (1-u)(1+\beta u)^2/u^2(1+bu)\ . \qquad \cdots\cdots\cdots(15)$$

Integrating Eq. (14) from the electron-injecting contact ($x=0$) to the hole-injecting contact ($x=L$), we obtain the total current equation as a function of the injected carrier density ratio $u=\delta n/\delta p$. To complete the solution, the boundary condition at both injection contacts must be applied. As it is evident from Fig. 1, the electric field intensity ε rises smoothly from zero at the electron-injecting contact to a maximum extremly close to the hole injecting conatct, i.e., $\varepsilon=0$ (cathode) and $\varepsilon=\varepsilon_{\max}$ at $x=L$ (anode). It is obvious that, in this case also, $\varepsilon=0$ corresponds to $u=1$. Integrating Eq. (14) under the above boundary conditions, we have

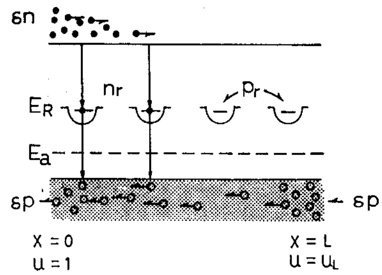


Fig. 1. Schematic features for carrier injection in a simplified model and boundary con ditions for the integration.

$$J = -q\langle v_p s_p\rangle N_R^2 L\cdot\left(\frac{\beta^2}{b}\right)\frac{1}{H(u_L)}\ , \qquad \cdots\cdots\cdots(16)$$

where

$$H(u_L) = \int_{u_L}^{1} h(u)\,du\ , \qquad \cdots\cdots\cdots(17)$$

and the subscript L in u denotes the value at $x=L$.

With Eqs. (12) to (16), the expression obtained for the applied voltage, $V=\int_0^L \varepsilon\, dx$, is

$$V = -\frac{\langle v_p s_p\rangle N_R L^2}{\mu_n}\cdot\beta\frac{F(u_L)}{H^2(u_L)}\ , \qquad \cdots\cdots\cdots(18)$$

where

$$F(u_L) = \int_{u_L}^{1} g(u)h(u)\,du\ . \qquad \cdots\cdots\cdots(19)$$

On account of deciding the direction of the electron injection to be positive, Eqs. (16) and (18) show negative. This fact means that the directions of the vector of the current and the voltage are opposite to the direction of the electron injection. The values of the integrals $H(u_L)$ and $F(u_L)$ can be dircetly obtained, for example, by the method of Gauss's integrations. Here, to estimate each physical term in the expressions of $H(u_L)$ and $F(u_L)$, we may evaluate by the well-known technique of expanding the integrands in partial fractions:

$$H(u)=-A\cdot ln\,(u_L)+B\cdot\left(\frac{1}{u_L}-1\right)+C\cdot\frac{1}{b}\,ln\left(\frac{1+b}{1+bu_L}\right)$$
$$+D\cdot\frac{(1-u_L)}{(1+b)(1+bu_L)}, \qquad \cdots\cdots\cdots(20)$$

$$F(u)=-P\cdot ln\,(u_L)+Q\cdot\left(\frac{1}{u_L}-1\right)+\frac{R}{2}\cdot\left(\frac{1}{u_L^2}-1\right)+S\cdot\frac{1}{b}\,ln\left(\frac{1+b}{1+bu_L}\right)$$
$$+T\cdot\frac{(1-u_L)}{(1+b)(1+bu_L)}+\frac{W}{2}\cdot\frac{(1-u_L)\{2+b(1+u_L)\}}{(1+b)^2(1+bu_L)^2}, \qquad \cdots\cdots\cdots(21)$$

where

$$\left.\begin{aligned}A&=2(\beta-b)-1\,, \qquad B=1\,,\\ C&=\frac{1}{b}(b-\beta)[b(1+2b)+\beta]\,,\\ D&=\frac{1}{b}(1+b)(b-\beta)^2\,,\end{aligned}\right\} \qquad \cdots\cdots\cdots(22)$$

and also,

$$\left.\begin{aligned}P&=3(\beta-b)\{\beta-2(1+b)+1\}\,, \qquad Q=3(\beta-b)-2\,,\\ R&=1\,,\\ S&=\frac{1}{b^2}(\beta-b)[(\beta-b)^2+3b(1+b)\{\beta(1-b)+2b^2\}]\,,\\ T&=\frac{1}{b^2}(\beta-b)(1+b)[(\beta+b)\{b(1-3b)-2\beta\}+2b(3b^2+\beta)]\,,\\ W&=\beta(\beta^2-6\beta+3)-3bP-3b^2Q-b^3R-S-T\,.\end{aligned}\right\} \qquad \cdots\cdots\cdots(23)$$

As is obvious from Eqs. (20) to (23), $H(u_L)$ and $F(u_L)$ are integral functions with variables of the recombination capture rate $\beta=\langle v_n s_n\rangle/\langle v_p s_p\rangle$ and the mobility ratio $b=\mu_n/\mu_p$. Substituting Eqs. (20) and (21) into Eqs. (16) and (18), and solving these equations with respect to $u_L=(\delta n/\delta p)_{x=L}$ which is u at the hole injection electrode ($x=L$), then we can find the influence of β and b on the voltage-current characteristics.

Now, for convenience, in Eqs. (16) and (18), we set as follows:

$$J_1=-q\langle v_p s_p\rangle N_R^2 L(A/\mathrm{cm}^2)\,, \qquad \cdots\cdots\cdots(24)$$
$$V_1=-\langle v_p s_p\rangle N_R L^2/\mu_n(V)\,, \qquad \cdots\cdots\cdots(25)$$

and indicating the voltage ($|V|/|V_1|$)-current ($|J|/|J_1|$) characteristics in a diagram, so we get Figs. 2 to 4 as results. Fig. 2 shows an influence of the capture ratio β in the case of $b=50$, and we find that the negative resistance region increases with the increase of β. Then Figs. 3 and 4 show an effect of the mobility ratio b with constant β; 10^2 and 10^3 respectively. They show that the smaller b of a sample is, the more extensive the negative resistance region becomes.

