Some Contributions to the Theory of Electrostatic Field

By

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The problem of disc electrodes is discussed. Riemann's solution and the solution by the method of images are shown to be unacceptable, and a method based on an integral equation is devised to acquire a more satisfactory solution. In the second half of this paper, the problem of two tangent spheres is studied, using tangent-sphere coordinates. Potential distirbutions due to charged tangent spheres and also due to tangent spheres in an external electric field are analytically obtained.

Introduction

Although any problems of electrostatic fields can be reduced to the Laplace-type equations, rigorous solutions to these problems are in most cases extremely difficult. The motivation behind the present paper is to give some examples, if not typical, that show how one can get around such difficulties.

The first problem to be treated is a problem of disc electrodes. Riemann's solution quoted in the well-known book of Gray and Mathews [1] is shown to be incorrect, since it leads to a physical contradiction. Even the method of images is found misleading for this problem. After indicating these pitfalls, we then proceed to obtain a more satisfactory solution.

The other problem discussed in this paper is a problem of tangent spheres that are electrically conducting. Particular coordinates suited to this problem are introduced, and solutions are presented for the case of two tangent charged spheres and also for the case of tangent spheres immersed in an external electric field. The result obtained may find some application in the theory of electrostatic separation of ores.

Numerical results are scheduled for a companion paper.

2. Disc Electrodes Problem

2.1 Statement of the Problem

As shown in Fig. 1, we consider a plate of thickness $2h$ and of finite electrical

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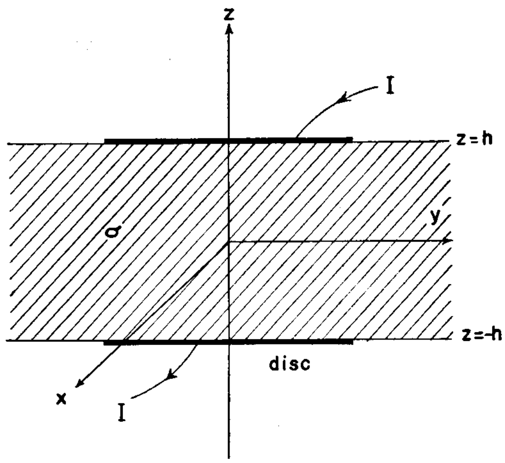

Fig. 1. Setup of the problem of two disc electrodes. Two disc electrodes are mounted coaxially on a conducting plate (shaded) of infinite extent.

conductivity σ. This plate is bounded by vacuum at $z=\pm h$. We use the cylindrical coordinates of Fig. 1. On this plate of infinite extent, we coaxially arrange two identical perfectly-conducting discs of radius a as electrodes. Through these electrodes we force a constant electric current I to flow, and we attempt to find the distribution of electric potential V in the conducting plate.

The boundary conditions to be invoked at $z=\pm h$ are:

$$\frac{\partial V}{\partial z}=0, \qquad (r>a); \tag{1}$$

$$V=\pm V_0, \qquad (r<a). \tag{2}$$

Here V_0, a constant, is the potential of the electrode at $z=h$, and $-V_0$ is that of the other electrode at $z=-h$.

Condition (1) follows from the fact that at the plate surface away from the disc there is no electric current flowing out into vacuum. Condition (2) is simply the requirement that the perfectly conducting electrode be an equipotential body.

2.2 Riemann's Solution and its Fallacy

Gray and Mathews [1] quoted in their book on Bessel functions Riemann's solution to the present disc-electrode porblem. The argument is as follows.

