A Variational Principle for Transport Phenomena II Applications to Several Problems

By

Masakatsu HIRAOKA* and Kanya TANAKA*

(Received June 29, 1968)

The variational principle formulated in the previous paper was applied to several problems in transport phenomena. The problems of momentum, heat and mass transfer of which exact solutions have been well known were treated and the approximate solution was obtained in order to distinguish the characteristics of the variational principle. It was indicated that the approximate solutions by the variational principle agreed with the exact solutions for the quantities related to the average values in a space or the quantities connected with the functional (flow rate, Nusselt number, Sherwood number), even when the grade of approximation for the profiles of velocities, temperature and concentration was not so good. The method described in this paper could apply to more complex problems which could not be solved exactly.

Introduction

Advantage of describing a boundary value problem by a equivalent variational prinicple is that it offers a powerful tool for solving the boundary value problems. In order to solve the boundary value problems by use of the variational principle, we use Ritz's or Galerkin's direct method with Lagrange-Biot's equation derived in the previous paper[1].

It is the purpose of this paper to indicate how to use our variational prinicple as the approximate solving method of the boundary value problems and how the reciprocal formula with the maximum principle could be used to estimate the errors in the approximate calculations.

In this paper, we treat the problems that the exact solutions have already been well known to distinguish the questions.

1. Unsteady Flow in Circular Tube

Let us consider the change of velocity profile and flow rate with time from

* Department of Sanitary Engineering

** Doshisha University, Japan

when the pressure gradient $(p_0-p_l)/l$ was added to Newtonian fluid (constant μ and ρ) in a circular tube of radius R and length l at $t=0$ [2].

Equation of motion for this problem is written as follows.

$$\rho\frac{\partial v_z}{\partial t} = \frac{p_0-p}{l}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{1}$$

$$\text{I. C.} \quad : \quad \text{at } t=0 \quad v_z=0 \ (0\leqq r\leqq R) \tag{2}$$

$$\text{B. C. 1}: \quad \text{at } r=0 \quad v_z=\text{finite} \tag{3}$$

$$\text{B. C. 2}: \quad \text{at } r=R \quad v_z=0 \tag{4}$$

By introduction of dimensionless variables

$$\phi=\frac{v_z}{(p_0-p_l)R^2/4\mu l}, \quad \xi=\frac{R}{r}, \quad \tau=\frac{\mu t}{\rho R^2} \tag{5}$$

Eqs. (1)～(4) are rewritten in the next form.

$$\frac{\partial\phi}{\partial\tau}=4+\frac{1}{\xi}\cdot\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}\right) \tag{6}$$

$$\text{I. C.} \quad : \quad \text{at } \tau=0, \quad \phi=0 \ (0\leqq\xi\leqq 1) \tag{7}$$

$$\text{B. C. 1}: \quad \text{at } \xi=0, \quad \phi=\text{finite} \tag{8}$$

$$\text{B. C. 2}: \quad \text{at } \xi=1, \quad \phi=0 \tag{9}$$

The solution of this problem has been obtained by use of the method of separation of variables or Hankel transform in the next form.

$$\phi(\xi,\tau)=(1-\xi^2)-8\sum_{i=1}^{\infty}\frac{J_0(\alpha_i\xi)}{\alpha_i^3J_1(\alpha_i)}e^{-\alpha_i^2\tau} \tag{10}$$

where, α_i are the real roots of Bessel's function of first kind $J_0(\alpha)=0$; $\alpha_1=2.3038$ $\alpha_2=5.5201$, $\alpha_3=8.6537$..... .

The flow rate can be calculated from the velocity profile

$$Q_V=\int_0^{2\pi}d\theta\int_0^R v_z(r)\cdot rdr=2\pi\int_0^R v_z(r)rdr \tag{11}$$

Now, let us define a dimensionlees flow rate

$$\Lambda=\frac{Q_V}{\pi(p_0-p_l)R^4/8\mu l}=4\int_0^1\phi\xi d\xi \tag{12}$$

By substituiton of Eq. (10) into this equation, we get

$$\Lambda(\tau)=1-32\sum_{i=1}^{\infty}\frac{e^{-\alpha_i^2\tau}}{\alpha_i^4} \tag{13}$$

When τ goes to infinitive, the state becomes steady and Λ approaches 1, which expresses Hagen-Poiseuille's law. We could obtain the exact solution (Eq. (10)) by use of the variational principle, but we try to get an approximate solution and to estimate the error in the approximate solution here. The functional Eq. (8) in the previous paper can be written as follows: because $\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}=0$, $(\nabla\cdot\boldsymbol{v})=0$, $\boldsymbol{g}=0$, so

$$I[v_z] = \int_0^l dz\int_0^{2\pi} d\theta\int_0^R\left[\rho\frac{\partial\bar{v}_z}{\partial t}v_z+\frac{\mu}{2}\left(\frac{\partial v_z}{\partial r}\right)^2\right]rdr$$

$$+\int_0^{2\pi} d\theta\int_0^R(-p_0)v_z rdr+\int_0^{2\pi} d\theta\int_0^R p_l v_z rdr$$

$$= 2\pi l\int_0^R\left[\rho\frac{\partial\bar{v}_z}{\partial t}+\frac{\mu}{2}\left(\frac{\partial v_z}{\partial r}\right)^2-\left(\frac{p_0-p_l}{l}\right)v_z\right]rdr \tag{14}$$

The functional I$[v_z]$ has a unit of power [erg/sec]. Then, we change this equation in the dimensionless form

$$\mathcal{J}[\phi] = \frac{I[v_z]}{\pi(\Delta p)^2R^4/8\mu l} = \int_0^1\left[\frac{\partial\bar{\phi}}{\partial\tau}\phi+\frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2-4\phi\right]\xi d\xi \tag{15}$$

We assume the next form as the admissible function by considering that the steady solution becomes $\phi(\xi,\infty)=1-\xi^2$

$$\phi(\xi,\tau) = q(\tau)\,(1-\xi^2) \tag{16}$$

where, unknown function $q(\tau)$ is the generalized coordinates. Lagrange-Biot's equation which determine $q(\tau)$ can be wirtten as

$$\frac{\partial U}{\partial q}+\frac{\partial D}{\partial\dot{q}} = Q^{S_\tau} \tag{17}$$

where,

$$U = \int_0^1\frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2\xi d\xi \tag{18}$$

$$D = \int_0^1\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2\xi d\xi \tag{19}$$

$$Q^{S_\tau} = \int_0^1 4\left(\frac{\partial\phi}{\partial q}\right)\xi d\xi \tag{20}$$