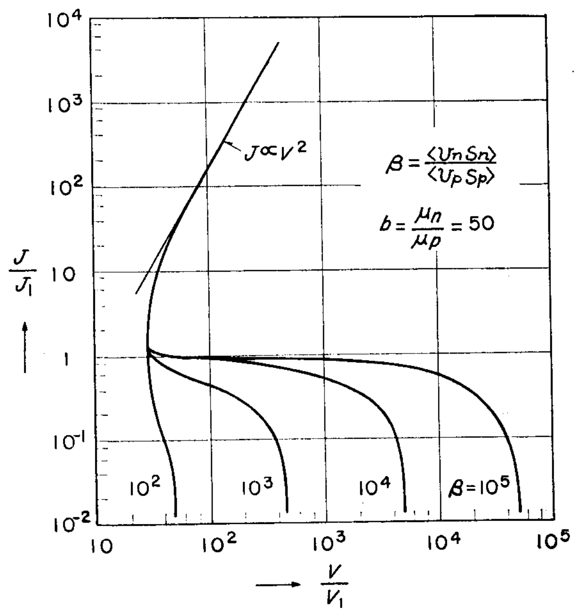


Fig. 2. Computed current-voltage characteristics with various values of the capture rate $\beta = \langle v_n s_n \rangle / \langle v_p s_p \rangle$.

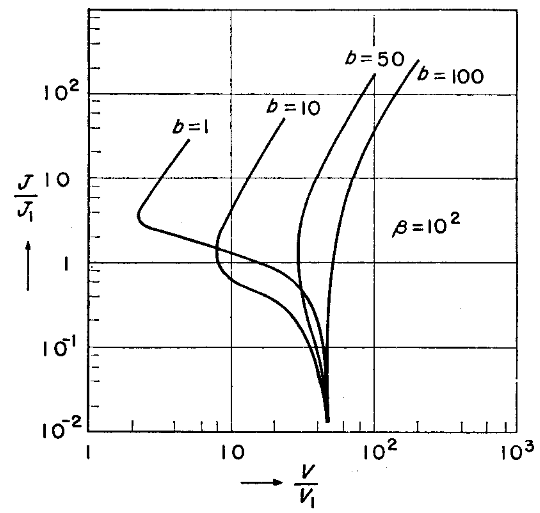


Fig. 3. Current-voltage characteristics as functions of the mobility ratio $b = \mu_n / \mu_p$ with $\beta = 10^2$.

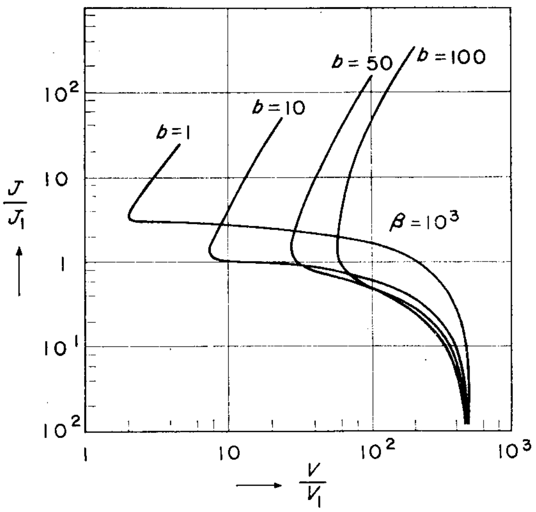


Fig. 4. Current-voltage characteristics as functions of the mobility ratio $b=\mu_e/\mu_p$ with $\beta=10^3$.

3.2 Voltage Current Characteristics for Normal Extrinsic Semiconductors ($P_0 \neq N_0$)

In the above considerations, we have discussed under the assumptions that the impurity with ionized free carriers are negligible in comparison with the injected carriers and just as equal to each other ($N_0=P_0$). For extrinsic semiconductors, however, since the assumptions are not valid, we may not adapt the expressions derived in the preceding section to the case of semiconductors in which the electron and hole distributions are non-degenerate. In this section, therefore, we shall study double injection under conditions that the density of free carriers to be present initially is sufficiently high to permit charge neutralization of the injected carriers.

Usually they present a conductivity; $P_0>N_0$ (p-type) or $P_0<N_0$ (n-type). Here, for convenience, we deal with the former case of p-type semiconductors, and concerning about the injected electrons, we regard the capture probability ratio as positive, i.e. $\beta>0$. First taking $\phi=N_0/P_0$ in the fundamental Eq. (8), so we get $\sigma_{n0}/\sigma_{p0}=b\phi$. Next, dividing δn and δp by P_0, and using the dimensionless density $\nu=\delta n/P_0$ and $\pi=\delta p/P_0$, Eq. (8) is reduced to the form:

$$\frac{\partial}{\partial x}(\nu\varepsilon)-\phi\frac{\partial}{\partial x}(\pi\varepsilon)=(1+b\phi)\frac{r}{\mu_n P_0}\,. \qquad \cdots\cdots\cdots(26)$$

Here, after the previous method of solution,

$$\nu=\frac{\beta u^2}{(1-u)(1+\beta u)}\mathcal{R}\,, \qquad \cdots\cdots\cdots(27)$$

$$\pi = \frac{\beta u}{(1-u)(1+\beta u)} \mathcal{R} , \qquad \cdots\cdots\cdots(28)$$

$$\varepsilon = \frac{J}{q\mu_p P_0}\left\{1+b\phi+\frac{\beta u(1+bu)}{(1-u)(1+\beta u)}\mathcal{R}\right\}^{-1} , \qquad \cdots\cdots\cdots(29)$$

where $\mathcal{R} = N_R/P_0$.

Substituting Eqs. (27), (28) and (29) into Eq. (26), we get the following equation as a result corresponding to Eq. (14) in the previous section:

$$h_0(u)\cdot\frac{\partial u}{\partial x} = \frac{(1+b\phi)^2}{Jb}\cdot q\langle v_n s_n\rangle P_0 N_R , \qquad \cdots\cdots\cdots(30)$$

where

$$h_0(u) = \frac{(1-u)(1+\beta u)^2[\{Q(1+b\phi)+\beta(1-\phi)-1\}u^2+2u-\phi]}{u^2\{(bQ-\beta)u^2+(Q+\beta-1)u+1\}^2} , \qquad \cdots\cdots\cdots(31)$$

with $Q = \beta\mathcal{R}/(1+b\phi) = \beta N_R(P_0+bN_0)$.