In place of condition (2), the condition

$$\frac{\partial V}{\partial z}=\frac{I}{2\pi\sigma a}\frac{1}{\sqrt{a^2-r^2}} \qquad (z=\pm h,\ r<a) \tag{3}$$

is considered. Condition (1) must of course be retained. The above condition (3) results from the assumption that the normal component of the electric field at the electrode surface is identical with that of the single-electrode case where a single disc electrode, providing a current source of $2I$, is immersed in an unlimited medium of conductivity σ. In view of the symmetrical setup of the problem, one can assume

$$V=\int_0^\infty L(\lambda)\sinh\lambda z\, J_0(\lambda r)\,d\lambda\,, \tag{4}$$

where $L(\lambda)$ is to be determined by the boundary conditions. By means of the formula

$$\int_0^\infty \sin\lambda a\, J_0(\lambda r)\,d\lambda=\begin{cases}0 & (r>a)\\ \dfrac{1}{\sqrt{a^2-r^2}} & (r<a)\end{cases} \tag{5}$$

one finds that $L(\lambda)$ assumes the form

$$L(\lambda)=\frac{I}{2\pi\sigma a}\,\frac{1}{\cosh\lambda h}\,\frac{\sin\lambda a}{\lambda}\,. \tag{6}$$

Thus, the solution is:

$$V=\frac{I}{2\pi\sigma a}\int_0^\infty \frac{\sinh\lambda z}{\cosh\lambda h}\,\frac{\sin\lambda a}{\lambda}\,J_0(\lambda r)\,d\lambda\,. \tag{7}$$

This solution certainly fulfils conditions (1) and (3), whereas it does not meet the requirement (2), i.e., V=constant at the electrode surface. Therefore, Riemann's solution (7) does not describe the actual physical situation of two disc electroces facing each other. The fallacy of Riemann's solution arises from the inappropriate condition (3), and one should start with conditions (1) and (2). Note that in the book by Gray and Mathews [1] Riemann's solution is quoted as if it were an exact solution to the present problem of two disc electrodes.

2.3 Method of Images

Suppose that an infinite number of disc electrodes, identical in shape, are arranged in an infinite space of conductivity σ. The arrangement is shown in Fig. 2. We assign current intensities $\pm 2I$ to each of these electrodes, so that the potential distribution in $-h<z<h$ (shaded in Fig. 2) can be regarded as the same as that in the conducting region of Fig. 1. The potential due to one of the electrodes e_n (n=0, 1, 2, $\cdots$) is given by

$$v_n=\frac{(-1)^n 2I}{4\pi\sigma a}\int_0^\infty e^{-(2n+1)\lambda h+\lambda z}J_0(\lambda r)\sin\lambda a\,\frac{d\lambda}{\lambda}\,. \tag{8}$$

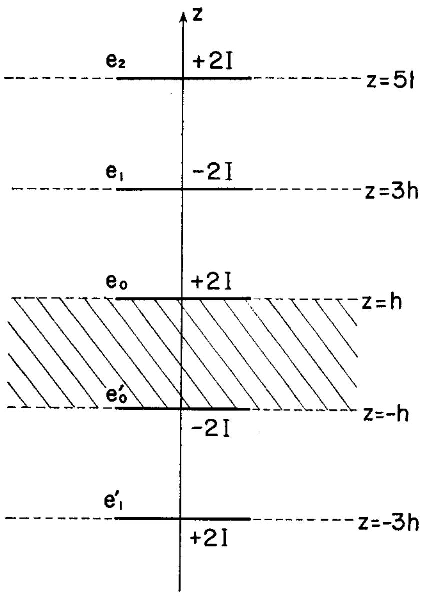

Fig. 2. Method of images for the problem of two disc electrodes. Potential distribution in the shaded region is identical with that in the conducting region of Fig. 1.