From Eq. (7),

$$\text{I. C.}:\quad \text{at } \tau=0,\quad q=0 \tag{21}$$

When we calculate Eq. (18~(20) by use of Eq.(16), the results are obtained as follows

$$U = \int_0^1\frac{1}{2}q^2(-2\xi)^2\xi d\xi = \frac{1}{2}q^2 \tag{22}$$

$$D = \int_0^1 \frac{1}{2}\dot{q}^2(1-\xi^2)^2\xi d\xi = \frac{1}{12}\dot{q}^2 \tag{23}$$

$$Q^{S_\tau} = \int_0^1 4(1-\xi^2)\xi d\xi = 1 \tag{24}$$

By substitution of these quantities into Eq. (17), we get

$$\frac{1}{6}\cdot\frac{dq}{d\tau}+q = 1 \tag{25}$$

When we solve this equation with the initial condition Eq. (21),

$$q(\tau) = 1-e^{-6\tau} \tag{26}$$

are obtained. Therefore, an approximate solution can be expressed by the next form

$$\phi(\xi, \tau) = (1-e^{-6\tau})(1-\xi^2) \tag{27}$$

The variations of flow rate for this velocity profile are calculated from Eq. (12)

$$A(\tau) = 4\int_0^1(1-e^{-6\tau})(1-\xi^2)\xi d\xi = 1-e^{-6\tau} \tag{28}$$

The obtained results are shown in Fig. 1 and Fig. 2, in which the exact solutions are drawn by the dotted line and the approximate solutions are drawn by the solid line.

Let us compare the approximate solution botained above with the exact solution. For the velocity profile, the approximate solutions perfectly coincide with the exact

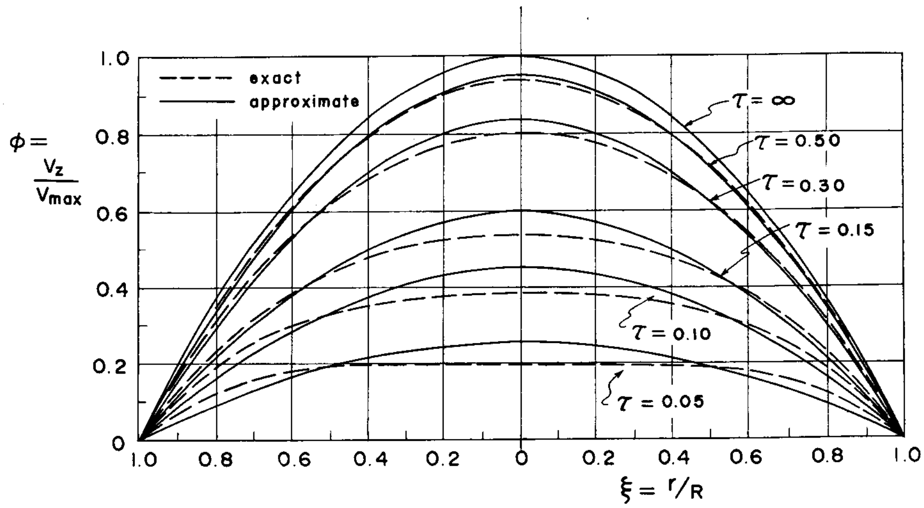

Fig. 1 Variation of Velocity Distribution in Circular Duct

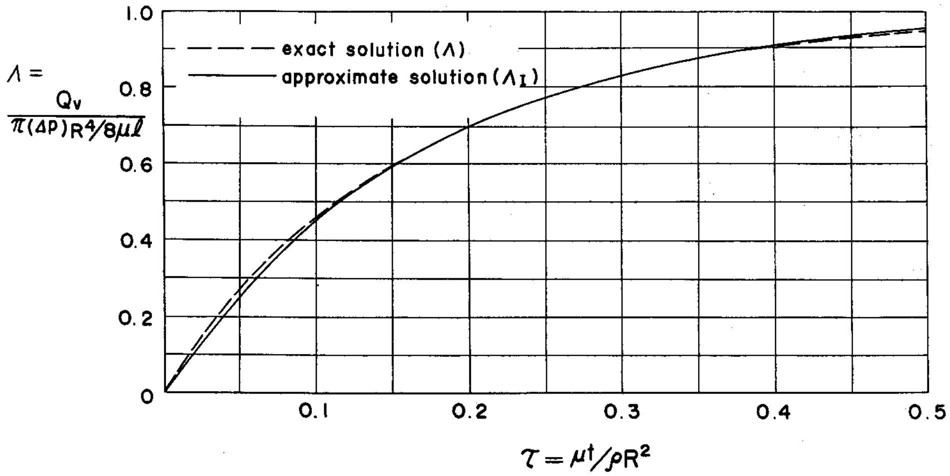

Fig. 2 Variation of Flow Rate

solution only at $\tau=0$ and $\tau=\infty$, and the exact solutions are more flat than the approximate solutions in the ragne $0<\tau<\infty$ as shown in Fig. 1. But it is recognized that the agreement on the flow rates is better than that of the velocity profile. Let us consider further on this point. Some considerations on this point will be given as follows.