Integrating Eq. (30) over the sample, we can obtain the equation of total currents in semiconductors as a function of $u_L(=\delta n/\delta p)$:

$$J = -q\langle v_p s_p\rangle N_0 N_R L\cdot\beta\left\{\frac{(1+b\phi)^2}{b\phi}\right\}\frac{1}{H_0(u_L)} , \qquad \cdots\cdots\cdots(32)$$

where the value of $H_0(u_L)$ is

$$H_0(u_L) = \int_{u_L}^{1} h_0(u)du . \qquad \cdots\cdots\cdots(33)$$

As for the terminal voltage V, we can seek for it with $V=\int_0^L \varepsilon\, dx$. Using Eqs. (29), (30) and (32), we can get the following equation in the same manner of the previous method:

$$V = -\frac{\langle v_p s_p\rangle N_R L^2}{\mu_n}\cdot\beta(1+b\phi)\frac{F_0(u_L)}{H_0^2(u_L)} . \qquad \cdots\cdots\cdots(34)$$

In Eq. (34),

$$F_0(u_L) = \int_{u_L}^{1} q_0(u)h_0(u)du , \qquad \cdots\cdots\cdots(35)$$

and $g_0(u)$ $h_0(u)$ becomes as follows:

$$g_0(u)\ h_0(u) = \frac{(1-u)^2(1+\beta u)^3[\{Q(1+b\phi)+\beta(1-\phi)-1\}u^2+2u-\phi]}{u^2\{(bQ-\beta)u^2+(Q+\beta-1)u+1\}^3} . \qquad \cdots(36)$$

It is not always easy to obtain values of $H_0(u_L)$ and $F_0(u_L)$, but taking $u=10^U$

in the functions (31) and (36), and considering that the integration limit is $-12 < U < 0$, i.e., $10^{-12} < u < 1$, we can estimate with Gauss's integration method by a computer. As is obvious from (31) and (36), the functions $H_0(u_L)$ and $F_0(u_L)$ clearly include the variables β, b, Q and ϕ, namely

β : recombination rate

b : mobility ratio

N_R, P_0 and N_0: the densities of the recombination centers, free holes and free electrons respectively.

Therefore, we can find how the current J and the voltage V are effected by these parameters.

Here, we deal with the effects of double injection in the exhaustion region in which the thermally-generated free carriers can be neglected. Hence, the density of the free electrons N_0 $(=N_d)$ is almost compensated by the free holes P_0 $(=N_a)$ at low temperature (77°K), so virtually we can approximate $P_0 \gg N_0$. In the following estimation, we shall take now $N_0 = 10^2$ cm^{-3} as the reserved electron density and $P_0 = 10^{13}$ to 10^{16} cm^{-3} which are appropriate in the case of InSb.

Under these conditions, rewriting the right hands of Eqs. (32) and (34) in consideration of $b\phi \ll 1$ since $b < 10^2$, we get following equations:

$$\frac{|J|}{|J_1|} \approx \beta\left(2+\frac{P_0}{bN_0}\right)\cdot\frac{1}{H_0(u_L)} \approx \frac{\beta}{b\phi}\cdot\frac{1}{H_0(u_L)}, \qquad \cdots\cdots\cdots(37)$$

$$\frac{|V|}{|V_1|} \approx \beta\cdot\frac{F_0(u_L)}{H_0^2(u_L)}, \qquad \cdots\cdots\cdots(38)$$

where J_1 and V_1 are defined as follows:

$$\left.\begin{aligned} J_1 &= -q\langle v_p s_p\rangle N_0 N_R L \quad (A/\mathrm{cm}^2)\,, \\ V_1 &= -\langle v_p s_p\rangle N_R L^2/\mu_n \quad (V)\,. \end{aligned}\right\} \qquad \cdots\cdots\cdots(39)$$

In the next section, we discuss in detail about the condition which is necessary to generation of the negative resistance and about the effect of each parameter which influences a magnitude of the negative resistance regime.

4. Influence of Each Parameter on A Negative Resistance

As is explained in the previous section, the values of the integral functions $H_0(u_L)$ and $F_0(u_L)$ are greatly changed by the capture rate β to injected electrons and holes, by the mobility ratio b, by the density of recombination center N_R and by the free hole density P_0.

According to the quantitative calculations, we shall now discuss how these significant terms have influence on the magnitude of the negative resistance regime.

4.1 Modified Recombination Density $\mathcal{R}=N_R/P_0$

The free hole density P_0 and the recombination center density N_R in a thermal equilibrium interchange their carriers each other. Therefore a modified recombination density $\mathcal{R}$, denoted as a ratio N_R/P_0, is one of the most important parameters which dominate the magnitude of the negative resistance. As an example, taking $b=50$, $\beta=10^4$ and $N_R=10^{14}$ cm^{-3}, we show the calculated results in Fig. 5 with P_0 from 10^{13} cm^{-3} to 10^{16} cm^{-3}.

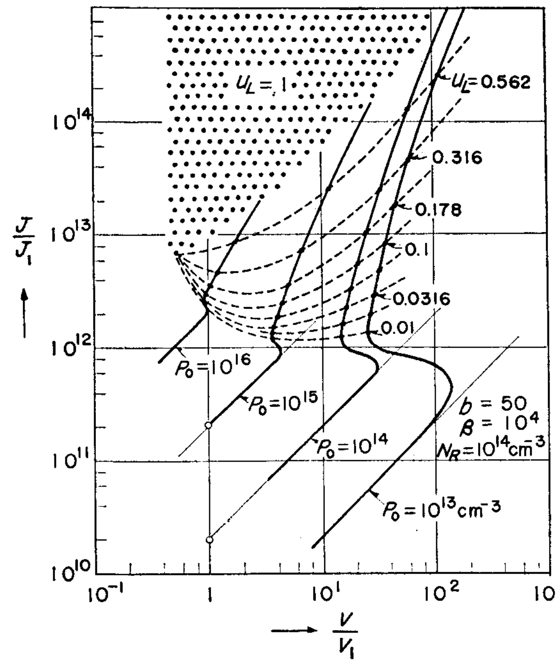


Fig. 5. Dependence of the ionized holes P_0. The dashed curves give the equal injection rate. The dot area shows an expression of perfect plasma; i.e., $u \approx 1$.

As shown in Fig. 5, it is clear that the negative resistance disappears when the free hole density P_0 overcomes the density of the recombination center, in other words, under the condition of $\mathcal{R} \leqslant 1$.

In the semiconductor regime of Fig. 5 the value of $u_L=\delta n/\delta p$ is shown. We find in Fig. 5 that the state of the perfect plasma ($u_L=1$) occurs in a much higher current region than the minimum voltage V_M. This contributes to the explanation of a plasma oscillation which will be mentioned in a following section. On the contrary, when N_R overcomes P_0, the negative resistance regime increases as shown in Fig. 6. Besides under the condition of $\mathcal{R}>10^3$, the current remarkably rises from the low-level where P_0 is dominant, to the threshold voltage V_{TH} at which the negative resistance occurs.

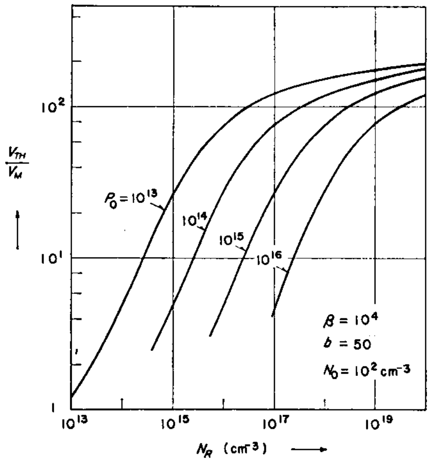


Fig. 6. Influcnce on the voltage ratio V_{TH}/V_M with increase of the recombination densisy N_R.