Similarly, the potential due to one of the electrodes e_n' ($n=0, 1, 2, \cdots$) is given by

$$v_n' = \frac{(-1)^{n+1}2I}{4\pi\sigma a}\int_0^\infty e^{-(2n+1)\lambda h-\lambda z} J_0(\lambda r) \sin \lambda a \frac{d\lambda}{\lambda}\,. \tag{9}$$

Therefore, the potential V in the region $-h<z<h$ is:

$$\begin{aligned} V &= \sum_{n=0}^{\infty} (v_n+v_n') \\ &= \frac{I}{2\pi\sigma a}\sum_{n=0}^{\infty}(-1)^n\int_0^\infty (e^{-(2n+1)\lambda h+\lambda z} - e^{-(2n+1)\lambda h-\lambda z}) \\ &\quad \times J_0(\lambda r) \sin \lambda a \frac{d\lambda}{\lambda} \\ &= \frac{I}{2\pi\sigma a}\int_0^\infty \frac{\sinh \lambda z}{\cosh \lambda h} J_0(\lambda r) \sin \lambda a \frac{d\lambda}{\lambda}\,. \end{aligned} \tag{10}$$

Thus, here again, the simple application of the method yeilds an incorrect result, since in Eq. (10) we have recovered Eq. (7).

The fallacy in the method of images, as well as in Riemann's solution previously discussed, lies in the neglect of mutual interactions between the electrodes. Solutions (7) and (10) indicate that the superposition principle breaks down in the present problem. Each of the electrodes does not contribute to the resultant potential as if it were in isolation, but rather, these electrodes mutually rearrange the potential distribution in between the electrodes in order that condition (2) be not violated.

2.4 Proposed Exact Solution by an Integral Equation

To determine the function $L(\lambda)$ of Eq. (4), we start with the exact boundary conditions at $z=\pm h$.

$$V = \pm V_0 \qquad \text{for} \qquad r<a\,, \tag{11}$$

$$\frac{\partial V}{\partial z} = 0 \qquad \text{for} \qquad r>a\,. \tag{12}$$

Also we perform a scaling of variables as follows:

$$\left.\begin{aligned} &\lambda = u/a\,, \quad z = a\zeta\,, \quad r = a\rho\,, \quad h = at\,, \\ &V/V_0 = U\,, \quad L(\lambda)/(aV_0) = B(u) \end{aligned}\right\} \tag{13}$$

Eq. (4) then takes the form

$$U = \int_0^\infty B(u) \sinh \zeta u J_0(\rho u) du\,, \tag{14}$$

while the boundary conditions (11) and (12) are expressed as

$$\int_0^\infty B(u) \sinh tu\, J_0(\rho u) du = 1 \qquad \text{for} \quad 0<\rho<1\,, \tag{15}$$

$$\int_0^\infty B(u) u \cosh tu\, J_0(\rho u) du = 0 \quad \text{for} \quad 1<\rho\,. \tag{16}$$

Conditions (15) and (16) may further be simplified to

$$\int_0^\infty F(u) J_0(\rho u) du = g(\rho) \qquad \text{for} \quad 0<\rho<1\,, \tag{17}$$

$$\int_0^\infty F(u) u J_0(\rho u) du = 0 \qquad \text{for} \quad 1<\rho\,, \tag{18}$$

where

$$\left.\begin{aligned} g(\rho) &= 1+g_1(\rho)\,, \\ g_1(\rho) &= \int_0^\infty B(u) e^{-tu} J_0(\rho u) du\,, \\ F(u) &= B(u) \cosh tu\,. \end{aligned}\right\} \tag{19}$$

Eqs. (17) and (18) may be called dual integral equations, the solution of which is given by Busbridge [2] as follows:

$$F(u) = \frac{2}{\pi} \cos u \int_0^1 \frac{yg(y)}{\sqrt{1-y^2}} dy + \frac{2}{\pi} \int_0^1 \frac{ydy}{\sqrt{1-y^2}} \int_0^1 g(xy) xu \sin(xu) dx \,. \tag{20}$$

Considering

$$\left(\cos u + \int_0^1 xu \sin(xu) dy\right) \int_0^1 \frac{ydy}{\sqrt{1-y^2}} = \frac{\sin u}{u} \tag{21}$$

and using $g_1(\rho)$, Eq. (20) is written as

$$F(u) = \frac{2}{\pi} \frac{\sin u}{u} + \frac{2}{\pi} \cos u \int_0^1 \frac{yg_1(u)}{\sqrt{1-y^2}} dy + \frac{2}{\pi} \int_0^1 \frac{ydy}{\sqrt{1-y^2}} \int_0^1 g_1(xy) xu \sin(xu) dx \,. \tag{22}$$