The stress tensor is expressed as

$$\tau_{rz} = -\mu\frac{\partial v_z}{\partial r} \tag{29}$$

and can be written in the dimensionless form

$$\mathcal{T}_{\xi z} = \frac{\tau_{rz}}{(\Delta p)R/4l} = -\frac{\partial\phi}{\partial\xi} \tag{30}$$

Eqs. (1) and (6) are rewritten as follows by use of these quantities.

$$\rho\frac{\partial v_z}{\partial t} = \frac{\Delta p}{l} - \frac{1}{r}\cdot\frac{\partial}{\partial r}(r\tau_{rz}) \tag{31}$$

$$\frac{\partial\phi}{\partial\tau} = 4 - \frac{1}{\xi}\cdot\frac{\partial}{\partial\xi}(\xi\mathcal{T}_{\xi z}) \tag{32}$$

The stationary value of the functional is obtained from Eq. (59) in the previous paper.

$$\begin{aligned} I_0 &= -\frac{1}{2}\int_0^l dz\int_0^{2\pi}d\theta\int_0^R\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\tau_{rz})v_z r dr \\ &= -\pi l\int_0^R\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\tau_{rz})v_z r dr \end{aligned} \tag{33}$$

We can write Eq. (33) in the dimensionless form

$$\mathcal{I}_0 = \frac{I_0}{\pi(\Delta p)^2R^4/8\mu l} = -\frac{1}{2}\int_0^1 \frac{1}{\xi}\cdot\frac{\partial}{\partial\xi}(\xi\mathcal{T}_{\xi z})\phi\xi d\xi \tag{34}$$

Substitution of Eq. (32) leads to

$$\mathcal{I}_0 = \frac{1}{2}\int_0^1 \frac{\partial\phi}{\partial\tau}\phi\xi d\xi - \frac{1}{2}\int_0^1 4\phi\xi d\xi = \mathcal{K} - \frac{1}{2}\Lambda \tag{35}$$

where,

$$\mathcal{K} = \frac{1}{2}\int_0^1 \frac{\partial\phi}{\partial\tau}\phi\xi d\xi \tag{36}$$

From Eq. (35), we can see that the stationary value of functional $\mathcal{I}_0$ is a quantity connected with the flow rate Λ and at $\tau=0$, or ∞, $\mathcal{I}_0=-\frac{1}{2}$. The approximate value of $\mathcal{K}$ is obtained from Eq. (36) by use of Eq. (27).

$$\mathcal{K}' = \frac{1}{2}\int_0^1 6e^{-6\tau}(1-e^{-6\tau})(1-\xi^2)^2\xi d\xi = \frac{1}{2}(e^{-6\tau}-e^{-12\tau}) \tag{37}$$

Then, the approximate value of $\mathcal{I}_0$ is calculated from Eq. (15) by use of Eq. (27).

$$\mathcal{I}_0' = -\frac{1}{2}(1-2e^{-6\tau}+e^{-12\tau}) \tag{38}$$

From Eq. (35), we get the approximate value of flow rate Λ_I from the next equation

$$\mathcal{I}_0' = \mathcal{K}' - \frac{1}{2}\Lambda_I \tag{39}$$

which coincides with Eq. (28).

Let us apply the reciprocal variational principle in order to find the grade of approximation about the minimum value of functional.

Eq. (56) in the previous paper can be written in this case as follows.

$$H[\tau_{rz}] = -\int_0^l dz\int_0^{2\pi} d\theta\int_0^R \frac{1}{2\mu}\tau_{rz}^2 rdr = -\frac{\pi l}{\mu}\int_0^R \tau^2_{rz}\cdot rdr \tag{40}$$

By introduction of the dimensionless variables, we get

$$\mathcal{H}[\mathcal{T}_{\xi z}] = \frac{H[\tau_{rz}]}{\pi(\Delta p)^2R^4/8\mu l} = -\frac{1}{2}\int_0^1 \mathcal{T}^2_{\xi z}\cdot\xi d\xi \tag{41}$$

If the approximate value $\mathcal{H}_0'$ of maximum value $\mathcal{H}_0$ could be obtained, the next relation should be valid from the maximum and minimum principle.

$$\mathcal{H}_0' < \mathcal{H}_0 = \mathcal{I}_0 < \mathcal{I}_0' \tag{42}$$

and by comparion of $\mathcal{I}_0'$ and with $\mathcal{H}_0'$, we can check the grade of approximation.

The argument function $\mathcal{T}_{\xi z}$ in Eq. (41) must satisfy the dimensionless form of admissibility condition

$$\frac{\partial \phi}{\partial \tau} = 4 - \frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi \mathcal{T}_{\xi z}) \tag{43}$$

and then, substitution of Eq. (27) into Eq. (43) and use of the condition that $\mathcal{T}_{\xi z}$ is finite at $\xi = 0$ leads to

$$\mathcal{T}_{\xi z} = 2\xi - 3e^{-6\tau}\xi\left(1 - \frac{\xi^2}{2}\right) \tag{44}$$

By substitution into Eq. (41), we get

$$\mathcal{H}_0' = -\frac{1}{2}\left(1 - 2e^{-6\tau} + \frac{33}{32}e^{-12\tau}\right) \tag{45}$$

Comparison between Eqs. (38) and (45) indicates

$$\mathcal{I}_0' - \mathcal{H}_0' = \frac{1}{64}e^{-12\tau} \leqq \frac{1}{64} \tag{46}$$

Eq. (35) becomes $\mathcal{H}_0 = \mathcal{K} - \frac{1}{2}\Lambda$, because $\mathcal{I}_0 = H_0$ at the stationary state.

If we define the approximate value Λ_H by

$$\mathcal{H}_0' = \mathcal{K}' - \frac{1}{2}\Lambda_H \tag{47}$$

the next equation is obtained from Eqs. (37) and (45).

$$\left.\begin{aligned} \Lambda_H &= 2(\mathcal{K}' - \mathcal{H}_0') \\ &= 1 - e^{-6\tau} + \frac{1}{32}e^{-12\tau} \\ &= \Lambda_I + \frac{1}{32}e^{-12\tau} \end{aligned}\right\} \tag{48}$$

Table 1. Comparison of Λ and it's approximate value Λ_I and Λ_H

τ	Λ	Λ_I	$100(\Lambda_I - \Lambda)/\Lambda$ [%]	Λ_H	$100(\Lambda_H - \Lambda)/\Lambda$ [%]
0	0	0	0	0	0
0.05	0.2759	0.2592	−6.05	0.2764	+0.181
0.10	0.4617	0.4512	−2.27	0.4606	−0.238
0.15	0.5981	0.5934	−0.786	0.5986	+0.0836
0.20	0.6990	0.6988	−0.030	0.7016	+0.372
0.30	0.8312	0.8347	+0.421	0.8352	+0.481
0.40	0.9053	0.9093	+0.442	0.9096	+0.475
0.50	0.9469	0.9502	+0.349	0.9503	+0.359
	1.0000	1.0000	0	1.0000	0

By use of Eqs. (35), (39), (42) and (47), the next relations are obtained

$$\Lambda_I + 2(\mathcal{K} - \mathcal{K}') < \Lambda < \Lambda_H + 2(\mathcal{K} - \mathcal{K}') \tag{49}$$

$$\Lambda_I < \Lambda_H \tag{50}$$

The compariosn of the numerical values of flow rate Λ and it's approximate values Λ_I and Λ_H are shown in Table 1. The agreement of exact solutions and approximate solutions are highly excellent in respect to the flow rates.