This characteristic form is enough to explain Holonyak's experimental results in a *p-Si-n* diode which is heavily *Cu*-doped *Ge*[6].

4.2 Caputure Rate $\beta=\langle v_n s_n\rangle/\langle v_p s_p\rangle$

First, we consider how the terminal voltage given by Eq. (38) is changed by the capture rate β, so we get Fig. 7 as a function of u_L which is the injection ratio at $x=L$. In Fig. 7, for convenience, considering $P_0=10^{13}\,\mathrm{cm}^{-3}$, $N_R=10^{14}\,\mathrm{cm}^{-3}$ (i.e., $N_R/P_0=10$) and $b=50$, we calculated the terminal voltage with variables of β from 10 to 10^5. Fig. 7 clearly shows that, according to the increase of u_L $(=\delta n/\delta p)$ at the hole injection contact, the terminal voltage $|V|/|V_1|$ at first monotonically increases and becomes maximum at a certain critical value. And then over this critical value it decreases by degrees. This implies the phenomenon of the negative resistance, and this maximum voltage indicates a critical threshold voltage at which negative resistance occurs in a bulk semiconductor. When u_L increase still more, $|V|/|V_1|$ has a minimum value. This state occurs when the recombination center is perfectly occupied with injected electrons, and approaching plasma state $(u_L=1)$, the diode begins to exhibit positive the characteristics of a resistance with current increasing.

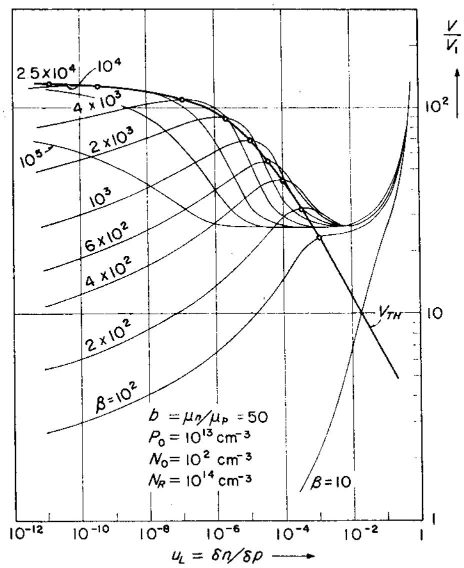


Fig. 7. Relation of the dimensionless voltage $|V|/|V_1|$ vs. the injection carrier rate $u_L=\delta_n/\delta_p$ with various values of $\beta(\beta<0)$.

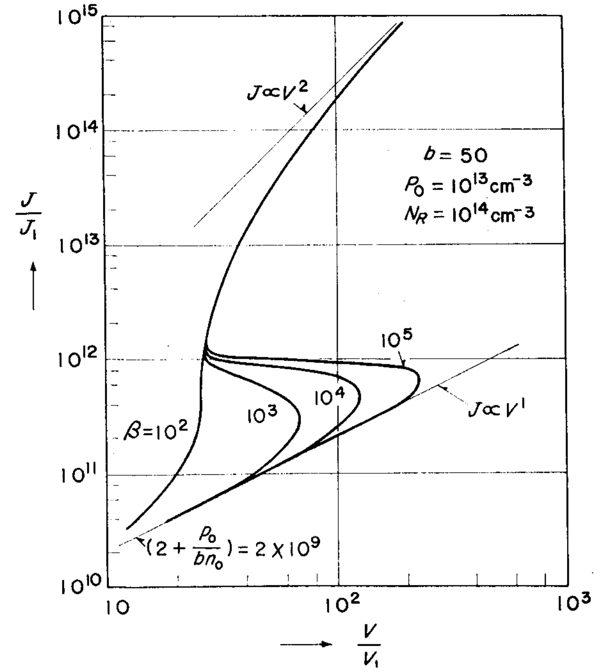


Fig. 8. Role of the capture rate β for a high mobility semiconductor (e.g., InSb).

Next, we discuss the effect of β on the negative resistance above mentioned. Fig. 7 clearly shows that, in order to obtain a remarkable negative resistance region, it is necessary that β is larger than a certain value. When β increases, the critical threshold voltage becomes greater and moves to the side of low injection level, that is, to the small value of u_L.

To confirm the effect of β more accurate, calculating the voltage-current characteristics, by Eqs. (37) and (38), Fig. 8 can be obtained. This calculated result is very different from Lampert's one in some points. For example:

(1) The behaviour of curve at low-injection level where ionized hole P_0 is dominant.

(2) The value of the critical threshold voltage V_{TH} is much smaller than the result calculated under condition of $P_0=N_0\ll\delta n$, δp (c.f. Fig. 3).

(3) In the semiconductor regime over the minimum voltage V_M, current J is not always proportional to square of voltage V in the case of $b=50$.

About (3) we shall discuss in detail in the following section.

4.3 Mobility Ratio $b=\mu_n/\mu_p$

In order to examine the effect of mobility ratio b on voltage-current characteristics, setting $\beta=10^4$, $P_0=10^{13}$ cm^{-3}, $N_R=10^{14}$ cm^{-3} in Eqs. (37) and (38), and estimating those equations, we get Fig. 9. This figure shows that in the limited case of samples with a small mobility, such as *Ge* or *Si*, the current increases in

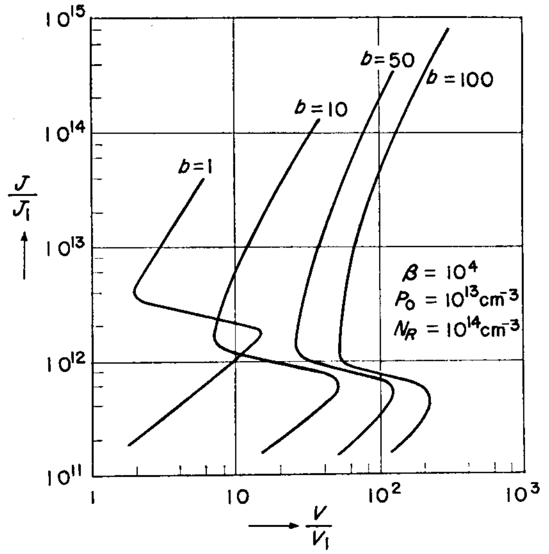


Fig. 9. Role of the mobility ratio $b=\mu_n/\mu_p$ with a constant value $\beta=10^4$ and $(N_R/R_0)=10$.

proportion to the square of voltage, i.e. $J \propto V^2$ in the semiconductor regime beyond the minimum voltage V_M, as pointed out by Lampert.