After some algebra and with the use of formulas such as

$$\int_0^1 J_0(zy) \frac{ydy}{\sqrt{1-y^2}} = \frac{\sin z}{z} \,,$$

$$\int \sin xv \sin xu \, dx = \frac{1}{2}\left[\frac{\sin(v-u)x}{v-u} - \frac{\sin(v+u)x}{v+u}\right],$$

we finally arrive at the following Fredholm integral equation of the second kind:

$$B(u) = f(u) + \int_0^\infty B(\lambda) K(\lambda, u) d\lambda \,, \tag{23}$$

where

$$f(u) = \frac{2}{\pi} \frac{\sin u}{u \cosh tu} \,, \tag{24}$$

$$K(\lambda, u) = \frac{1}{\pi} \frac{e^{-t\lambda}}{\cosh tu} \left[\frac{\sin(\lambda+u)}{\lambda+u} + \frac{\sin(\lambda-u)}{\lambda-u}\right]. \tag{25}$$

$B(u)$ given as the solution of the single intergral equation (23) is the solution of the dual integral equations (15) and (16).

Once $B(u)$, hence $L(\lambda)$, is known, the potential is given by Eq. (4), and the electric field component normal to the electrode surface is calculated from

$$E_n = -(E_z)_{z=h} = \int_0^\infty L(\lambda) \cosh \lambda h J_0(\lambda r) \lambda d\lambda \,. \tag{26}$$

Obviously,

$$I = \sigma \int_0^a E_n 2\pi r dr = 2\pi\sigma a V_0 \int_0^\infty B(u) \cosh tu J_1(u) du \,. \tag{27}$$

The resistance, R, between the two electrodes is known from $R=2V_0/I$.

One feasible way of solving Eq. (23) is the successive approximation. Namely, since $B(u)$ is expected to differ slightly from $f(u)$ we may take $f(u)$ for the zeroth order solution. Expanding $B(u)$ as

$$B(u) = f(u)+B_1(u)+B_2(u)+\cdots , \tag{28}$$

we have

$$B_n(u) = \int_0^\infty B_{n-1}(\lambda)K(\lambda, u)du , \qquad (n=1, 2, \cdots) , \tag{29}$$

where $B_0(u)=f(u)$. Riemann's solution previously discussed merely provides the zeroth order approximation to the present problem. Unless we calculate the residual sum $B_1+B_2+\cdots+B_n+\cdots$ of (28), we cannot obtain a physically consistent solution.

Numerical solutions along this line will be presented in a separate paper in the near future.

3. Problem of Tangent Conducting Spheres

3.1 Tangent Sphere Coordinates and Laplace's Equation

For boundary-value problems of two similar spheres mutually in contact, tangent-sphere coordinates [3] are the most appropriate. Orthogonal coordinate system is defined, as shown in Fig. 3, by a set of three surfaces, i.e., a toroidal surface

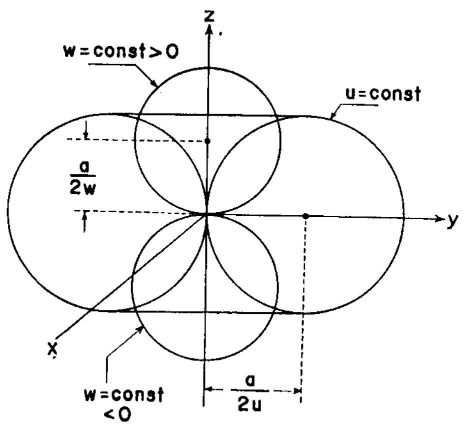

Fig. 3. Tangent-sphere coordinates.