2. An Application to Boundary-Layer Theory

Let us consider the boundary layer near the surface of a flat plate immersed in a fluid stream. This is a classical prolbem studied by H. Blasius (1908)[3].

The two-dimensional, steady-state equation of continuity and the x-component of the equation of motion are written as follows

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{51}$$

$$\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial y} = 0 \tag{52}$$

$$\text{B. C. 1}: \quad \text{at } y = 0, \quad v_x = v_y = 0 \tag{53}$$

$$\text{B. C. 2}: \quad \text{at } y = \infty, \quad v_x = v_\infty \tag{54}$$

We solve Eq. (52) for v_y and substitute the result into Eq. (51) to get

$$v_x \frac{\partial v_x}{\partial y} - \left(\int_0^y \frac{\partial v_x}{\partial x} dy \right) \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{55}$$

The functional for this problems can be written as

$$I[v_x] = \int_0^\infty \left[\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} v_x - \left(\int_0^y \frac{\partial \bar{v}_x}{\partial x} dy \right) \frac{\partial \bar{v}_x}{\partial y} v_x + \frac{\nu}{2} \left(\frac{\partial v_x}{\partial y} \right)^2 \right] dy$$

By putting

$$\frac{v_x}{v_\infty} = \varphi(\eta), \quad \text{wher} \quad \eta = \frac{y}{\delta_V(x)} \tag{56}$$

Lagrange-Biot's equation which have the generalized coordinates δ_V can be obtained from Eq. (143) in the previous paper ($D=0$, $Q^p = Q^g = Q^{S_\tau}$)

$$\frac{\delta U}{\delta \delta_V} = Q^V \tag{57}$$

where,

$$U = \int_0^\infty \frac{1}{2} \mu \left(\frac{\partial v_x}{\partial y} \right)^2 dy = \int_0^{\delta_V} \frac{\mu v_\infty}{2} \left(\frac{\partial \varphi}{\partial y} \right)^2 dy = \frac{\mu v_\infty^2}{2\delta_V} \int_0^1 (\varphi')^2 d\eta \tag{58}$$

$$
\begin{aligned}
Q^V &= -\rho\int_0^\infty\left[v_x\frac{\partial v_x}{\partial x}-\left(\int_0^y\frac{\partial v_x}{\partial x}dy\right)\frac{\partial v_x}{\partial y}\right]\frac{\partial v_x}{\partial \delta_V}dy \\
&= -\rho v_\infty^3\int_0^{\delta_V}\left[\varphi\frac{\partial \varphi}{\partial x}-\left(\int_0^y\frac{\partial \varphi}{\partial x}dy\right)\frac{\partial \varphi}{\partial y}\right]\frac{\partial \varphi}{\partial \delta_V}dy \\
&= -\rho v_\infty^3\cdot\frac{1}{\delta_V}\cdot\frac{d\delta_V}{dx}\int_0^1\left[\varphi\eta-\int_0^\eta\varphi'\xi d\xi\right](\varphi')^2\eta d\eta
\end{aligned}
\tag{59}
$$

in which

$$
\int_0^\eta\varphi'\xi d\xi = [\varphi\xi]_0^\eta-\int_0^\eta\varphi d\xi = \varphi\eta-\int_0^\eta\varphi d\xi \tag{60}
$$

By use of Eq. (74)~(77), we get

$$
\delta_V\frac{d\delta_V}{dx} = \frac{\nu\int_0^1(\varphi')^2d\eta}{2v_\infty\int_0^1\left(\int_0^\eta\varphi d\xi\right)(\varphi')^2\eta d\eta} \tag{61}
$$

In this problem, we use the form of φ which was introduced by Pohlhausen[4]: he introduced the form that $\varphi(\eta)$ satisfies the following conditions.

$$
\left.
\begin{aligned}
&\text{at}\quad \eta=0,\quad \varphi(\eta)=0,\quad \varphi''(\eta)=0 \\
&\text{at}\quad \eta=1,\quad \varphi(\eta)=1,\quad \varphi'(\eta)=0,\quad \varphi''(\eta)=0
\end{aligned}
\right\}
\tag{62}
$$

$$
\varphi(\eta) = 2\eta-2\eta^3+\eta^4 \tag{63}
$$

Integration of Eq. (61) by use of Eq. (63) with the condition that at $x=0$, $\delta_V=0$ gives

$$
\delta_V = 5.625\sqrt{\frac{\nu x}{v_\infty}} \tag{64}
$$

By substitution of Eq. (56) into Eq. (55), we get

$$
\frac{d^2\varphi}{d\eta^2} = -\frac{v_\infty}{\nu}\delta_V\frac{d\delta_V}{dx}\left[\varphi\varphi'\eta-\left(\int_0^\eta\varphi'\xi d\xi\right)\varphi'\right] \tag{65}
$$

Integration of this equation with considering that $\varphi'(1)=0$ gives

$$
\begin{aligned}
\varphi'(0) &= \frac{v_\infty}{\nu}\delta_V\frac{d\delta_V}{dx}\int_0^1\left[\varphi\eta-\int_0^\eta\varphi'\xi d\xi\right]\varphi' d\eta \\
&= \frac{v_\infty}{\nu}\delta_V\frac{d\delta_V}{dx}\int_0^1\left(\int_0^\eta\varphi d\xi\right)\varphi' d\eta
\end{aligned}
\tag{66}
$$

The local friction coefficient of a flat plate can be calculated from the next equation

$$
f_{loc} = \frac{\tau_0}{\frac{1}{2}\rho v_\infty^2} = \frac{1}{\frac{1}{2}\rho v_\infty^2}\cdot\mu\left(\frac{\partial v_x}{\partial y}\right)_{y=0} \tag{67}
$$

and integrating from $x=0$ to l and dividing by l the average friction coefficient is obtained

$$f=\frac{1}{l}\int_0^l f_{loc}(x)dx \tag{68}$$

Then, we can calculate the average friction coefficient from Eq. (68) with the relation of

$$\tau_0=\mu\left(\frac{\partial v_x}{\partial y}\right)_{y=0}=\frac{\mu v_\infty}{\delta_V}\varphi'(0) \tag{69}$$

Eq. (67) and Eq. (66). Theresult is

$$f_V=1.321\sqrt{\frac{\nu}{v_\infty l}} \tag{70}$$

The comparison with the results of Blasiu's exact solution, the approximate solution by pohlhausen and by us are shown in Table 2.