For samples with a large mobility ratio as in the case of InSb, however, this relation is not appropriate, that is, current J rises steeply in proportion to the third or the fifth power of voltage.

On the other hand, the magnitude of the negative resistance regime is hardly influenced by b as the value of β is more than 10^3, but it remarkably decreases according to the increase of b, if $\beta < 10^3$.

5. Correspondence of the Theoretical Values and the Experimental Results

To obtain an experimental confirmation of the present theory, we have compared it with our experimental results in some n^+pp^+ diodes fabricated on *Cu*-doped InSb (p-type). The ambient temperature of all these diodes is 77°K at which the region is an exhausion one.

Fig. 10 shows a negative resistance which has been observed in the forward characteristics of a typical InSb diode. For this diode the thickness of the p base is 3 mm, and the resistivity of the base at 77°K is 11 ohm-cm. The n^+ contact for

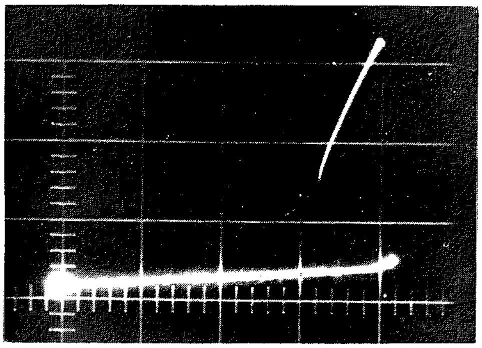


Fig. 10. One of the observed negative resistance in the forward direction in n^+pp^+-InSb at 77°K.
Horizontal scale : 2.5 V/div.,
Vertical scale : 2.0 mA/div..

electron injection was made by alloying to the base indium (In) sphere containing a small percentage of tellurium (Te). For the p^+ ohmic contact was alloyed with indium sphere containing of zinc (*Zn*). After alloying, the surfaces were etched with CP_4 till the diameter becomes about 0.2 mm.

In Fig. 11 we show the voltage-current characteristics which were obtained by our experiments. The carrier densities of these samples 1 and 2 are 1.6×10^{14} cm^{-3} and 6×10^{14} cm^{-3} respectively.

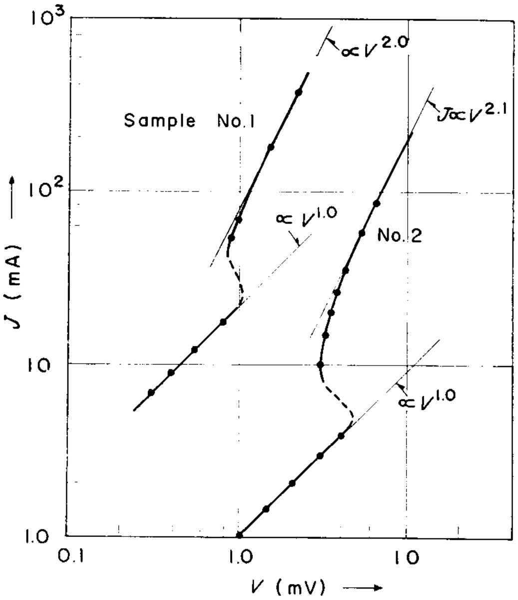


Fig. 11 Two typical current-voltage characteristics measured in n^+pp^+-InSb diodes, on a log-log plot.

This experimental results are enough to show that our calculating analysis mentioned in the previous section is very resonable.

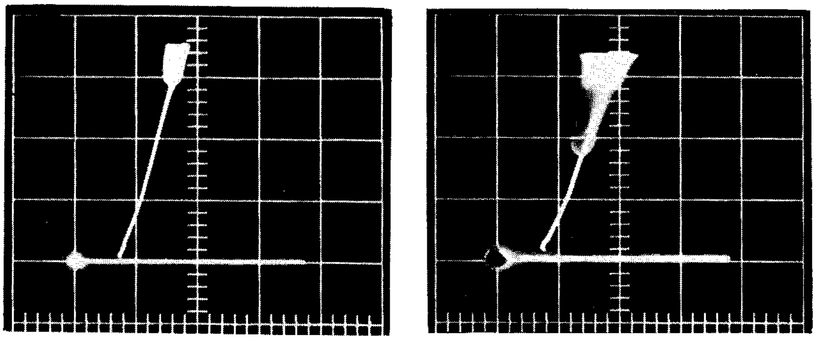
From these experimental results we know as follows:

(1) In general the negative resistance is not observed when the density of the free holes in the base goes over 10^{15} cm^{-3}. Since the degenerate density in InSb is 3.4×10^{15} cm^{-3} at 77°K, the behavior as a semiconductor is not observed beyond this value.

(2) The magnitude of the negative resistance regime is obviously dependent on the ratio of the recombination density N_R and the free hole density P_0, that is, the modified recombination density $\mathcal{R}(=N_R/P_0)$. However, it is difficult in practice to control separately the densities of N_R and P_0. Because, in the majority of cases, the free hole density also increases as the doped impurity for the recombination increases.

(3) In these diodes with a large value of b, the rise of the current in a high injection level over V_M scarcely does not change irrespective of the base resistance, that is, of the equilibrium density of holes.

Hitherto the point of V_M was regarded as the state of complete plasma. In the previous section 4.1, however, we pionted out that the state of complete plasma, where the value of the ratio $\delta n/\delta p$ is 1, is not on the point of V_M but in the high current region over that point. We predicted on the phenomenon of oscillation according to the instability of plasma in section 4.1. Figs. 12 (*a*) and (*b*) show plasma oscillations in the case that the direction of the electric field is parallel to the magnetic field and the field intensity is 3,500 Gauss in Fig. (a) and 4,500 Gauss in Fig. (b) respectively. The critical value at which the oscillation occurs approaches to V_M with the increase of the magnetic field. Fig. 12 (b) shows that the oscillation occurs even in the region of an incomplate plasma ($u_L<1$). It may



(a) (b)

Fig. 12. Two typical plamsa oscillation modes observed in a magnetic field with a n^+pp^+-InSb diode at 77°K.
(a) : B=3,500 Guass, (b): B=4,500 Gauss.
Horizontal scale : 2.0 V/div.,
Vertical scale : 50 mA/div..

be considered that the oscillation condition, i.e., $\omega_p\tau(\mu B)=1$, is satisfied with the increase of the magnetic field. Besides as shown in Fig. 12 (b), it is clear that the mode of the oscillating wave is a relaxation oscillation in this state of the incomplete plasma, and it is clearly distinguished from the mode of sine wave in the state of a comparatively complete plasma. This fact implies that the uniform plasma wave may be disturbed by the excess holes in the base crystal. This tendency may be accounted for by the dotted area shown in Fig. 5.