u=constant (with no *peephole* at the center), two tangent spheres (w=constant>0 for the upper sphere, and w=constant<0 for the lower sphere), the centers of which lie on the z-axis, and a plane (φ=constant), comprising the z-axis, to determine the azimuthal angle φ. The relation between the Cartesian coordinates (x, y, z) and the tangent-sphere coordinates (u, w, φ) is:

$$\left.\begin{aligned} x &= \frac{au \cos \varphi}{u^2+w^2}\,, \\ y &= \frac{au \sin \varphi}{u^2+w^2}\,, \\ z &= \frac{aw}{u^2+w^2}\,. \end{aligned}\right\} \tag{30}$$

Here a is a constant that represents the radius of a reference spherical surface.

Laplace's equation in terms of (u, w, φ) is

$$\frac{1}{u}\frac{\partial}{\partial u}\left(\frac{u}{u^2+w^2}\frac{\partial V}{\partial u}\right)+\frac{\partial}{\partial w}\left(\frac{1}{u^2+w^2}\frac{\partial V}{\partial w}\right)+\frac{1}{u^2(u^2+w^2)}\frac{\partial^2 V}{\partial \varphi^2}=0 \tag{31}$$

which is known to be R-separable [4]. Namely, decomposing the unknown function V into products such as

$$V(u, w, \varphi) = \sqrt{u^2+w^2}\,U(u)\,W(w)\,\Phi(\varphi)\,, \tag{32}$$

we have three ordinary differential equations one of which is Bessel's equation. Thus, the general solution V that takes finite values along the z-axis is given by

$$\begin{aligned} V(u, w, \varphi) &= \sqrt{u^2+w^2}\sum_{m=0}^{\infty}(a_m \cos m\varphi + b_m \sin m\varphi) \\ &\quad \times \int_0^{\infty}(L \cosh \lambda w + M \sinh \lambda w) J_m(\lambda u)\,d\lambda\,, \end{aligned} \tag{33}$$

where a_m and b_m are arbitrary constants, and L and M are arbitrary functions of λ or arbitrary constants. In case of axial symmetry, we have, for (33), a much simpler expression as

$$V(u, w) = \sqrt{u^2+w^2}\int_0^{\infty}(L \cosh \lambda w + M \sinh \lambda w) J_0(\lambda u)\,d\lambda\,. \tag{34}$$

3.2 Two Conducting Charged Spheres

Here we consider the case of two identical tangent spheres, as shown in Fig. 4, placed in vacuum. Each of these conducting spheres is given charge $Q_1(=Q/2)$. Using Eq. (34) and considering the potential V is an even function of w, we have

$$V = \sqrt{u^2+w^2}\int_0^{\infty} L(\lambda) \cosh \lambda w\, J_0(\lambda u)\,d\lambda\,. \tag{35}$$

To determine $L(\lambda)$ we assume the potential V at the spherical surfaces $(w=\pm w_0;$

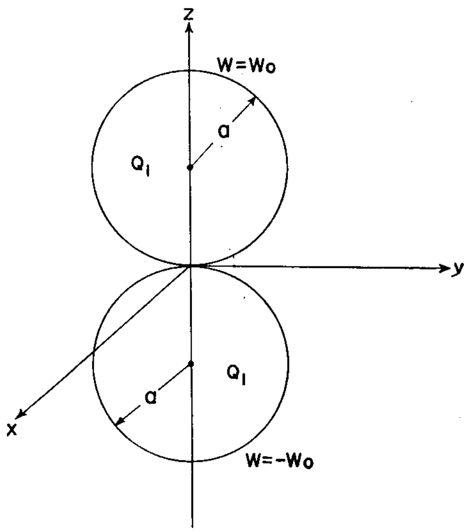

Fig. 4. Two identical conducting spheres mutually in contact. The radius of the sphere is *a*. Each of the spheres has electrostatic charge Q_1 ($=Q/2$).