Table 2 Comparison between rigorous and approximate solutions

	Exact Solution by Blasius	Approximate Solution by Pohlhausen	Approximate Solution by Authors
Thickness of Boundary Layer	$\delta_B \doteqdot 5.0\sqrt{\frac{\nu x}{v_\infty}}$	$\delta_p=5.85\sqrt{\frac{\nu x}{v_\infty}}$	$\delta_V=5.625\sqrt{\frac{\nu x}{v_\infty}}$
Average friction coefficient	$f_B=1.328\sqrt{\frac{\nu}{v_\infty l}}$	$f_p=1.372\sqrt{\frac{\nu}{v_\infty l}}$	$f_V=1.321\sqrt{\frac{\nu}{v_\infty l}}$
Error to f_B	0%	+3.32%	−0.527%

3. Graetz-Nusselt's Problem about Heat transfer in Circular Duct

Let us consider the problem that heat is tranfered from the wall to the fluid of laminar flow as illustrated in Fig. 3. This problem was initially treated by Graetz (1883) and then was researched by Nusselt (1910), so it was called the Graetz-Nusset's problem and has been studied with various boundary conditions[5]. Initially,

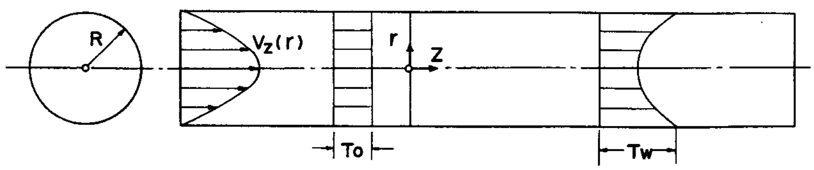

Fig. 3 Temperature Distribution in Circular Duct

we assume that the fluid and the wall have the same uniform temperature To and the laminar velocity profiles are completed:

$$v_z = v_{\max}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{71}$$

Now, let us consider the variations of temperature profile and heat flux from when the temperature of wall suddenly change. The basic equation for this problem can be written as follows

$$\rho\hat{C}_p v_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = \frac{\lambda}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{72}$$

$$\left.\begin{array}{lll} \text{B. C. 1}: & \text{at } z=0, & T=T_0 \\ \text{B. C. 2}: & \text{at } r=R, & T=T_W \\ \text{B. C. 3}: & \text{at } r=0, & T=\text{finite} \end{array}\right\} \tag{73}$$

By introduction of the dimensionless variables

$$\theta = \frac{T-T_0}{T_W-T_0}, \quad \xi = \frac{r}{R}, \quad \tau = \frac{\lambda\cdot z}{\rho\hat{C}_p v_{\max}R^2} \tag{74}$$

We get

$$(1-\xi^2)\frac{\partial\theta}{\partial\tau} = \frac{1}{\xi}\cdot\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) \tag{75}$$

$$\left.\begin{array}{lll} \text{B. C. 1}: & \text{at } \tau=0, & \theta=0 \\ \text{B. C. 2}: & \text{at } \xi=1, & \theta=1 \\ \text{B. C. 3}: & \text{at } \xi=0, & \theta=\text{finite} \end{array}\right\} \tag{76}$$

The exact solution of this problem has already been obtained by use of the method of seperation of variables[5].

Here, we will try to obtain an approximate solution. The functional is written by use of Eq. (28) in the previous paper as follows.

$$I[\theta] = \int_0^{2\pi}d\theta\int_0^R\left[\rho\hat{C}_p v_z\frac{\partial\theta}{\partial z}\theta+\frac{\lambda}{2}\left(\frac{\partial\theta}{\partial r}\right)^2\right]rdr \tag{77}$$

By introduction of the dimensionless variables, we get

$$\mathcal{J}[\theta] = \frac{I[\theta]}{2\pi\lambda(\Delta T)^2} = \int_0^1\left[(1-\xi^2)\frac{\partial\theta}{\partial\tau}\cdot\theta+\frac{1}{2}\left(\frac{\partial\theta}{\partial\xi}\right)^2\right]\xi d\xi \tag{78}$$

Lagrange-Biot's equation can be written in the next form ($Q^H=Q^{S_q}=0$, $D=0$)

$$\frac{\partial U}{\partial q_i} = Q^V \tag{79}$$

where,

$$U = \int_0^1 \frac{1}{2}\left(\frac{\partial \theta}{\partial \xi}\right)^2 \xi d\xi \tag{80}$$

$$Q_i{}^V = -\int_0^1 (1-\xi^2)\frac{\partial \theta}{\partial \tau} \cdot \frac{\partial \theta}{\partial q_i} \xi d\xi \tag{81}$$

We must calculate the temperature by dividing into two parts depending on length of Z, because the temperature profile will change with time as illustrated in Fig. 4.