6. Discussion

As mentioned above, under the theoretical solutions by computer and the experimental results we explain that the magnitude of the negative resistance regime is mainly determined by the correlation between the capture rate $\beta = \langle v_n s_n \rangle / \langle v_p s_p \rangle$ and the modified recombination density $\mathcal{R} = N_R/P_0$. Here, at first, we examine how these two factors relate to the magnitude of the negative resistance.

Fig. 13 shows the relation between the ratio of the threshold voltage V_{TH} and the minimum voltage V_M (V_{TH}/V_M) and β with a variable of N_R/P_0. As shown in this figure, under condition that the free carrier density P_0 is equal to the density of the recombination center N_R, the value of the negative resistance is small even if β is large. And when $N_R/P_0 = 10$, V_{TH}/V_M increases in proportion to $\beta^{0.5}$, and it approaches to $V_{TH}/V_M \propto \beta$ according to the increase of N_R/P_0. On the other hand, Lampert[2] pointed out that, when $\beta > 0$, the relation between

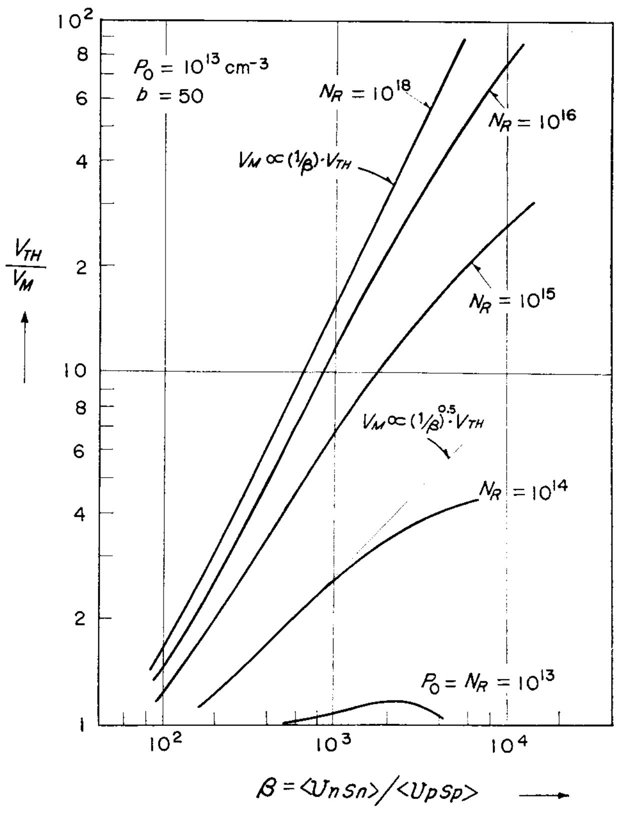


Fig. 13 Essential relation for normal semiconductors. The expression of $V_M \approx (1/\beta) \cdot V_{TH}$ pointed out by Lampert is appropriate, if $(N_R/P_0) > 10^4$

V_{TH} and V_M is combined with

$$V_M = (1/\beta) \cdot V_{TH} . \qquad \cdots\cdots\cdots(40)$$

This relation is not appropriate to normal semiconductors in which the free hole density P_0 is not negligible. Namely, using Eq. (40) we cannot confirm the experimental result which shows that the value of the negative resistance tends to decrease according to the increase of free holes in the bulk. Eq. (40) is valid only under the condition of $P_0 \ll N_R$, for example $P_0 = 10^{13}$ cm^{-3}, $N_R = 10^{18}$cm^{-3}.

Next, we consider the critical voltage V_{TH} and the minimum voltage V_M. At first, from the condition that the lifetime of the injected electrons is equal to the transit time, we get easily

$$V_{TH} = \langle v_n s_n \rangle N_R L^2 / 2\mu_n \qquad \cdots\cdots\cdots(41)$$

as the value of the critical threshold voltage.

In semiconductors, however, this threshold voltage does not practically conform to the experimental results, so we try to derive as follows.

The current in the low injection state at which δn and δp are negligible in comparison with N_0 and P_0 in the thermal equilibrium is given as follows:

$$J_0 = q(bN_0 + P_0)\mu_p V/L . \qquad \cdots\cdots\cdots(42)$$

When this current increases and the influence of the injected carriers is effective, the current coincides with Eq. (37). A point of the coincidence gives approximately the critical voltage. Namely, by setting $J_0 = J_1$, we get a following equation:

$$\frac{J}{J_1} = \left(2 + \frac{P_0}{bN_0}\right) \frac{V}{V_1} , \qquad \cdots\cdots\cdots(43)$$

where J_1 and V_1 show constants in Eq. (39).

As shown in Figs. 5 and 8, Eq. (43) is one which goes through the point where $J/J_1 = (2 + P_0/bN_0)$ and $|V|/|V_1| = 1$ and has a slope of 45°. As a matter of fact, the cross point of $J_0 = J$ shows the critical voltage, but we cannot get an accurate vlaue unless $N_R \gg P_0$.

Next we consider the minimum voltage V_M.This point of the minimum voltage V_M is the state in wnich the recombination center is completerly filled with the injected electrons. As shown in Fig. 7, u_L in this case exsists $10^{-6} < u_L < 1$. We extend the integral functions of $H_0(u_L)$ and $F_0(u_L)$ given by Eqs. (33) and (35) into partial fractions, and examine which terms in these fractions are dominant. We take appropriate values to P_0, N_R, b and β etc., and examine near the characteristics V_{TH} and V_M. When β is sufficiently large ($\beta > 10^4$), we can approximate $H_0(u_L)$ and $F_0(u_L)$ with $u_L \ll 1$ as follows (see Appendix):

$$H_0(u_L) \approx \frac{D_H}{2l^2\beta^2} \cdot \frac{1}{(u_L+r)^2-s}, \quad \cdots\cdots\cdots(44)$$

$$F_0(u_L) \approx \frac{F_F}{4l^2\beta^3} \cdot \frac{1}{\{(u_L+r)^2-s^2\}^2}, \quad \cdots\cdots\cdots(45)$$

where D_H, F_F, l, r and s are defined in Appendix.

When we calculate V by using these $H_0(u_L)$ and $F_0(u_L)$, we get a simple result as follows:

$$\frac{V}{V_1} = b \quad \text{(becomes constant)}, \quad \cdots\cdots\cdots(46)$$

where V is equal to V_M in this case.
The values in Fig. (7) are calculated as $b=50$ (for InSb). Fig. (7) shows the value of V/V_1 to be constant ($=28$) in the region of $10^{-6} < u_L < 1$. Accordingly Eq. (46) is satisfactory.