w_0 is taken to be $\frac{1}{2}$ hereafter) to be V_0. Thus, at the spherical surfaces, we must have

$$\frac{V_0}{\sqrt{u^2+w_0^2}} = \int_0^\infty L(\lambda)\cosh\lambda w_0 J_0(\lambda u)d\lambda\,. \tag{36}$$

It follows from this that

$$L(\lambda) = V_0 e^{-\lambda w_0}\operatorname{sech}\lambda w_0\,.$$

Therefore, the required solution is

$$V(u,\,w) = V_0\sqrt{u^2+w^2}\int_0^\infty e^{-\lambda w_0}\frac{\cosh\lambda w}{\cosh\lambda w_0}J_0(\lambda u)d\lambda\,, \tag{37}$$

from which the electric field can be calculated by

$$E_u = -\frac{u^2+w^2}{a}\frac{\partial V}{\partial u}\,, \tag{38}$$

$$E_w = -\frac{u^2+w^2}{a}\frac{\partial V}{\partial w}\,. \tag{39}$$

To find the relationship between the given charge, Q_1, and the potential of the conducting spheres, V_0, we proceed to evaluate the electric field component, E_n, directed normal to, and outward from, the spherical surface. Concerning this field component, we have

$$
\begin{aligned}
E_n &= -E_w \qquad (w = \pm w_0) \\
&= \frac{V_0}{a} w_0 + \frac{V_0}{a}(u^2+w_0^2)^{3/2} \int_0^\infty e^{-\lambda w_0} \tanh \lambda w_0 J_0(\lambda u) \lambda d\lambda \,. \qquad (40)
\end{aligned}
$$

The total electric charge of the upper sphere is given by

$$
Q_1 = \varepsilon \oint E_n dS \,, \qquad (41)
$$

where

$$
dS = \frac{2\pi a^2 u}{(u^2+w_0^2)^2} du \,, \quad 0<u<\infty \,. \qquad (42)
$$

We finally arrive at the simple formula

$$
\begin{aligned}
Q_1 &= \frac{Q}{2} \\
&= 2\pi\varepsilon a V_0 \left[1 + \frac{1}{2} \int_0^\infty \frac{udu}{(u^2+w_0^2)^2} \right. \\
&\quad \left. +2 \sum_{n=1}^\infty (-1)^n \frac{2n+1}{2} \int_0^\infty \frac{udu}{(u^2+w_0^2)^{1/2}(u^2+(2n+1)^2 w_0^2)^{3/2}} \right] \\
&= 4\pi\varepsilon a V_0 \log_e 2 \qquad (43)
\end{aligned}
$$

where use is made of formulas such as

$$
\int_0^\infty e^{-p\lambda} J_0(\lambda q) d\lambda = \frac{1}{(p^2+q^2)^{1/2}} \,, \qquad (44)
$$

$$
\int_0^\infty e^{-p\lambda} J_0(\lambda q) \lambda d\lambda = \frac{p}{(p^2+q^2)^{3/2}} \,, \qquad (45)
$$

$$
\sum_{n=0}^\infty (-1)^n \frac{1}{n+1} = \log_e 2 \,. \qquad (46)
$$

Therefore,

$$
V_0 = \frac{Q}{8\pi\varepsilon a \log_e 2} \,, \qquad (47)
$$

and the electrostatic capacitance C of the two tangent spheres is given by

$$
C = \frac{Q}{V_0} = 4\pi\varepsilon a(2 \log_e 2) \,. \qquad (48)
$$

2.3 Two Tangent Spheres in an External Electric Field

Consider two identical conducting spheres of radius a mutually in contact and

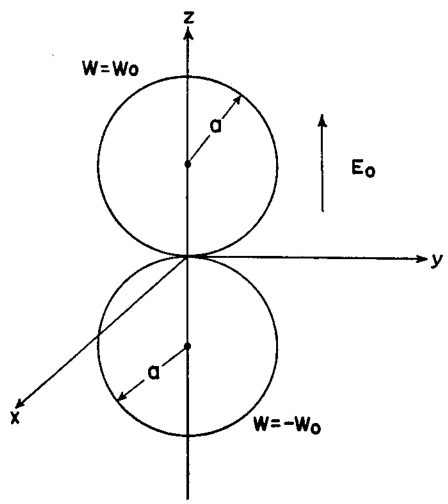

Fig. 5. Two tangent spheres immersed in a uniform electric field E_0. The direction of E_0 is along the z-axis.

immersed in a uniform electric field E_0 as shown in Fig. 5.