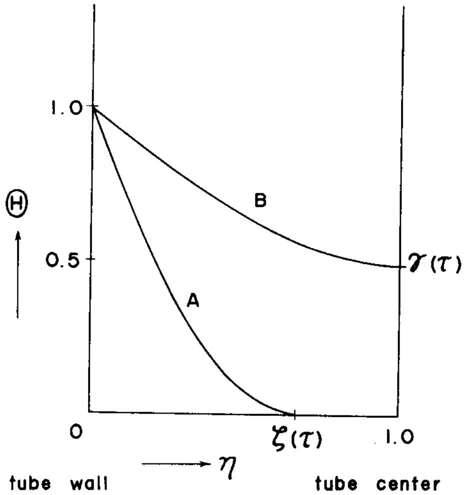

Fig. 4 Approximate Curve of Temperature Distribution

i) Z is small:

We assume the nexst form as the admissible function

$$\theta(\eta,\tau) = \begin{cases} \left[1-\dfrac{\eta}{\zeta(\tau)}\right]^2 & \text{for} \quad 0 \leqq \eta \leqq \zeta(\tau) \\ 0 & \text{for} \quad \zeta(\tau) \leqq \eta \leqq 1 \end{cases} \tag{82}$$

where, η is a dimensionless length from the wall.

$$\eta = \frac{R-r}{R} = 1-\xi \tag{83}$$

In Eq. (82), the penetration depth $\zeta(\tau)$ is selected as the generalized coordinates. Calculation of Eqs. (80) and (81) by inserting Eq. (82) gives

$$U = \int_0^1 \frac{1}{2}\left(\frac{\partial \theta}{\partial \eta}\right)^2 (1-\eta)d\eta = \frac{2}{\zeta^2}\int_0^\zeta \left(1-\frac{\eta}{\zeta}\right)^2 (1-\eta)d\eta = \frac{2}{3\zeta} - \frac{1}{6} \tag{84}$$

$$Q^V = -\int_0^1 \eta(2-\eta)\frac{\partial \theta}{\partial \tau}\frac{\partial \theta}{\partial \zeta}(1-\eta)d\eta$$

$$= -\frac{4\dot{\zeta}}{\zeta^4}\int_0^{\zeta}\left(1-\frac{\eta}{\zeta}\right)^2\eta^3(1-\eta)(2-\eta)d\eta \tag{85}$$

$$= -4\dot{\zeta}\left(\frac{1}{30}-\frac{\zeta}{35}+\frac{\zeta^2}{168}\right) \tag{86}$$

By substitution of these equations into Eq. (79), we get

$$\frac{dU}{d\zeta} = Q^V$$

and

$$\frac{d\tau}{d\zeta} = 6\zeta^2\left(\frac{1}{30}-\frac{\zeta}{35}+\frac{\zeta^2}{168}\right) \tag{87}$$

When we solve this equation with the boundary condition

$$\text{at} \quad \tau = 0, \quad \zeta = 0 \tag{88}$$

the solution is obtained as follows

$$\tau = \frac{\zeta^3}{15}\left(1-\frac{9}{14}\zeta+\frac{3}{28}\zeta^2\right) \tag{89}$$

We define the heat transfer coefficient at the wall

$$q_W = \lambda\frac{\partial T}{\partial r} = h(T_m-T_W) \tag{90}$$

The local heat transfer coefficient is expressed that

$$\left.\begin{aligned} Nu = \frac{D_0 h}{\lambda} &= \frac{q_W}{\lambda(T_m-T_W)/D_0} \\ &= \frac{2}{1-\Theta_m}\cdot\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} \\ &= -\frac{2}{1-\Theta_m}\cdot\left.\frac{\partial\Theta}{\partial\eta}\right|_{\eta=0} \end{aligned}\right\} \tag{91}$$

where $D_0=2R$ is a diameter of duct and T_m or Θ_m is a mean temperature defined by the next equation.

$$T_m = \frac{\int_0^{2\pi}d\theta\int_0^R Tv_z r dr}{\int_0^{2\pi}d\theta\int_0^R v_z r dr} \tag{92}$$

$$\Theta_m = \frac{\int_0^1\Theta\phi\xi d\xi}{\int_0^1\phi\xi d\xi} \tag{93}$$

in which

$$\phi(\xi) = \frac{v_z}{v_{\max}} = 1-\xi^2 \tag{94}$$

Calculation of Θ_m by inserting Eq. (82) gives

$$\Theta_m = \frac{\int_0^\zeta \left(1-\frac{\eta}{\zeta}\right)^2 \eta(1-\eta)(2-\eta)d\eta}{\int_0^1 (1-\xi^2)\xi d\xi} = \frac{2\zeta^2}{3}\left(1-\frac{3}{5}\zeta+\frac{1}{10}\zeta^2\right) \tag{95}$$

To get Nusselt number, we integrate Eq. (75)

$$\int_0^1 (1-\xi^2)\frac{\partial\Theta}{\partial\tau}\xi d\xi = \int_0^1 \frac{1}{\xi}\cdot\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right)\xi d\xi$$

$$= \left[\xi\frac{\partial\Theta}{\partial\xi}\right]_0^1 = \left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1}$$

$$\therefore \left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = \frac{d}{d\tau}\int_0^1 \Theta(1-\xi^2)\xi d\xi \tag{96}$$

By substitusion of Eq. (82), we get

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = \frac{d}{d\tau}\int_0^\zeta \left(1-\frac{\eta}{\zeta}\right)^2 \eta(1-\eta)(2-\eta)d\eta$$

$$= \zeta\left(\frac{1}{3}-\frac{3}{10}\zeta+\frac{1}{15}\zeta^2\right)\frac{d\zeta}{d\tau} \tag{97}$$

The Nusselt number can be obtained by substitution of Eqs. (87), (96) and (97) into Eq. (91)

$$Nu = \frac{1}{1-\frac{2\zeta^2}{3}\left(1-\frac{3}{5}\zeta+\frac{1}{10}\zeta^2\right)} \cdot \frac{\frac{1}{3}-\frac{3}{10}\zeta+\frac{1}{15}\zeta^2}{3\zeta\left(\frac{1}{30}-\frac{1}{35}\zeta+\frac{1}{108}\zeta^2\right)} \tag{98}$$

Peclet's number can be written as

$$P_{e_H} = R_e \cdot P_r = \frac{D_0\langle v_z\rangle\rho}{\mu} \cdot \frac{\rho\alpha}{\mu} = \frac{D_0\langle v_z\rangle}{\alpha} \tag{99}$$

where, $\alpha = \lambda/\rho\hat{C}_p$

The average velocity is known:

$$\langle v_z\rangle = \frac{\int_0^{2\pi} d\theta \int_0^R v_z r dr}{\int_0^{2\pi} d\theta \int_0^R r dr} = \frac{1}{2}v_{\max} \tag{100}$$

and then, we get from Eq. (74)

$$\tau = \frac{\alpha z}{v_{\max}\cdot R^2} = 2\cdot\frac{z}{D}\cdot\frac{\alpha}{D_0\langle v_z\rangle} = 2\left(\frac{z}{D_0}\right)\Big/P_{e_H} \tag{101}$$

The comparison of the exact solution with the approximate solution Eq. (98) is shown in Fig. 5 in which Nusselt numbers are plotted against $\left(\frac{z}{D_0}\right)/P_{e_H}$.