Then we substitute $V_1 = \langle v_p s_p \rangle N_R L^2/\mu_p$ into Eq. (46) above mentioned, and we get $V = \langle v_p s_p \rangle N_R L^2/\mu_p$. Therefore, we get the following equation:

$$\frac{L^2}{\mu_p V} = \frac{1}{\langle v_p s_p \rangle N_R}. \quad \cdots\cdots\cdots(47)$$

The left hand of Eq. (47) shows the transit time in which the hole crosses through the bulk when the voltage V is applied on both sides of the diode. And the right hand shows the lifetime of the free hole when the recombination centers are all filled with the injected electrons. Accordingly the state where $N_R = n_r$, means the situation in which V is equal to V_M.

From these results, we get V_M as follows:

$$V_M = \frac{\langle v_p s_p \rangle N_R L^2}{\mu_p}, \quad \text{at} \quad N_R = n_r. \quad \cdots\cdots\cdots(48)$$

From these considerations, it is clear that Eq. (48) is sufficiently significant in the physical mechanism.

7. Conclusion

When we try to analyze the phenomenon of the negative resistance based on the double injection into semiconductors, we cannot sufficiently explain the experimental result with the theory of Lampert's model.

With the theory in this paper, however, we can sufficiently explain the behavior of the negative resistance observed in semiconductors which have the recombination center, and get the conclusion that it is resonably satisfied with the experimental results. We conclude that the most important factors, which dominate the condition of the generation and the magnitude of the negative resistance, are the capture rate β of which effect has already been known qualitatively and the modified recombination density $\mathcal{R}=N_R/P_0$ named by us.

Consequently, we know that the equation of $V_M=(1/\beta)\cdot V_{TH}$ for the magnitude of the negative resistance pointed out by Lampert is appropriate only in the case that the recombination density N_R is sufficiently large in comparison with the free hole density P_0 ($\mathcal{R}>10^5$). In addition, we know that the mobility ratio $b=\mu_n/\mu_p$ influences greatly the rise of the characteristics in the high current region over V_M, i.e., semiconductor region. Besides, the effect of b on the magnitude of the negative resistance is almost invaluable in the case of $\beta>10^3$. For the bulk of $\beta<10^2$, as the value of b increases the tendency of the decrease of the negative resistance becomes remarkable. This fact coincides with the experimental result that the magnitude of the negative resistance observed in InSb is much smaller than that of *Ge* or *Si* which has a small mobility ratio. Furthermore, it is concluded that the approximate expressions of the threshold voltage V_{TH} in Eq. (41) and the minimum voltage V_M in Eq. (48) are available in the case that β and N_R/P_0 are comparatively large. If it is not so, the above expressions are lacking in accuracy. The authors are confident that the new treatment is resonable well to interpret the phenomenon of the negative resistance observed in semiconductors.

8. Acknowledgment

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Appendix

Assuming $P_0 \gg N_0$, so that $\phi = N_0/p_0 = o$ and $Q = \beta\mathcal{R}/(1+b\phi) = \beta N_R/p_0$. Eqs, (31) and (36) may therfore be reduced to

$$h_0(u) = \frac{(1-u)(1+\beta u)^2\{\beta(N_R/p_0+1)u+2\}}{u\{\beta(bN_R/p_0-1)u^2+\beta(N_R/p_0+1)u+1\}^2}, \quad \cdots\cdots\cdots(A1)$$

$$f_0(u) = \frac{(1-u)^2(1+\beta u)^3\{\beta(N_R/p_0+1)u+2\}}{u\{\beta(bN_R/p_0-1)u^2+\beta(N_R/p_0+1)u+1\}^3} \quad \cdots\cdots\cdots(A2)$$

where $f_0(u) = g_0(u)\cdot h_0(u)$.

Expanding (A1) and (A2) in partical fractions, we have

$$h_0(u) = \frac{A_H}{u} + \frac{B_H u + C_H}{l\beta u^2+m\beta u+1} + \frac{D_H u + E_H}{(l\beta u^2+m\beta u+1)^2}, \quad \cdots\cdots\cdots(A3)$$

$$f_0(u) = \frac{A_F}{u} + \frac{B_F u + C_F}{l\beta u^2+m\beta u+1} + \frac{D_F u + E_F}{(l\beta u^2+m\beta u+1)^2} + \frac{F_F u + G_F}{(l\beta u^2+m\beta u+1)^3}, \quad \cdots\cdots\cdots(A4)$$

where

$$A_H = 2, \quad B_H = -\frac{m}{l}\beta^2 - 2l\beta, \quad C_H = \frac{m}{l}\left(1+\frac{m}{l}\right)\beta - 2\left(\frac{m+1}{l}+m\right)\beta,$$

$$D_H = -\frac{m^2}{l}\left(1+\frac{m}{l}\right)\beta^3 + \left\{2m\left(\frac{m}{l}+1\right)+\frac{3m}{l}+2\right\}\beta^2 - (m+2l+4)\beta,$$

$$E_H = -\frac{m}{l}\left(1+\frac{m}{l}\right)\beta^2 + \left\{\frac{2(m+1)}{l} - m + 4\right\}\beta - 2,$$

and also

$$A_F = 2, \quad B_F = \frac{m}{l}\beta^2 - 2l\beta, \quad C_F = -2\frac{m}{l^2}\left(\frac{m}{l}+1\right)\beta^2 + \left(\frac{3m+2}{l^2} - 2m\right)\beta,$$

$$D_F = \frac{m}{l}\left(3\frac{m}{l}+1\right)\left(\frac{m}{l}+1\right)\beta^3 - \frac{2}{l}\left\{3m+2+3\frac{m(m+1)}{l}\right\}\beta^2 + \left\{\frac{3(m+2)}{l} - 2l\right\}\beta,$$

$$E_F = -\frac{m^2}{l^2}\left(1+\frac{m}{l}\right)^2\beta^3 + \frac{1}{l}\left\{3m+2+\frac{2m(3m+4)}{l}+\frac{3m(m+2)}{l^2}\right\}\beta^2 - \left\{\frac{1}{l^2}(3m^2+12m+4) + \frac{6(m+2)}{l} + 2m\right\}\beta + \frac{m+6}{l},$$

$$F_F=\frac{m^3}{l^2}\left(1+\frac{m}{l}\right)^2\beta^4-\frac{m}{l}\left\{3(m+1)+\frac{2m(3m+4)}{l}+\frac{m^2(3m+5)}{l^2}\right\}\beta^3$$
$$+\left\{3(m+2)+\frac{2}{l}(3m^2+9m+2)+\frac{m}{l^2}(3m^2+12m+5)\right\}\beta^2$$
$$-\left\{2(m+l+6)+\frac{1}{l}(m^2+9m+6)\right\}\beta+2\,,$$
$$G_F=\frac{m^2}{l^2}\left(1+\frac{m}{l}\right)^2\beta^3-\frac{1}{l}\left\{(3m+2)+\frac{6}{l}m(m+1)+\frac{m^2(3m+4)}{l^2}\right\}\beta^2$$
$$+\left\{\frac{1}{l^2}(3m^2+9m+2)+\frac{6(m+2)}{l}-m+6\right\}\beta-\left(\frac{m+6}{l}+4\right),$$

with $l=bN_R/p_0-1$, $m=N_R/p_0+1$.