The potential distribution V of our interest can be assumed

$$V = V_c + V^* , \tag{49}$$

where V_c is the potential in the absence of the spheres, i.e.,

$$V_c = -zE_0 = -\frac{aw}{u^2+w^2} E_0 . \tag{50}$$

In view of the symmetry with respect to the xy-plane, the induced potential V^* is given by

$$V^* = \sqrt{u^2+w^2}\int_0^\infty M(\lambda) \sinh \lambda w \, J_0(\lambda u) d\lambda . \tag{51}$$

The function $M(\lambda)$ is to be determined by the boundary condition on the spherical surfaces, i.e.,

$$V_c + V^* = 0 \quad \text{at} \quad w = \pm w_0 \; (=\pm\tfrac{1}{2}) . \tag{52}$$

By means of formula (45), V_c of Eq. (50) is written as

$$V_c = \mp a E_0 \sqrt{u^2+w^2}\int_0^\infty e^{\mp\lambda w} J_0(\lambda u)\lambda d\lambda , \tag{53}$$

so that

$$M(\lambda) = aE_0 \frac{\lambda e^{-\lambda w_0}}{\sinh \lambda w_0} . \tag{54}$$

Thus the solution we need is found as follows:

$$V = -aE_0\sqrt{u^2+w^2}\int_0^\infty \left[e^{-\lambda w} - e^{-\lambda w_0}\frac{\sinh \lambda w}{\sinh \lambda w_0}\right] J_0(\lambda u)\lambda d\lambda\,, \qquad (w>0)\,, \qquad (55)$$

from which the electric field components E_u and E_w can be calculated using Eqs. (38) and (39).

Normal component of the electric field on the sphere ($w=w_0=\frac{1}{2}$) is given by

$$\begin{aligned} E_n &= -(E_w)_{w=w_0} \\ &= E_0(u^2+w_0^2)^{3/2}\int_0^\infty e^{-\lambda w_0}(1+\coth \lambda w_0)\,J_0(\lambda u)\lambda^2 d\lambda \\ &= 2E_0(u^2+w_0^2)^{3/2}\left[3\sum_{n=0}^{\infty}\frac{(2n+1)^2 w_0^2}{(u^2+(2n+1)^2 w_0^2)^{5/2}}\right. \\ &\quad \left. -\sum_{n=0}^{\infty}\frac{1}{(u^2+(2n+1)^2 w_0^2)^{3/2}}\right]. \end{aligned} \qquad (56)$$

Since the present tangent spheres are not initially given any electric charge, the total induced charge on the two spheres vanishes. However, it will be of some interest to know the amount of the induced charge on either of these two contacting spheres. This may be of some help when one estimates the efficiency of the method of electrostatic separation of ores, namely, by what electric field conducting particles can most effectively be separated from non-conducting particles. When the induced electric charge on the upper sphere is Q_i, then the induced charge on the lower sphere is $-Q_i$. For this Q_i, we have

$$Q_i = 2\pi\varepsilon a^2\int_0^\infty \frac{E_n u du}{(u^2+w_0^2)^2}\,. \qquad (57)$$

Using Eq. (56) for the above equation (57), where $w_0=\frac{1}{2}$, we have the final result that

$$\begin{aligned} Q_i &= 4\pi\varepsilon a^2 E_0\sum_{n=1}^{\infty}\frac{1}{n^2} \\ &= \frac{2}{3}\pi^3\varepsilon a^2 E_0\,. \end{aligned} \qquad (58)$$

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