If we can consider a special case when $\zeta \ll 1$

$$Nu \doteqdot \frac{10}{3\zeta} \doteqdot \frac{10}{3}\cdot(15\tau)^{-1/3} = 1.075\left(\frac{z/D_0}{P_{e_H}}\right)^{-1/3} \tag{102}$$

The exact solution by Sellars, Tribus and Klein[6] is that

$$Nu = 1.077\left(\frac{z/D_0}{P_{e_H}}\right)^{-1/3} \quad \text{for} \quad \frac{z/D_0}{P_{e_H}} \leq 0.005 \tag{103}$$

The agreement of both equations is excellent.

ii) Z is large:

We assume the admissible function that

$$\theta(\eta, \tau) = [1-\gamma(\tau)](1-\eta)^2+\gamma(\tau) \tag{104}$$

If we select $\gamma(\tau)$ as the generalized coordinates, Lagrange-Biot's equation can be written as follows.

$$\frac{\partial U}{\partial \gamma} = Q^V \tag{105}$$

where,

$$\left.\begin{aligned} U &= \int_0^1 \frac{1}{2}\cdot 4(1-\gamma)^2(1-\eta)^3 d\eta = \frac{1}{2}(1-\gamma)^2 \\ Q^V &= -\int_0^1 \dot{\gamma}\cdot\eta^3(2-\eta)^3(1-\eta)d\eta = -\frac{1}{8}\dot{\gamma} \end{aligned}\right\} \tag{106}$$

The equation that determines $\gamma(\tau)$ is

$$\frac{d\gamma}{d\tau} = 8(1-\gamma) \tag{107}$$

When we solve the above equation with the initial condition

$$\text{at} \quad \tau = \tau_0, \quad \gamma = 0 \tag{108}$$

the next solution is obtained

$$\gamma(\tau) = 1-e^{-8(\tau-\tau_0)} \tag{109}$$

Substitution into Eq. (104) gives

$$\theta(\eta, \tau) = e^{-8(\tau-\tau_0)}\cdot(1-\eta)^2+[1-e^{-8(\tau-\tau_0)}] \tag{110}$$

From Eq. (93), we get

$$\theta_m = \frac{1}{3}(1-\gamma)+\gamma \tag{111}$$

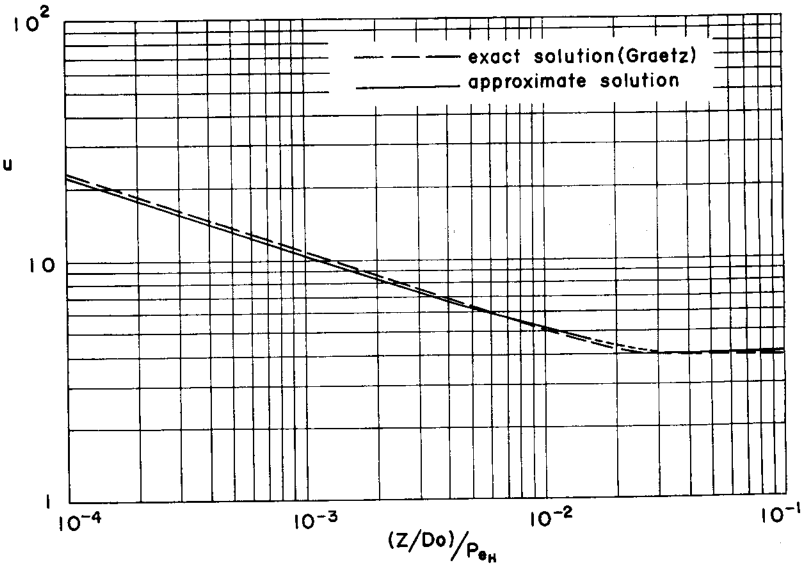

Fig. 5 Comparison of Heat Transfer Rate

From Eq. (96), we get

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=1} = \frac{d}{d\tau}\int_0^1 \theta(1-\xi^2)\xi d\xi = \frac{1}{6}\frac{d\gamma}{d\tau} = \frac{4}{3}(1-\gamma) \tag{112}$$

Therefore, we can obtain Nusselt number from Eq. (91)

$$Nu = 4 \tag{113}$$

This results are also plotted in Fig. 5.

4. Pigford's Problem about Mass Transfer in Wetted Wall

This well-known problem is formulated as follows

$$v_{\max}\left[1-\left(\frac{x}{B}\right)^2\right]\frac{\partial C_A}{\partial z} = \mathscr{D}\frac{\partial^2 C_A}{\partial x^2} \tag{114}$$

$$\left.\begin{array}{lll} \text{B. C. 1 :} & \text{at } z=0, & C=C_{A0} \\ \text{B. C. 2 :} & \text{at } x=0, & C_A=C_A{}^* \\ \text{B. C. 3 :} & \text{at } x=B, & \partial C_A/\partial x=0 \end{array}\right\} \tag{115}$$

By introduction of dimensionless variables

$$Y = \frac{C_A - C_{A0}}{C_A{}^* - C_{A0}}, \quad \xi = \frac{B}{x}, \quad \tau = \frac{\mathscr{D}z}{v_{\max}\cdot B^2} \tag{116}$$

We get

$$(1-\xi^2)\frac{\partial Y}{\partial \tau} = \frac{\partial^2 Y}{\partial \xi^2} \tag{117}$$

$$\left.\begin{array}{lll} \text{B. C. 1 :} & \text{at } \tau = 0, & Y = 0 \\ \text{B. C. 2 :} & \text{at } \xi = 0, & Y = 1 \\ \text{B. C. 3 :} & \text{at } \xi = 1, & \partial Y/\partial \xi = 0 \end{array}\right\} \tag{118}$$