Integrating (A3) and (A4) from $u=u_L$ to $u=1$, then $H_0(u_L)$ and $F_0(u_L)$ are given by

$$H_0(u_L)=\int_{u_L}^1 h_0(u)du$$
$$=-A_H\cdot ln\,u_L-\frac{B_H}{2l\beta s}\cdot ln\{(u_L+r)^2-s^2\}+\frac{C_H}{2l\beta s^2}\cdot ln\frac{u_L+r+s}{u_L+r-s}$$
$$+\frac{D_H}{2l^2\beta^2s^2}\cdot\frac{1}{(u_L+r)^2-s^2}+\frac{E_H}{2l^2\beta^2s^4}\cdot\frac{u_L+r}{(u_L+r)^2-s^2}-\frac{E_H}{4l^2\beta^2s^5}\cdot ln\frac{u_L+r+s}{u_L+r-s}$$
$$+\left[\frac{B_H}{2l\beta s}ln\{(1+r)^2-s^2\}+\frac{C_H}{2l\beta s^2}\cdot ln\frac{1+r-s}{1+r+s}-\frac{D_H}{2l^2\beta^2s^2}\cdot\frac{1}{(1+r)^2-s^2}\right.$$
$$\left.-\frac{E_H}{2l^2\beta^2s^4}\cdot\frac{1+r}{(1+r)^2-s^2}-\frac{E_H}{4l^2\beta^2s^5}\cdot ln\frac{1+r-s}{1+r+s}\right],\qquad\cdots\cdots\cdots\text{(A5)}$$

$$F_0(u_L)=\int_{u_L}^1 f_0(u)du$$
$$=-A_F\cdot ln\,u_L-\frac{B_F}{2l\beta s}\cdot ln\{(u_L+r)^2-s^2\}+\frac{C_F}{2l\beta s^2}\cdot ln\frac{u_L+r+s}{u_L+r-s}$$
$$+\frac{D_F}{2l^2\beta^2s^2}\cdot\frac{1}{(u_L+r)^2-s^2}+\frac{E_F}{2l^2\beta^2s^4}\cdot\frac{u_L+r}{(u_L+r)^2-s^2}-\frac{E_F}{4l^2\beta^2s^5}\cdot ln\frac{u_L+r+s}{u_L+r-s}$$
$$+\frac{F_F}{4l^3\beta^3s^3}\cdot\frac{1}{\{(u_L+r)^2-s^2\}^2}-\frac{G_F}{4l^3\beta^3s^7}\cdot\frac{u_L+r}{(u_L+r)^2-s^2}+\frac{G_F}{2l^3\beta^3s^6}\cdot\frac{u_L+r}{\{(u_L+r)^2-s^2\}^2}$$
$$+\frac{G_F}{8l^3\beta^3s^8}\cdot ln\frac{u_L+r+s}{u_L+r-s}+\left[\frac{B_F}{2l\beta s}ln\{(1+r)^2-s^2\}+\frac{C_F}{2l\beta s^2}ln\frac{1+r-s}{1+r+s}\right.$$
$$-\frac{D_F}{2l^2\beta^2s^2}\cdot\frac{1}{(1+r)^2-s^2}-\frac{E_F}{2l^2\beta^2s^4}\cdot\frac{1+r}{(1+r)^2-s^2}-\frac{E_F}{4l^2\beta^2s^5}\cdot ln\frac{1+r-s}{1+r+s}$$
$$-\frac{F_F}{4l^3\beta^3s^3}\cdot\frac{1}{(1+r)^2-s^2}+\frac{G_F}{4l^3\beta^3s^7}\cdot\frac{1+r}{(1+r)^2-s^2}-\frac{G_F}{2l^3\beta^3s^5}\cdot\frac{1+r}{\{(1+r)^2-s^2\}^2}$$

$$+\frac{G_F}{8l^3\beta^3s^8}\,ln\,\frac{1+r-s}{1+r+s}\Big],\qquad\cdots\cdots\cdots(A6)$$

where $r=m/2l,\ s^2=\left(\frac{m}{2l}\right)^2-\frac{1}{l\beta}$.

In the case of $P_0=10^{13}\,\mathrm{cm}^{-3}$, $N_R=10^{14}\,\mathrm{cm}^{-3}$, $b=50$ and $\beta=10^4$, V has a minimum value when $u_L=10^{-3}$. In regard to these values we examine the magnitude of each term in $H_0(u_L)$ and $F_0(u_L)$, and we get the approximating equations

$$H_0(u_L)=-A_H\cdot ln\,u_L-\frac{B_H}{2l\beta}\cdot ln\{(u_L+r)^2-s^2\}+\frac{D_H}{2l\beta}\cdot\frac{1}{(u_L+r-s)},\qquad\cdots\cdots\cdots(A7)$$

$$F_0(u_L)=\frac{F_F}{4l^2\beta^2}\cdot\frac{1}{\{(u_L+r)^2-s^2\}^2}.\qquad\cdots\cdots\cdots(A8)$$

These approximating equations are valid only in the case that the values of P_0, N_R, b and β are close to them as given above, so they are lacking in generality. Here, we deduce the approximate expression of V_M using the above quantity. The first, second and third terms in Eq. (A7) are 14, –12 and –200 respectively, so that $H_0(u_L)$ in (A7) can be roughly estimated as follows:

$$H_0(u_L)=\frac{D_H}{2l\beta}\cdot\frac{1}{(u_L+r)^2-s^2}.\qquad\cdots\cdots\cdots(A9)$$

This estimation, however, is not accurate, if β is not large ($\beta<10$). When we calculate by Eqs. (8) and (9), V becomes as follows:

$$\frac{V}{V_1}=\beta\cdot\frac{F_0(u_L)}{\{H_0(u_L)\}^2}=b\,,\qquad\cdots\cdots\cdots(A10)$$

where

$$D_H=-\frac{m^2}{l}\cdot\beta^3\,,\quad F_F=\frac{m^2}{l^2}\cdot\beta^4\,.$$

From the reason mentioned in Discussion, we regard V given by Eq. (A 10) as V_M. As V_1 equals to $\langle v_p s_p\rangle N_R L^2/\mu_n$, we have

$$V_M=\langle v_p s_p\rangle N_R L^2/\mu_p\,.\qquad\cdots\cdots\cdots(A11)$$