The exact solution has been obtained by Pigford[6]. We can write for velocity profile

$$\phi(\xi) = \frac{v_{\max}}{v_z} = 1-\xi^2 \tag{119}$$

and for average velocity

$$\langle\phi\rangle = \frac{\langle v_z\rangle}{v_{\max}} = \frac{1}{B}\int_0^B \frac{v_z}{v_{\max}}dx = \int_0^1 \phi(\xi)d\xi = \frac{2}{3} \tag{120}$$

Now, if we define the average concentration by

$$Y_b = \frac{\frac{1}{B}\int_0^B Y v_z dx}{\frac{1}{B}\int_0^B v_z dx} = \frac{\int_0^Y Y\phi d\xi}{\langle\phi\rangle} \tag{121}$$

the, rigorous solution of Pigford can be expressed as follows

$$Y_b = 1-\sum_{n=1}^{\infty} a_n e^{-b_n \tau} \tag{122}$$

in which the values of a_n and b_n are given in Table 3.

Table 3. Values of a_n and b_n

n	a_n	b_n
1	0.7857	5.121
2	0.1001	39.31
3	0.0360	105.6
4	0.0181	204.7

We can write the local Sherwood number by

$$Sh = \frac{Ne}{\mathcal{D}(C_A{}^*-C_{A0})/B} = -\left.\frac{\partial Y}{\partial \xi}\right|_{\xi=0} \tag{123}$$

where

$$Ni = -\mathcal{D}\left.\frac{\partial C_A}{\partial x}\right|_{x=0} \tag{124}$$

Average Shmidt number is calculated by

$$\bar{Sh} = \frac{1}{\tau}\int_0^\tau Sh d\tau \tag{125}$$

We can obtain a relation between Y_b and Sh by the following procedure. Integration of Eq. (117) with the boundary conditions Eq. (118) gives

$$\int_0^1 (1-\xi^2)\frac{\partial Y}{\partial \tau}d\xi = \int_0^1 \frac{\partial^2 Y}{\partial \xi^2}d\xi = \left[\frac{\partial Y}{\partial \xi}\right]_0^1$$

$$\therefore\ Sh = -\left.\frac{\partial Y}{\partial \xi}\right|_{\xi=0} = \frac{d}{d\tau}\int_0^1 Y\phi d\xi \tag{126}$$

Eq. (125) can be calculated by use of Eq. (121)

$$\left.\begin{aligned}\bar{Sh} &= \frac{1}{\tau}\left[\int_0^1 Y\phi d\xi\right]_{\tau=0}^{\tau=\tau} = \frac{1}{\tau}\int_0^1 Y\phi d\xi \\ &= \frac{1}{\tau}\cdot Y_b\cdot\langle\phi\rangle \\ &= \frac{2}{3\tau}\cdot Y_b\end{aligned}\right\} \tag{127}$$

Peclet number is defined by

$$P_{e_M} = R_e\cdot S_c = \frac{D_0\langle v_z\rangle\rho}{\mu}\cdot\frac{\mu}{\rho\mathcal{D}} = \frac{D_0\langle v_z\rangle}{\mathcal{D}} \tag{128}$$

then, a relation between τ and P_B is given

$$\tau = \frac{\mathcal{D}z}{v_{\max}B^2} = \frac{2}{3}\cdot\frac{(z/B)}{P_{e_M}} \tag{129}$$

Now, let us solve this problem approximately. The functional for this problems can be expressed as follows

$$I[C] = \int_0^B\left[v_z\frac{\partial\bar{C}}{\partial z}\cdot C + \frac{\mathcal{D}}{2}\left(\frac{\partial C}{\partial x}\right)^2\right]dx \tag{130}$$

By introduction of the dimensionless variables

$$\mathcal{I}[Y] = \frac{I[C]}{\mathcal{D}(\Delta C)^2/B} = \int_0^1\left[(1-\xi^2)\frac{\partial\bar{Y}}{\partial\tau}Y + \frac{1}{2}\left(\frac{\partial Y}{\partial\xi}\right)^2\right]d\xi \tag{131}$$

We assume the next admissibile function.

$$Y(\xi,\tau) = \begin{cases}\left[1-\dfrac{\xi}{\zeta(\tau)}\right]^2 & \text{for}\quad 0\leqq\xi\leqq\zeta(\tau) \\ 0 & \text{for}\quad \zeta(\tau)\leqq\xi\leqq 1\end{cases} \tag{132}$$

$\zeta(\tau)$ is a penetration depth of concentration profile and is the generalized coordinates. Then, we get Lagrange-Biot's equation by considering that $Q^M = Q^{S_N} = 0$, $D = 0$

$$\frac{\partial U}{\partial \zeta} = Q^V \tag{133}$$

where

$$\left.\begin{aligned} U &= \int_0^1 \frac{1}{2}\left(\frac{\partial Y}{\partial \xi}\right)^2 d\xi \\ Q^V &= -\int_0^1 \phi \frac{\partial Y}{\partial \tau} \cdot \frac{\partial Y}{\partial \zeta} d\xi = -\int_0^1 (1-\xi^2)\frac{\partial Y}{\partial \tau} \cdot \frac{\partial Y}{\partial \zeta} d\xi \end{aligned}\right\} \tag{134}$$

Calculation of these equations by use of Eq. (132) gives

$$U = \int_0^\zeta \frac{1}{2} \cdot \frac{4}{\zeta^2}\left(1-\frac{\xi}{\zeta}\right)^2 d\xi = \frac{2}{3\zeta} \tag{135}$$

$$Q^V = -\frac{4\dot{\zeta}}{\zeta^4}\int_0^\zeta (1-\xi^2)\cdot\xi^2\left(1-\frac{\xi}{\zeta}\right)^2 d\xi = -\frac{2\dot{\zeta}}{15\zeta}\left(1-\frac{2\zeta}{7}\right) \tag{136}$$

Substitution into Eq. (133) gives

$$\frac{d\zeta}{d\tau} = \frac{5}{\zeta\left(1-\frac{2\zeta}{7}\right)} \tag{137}$$

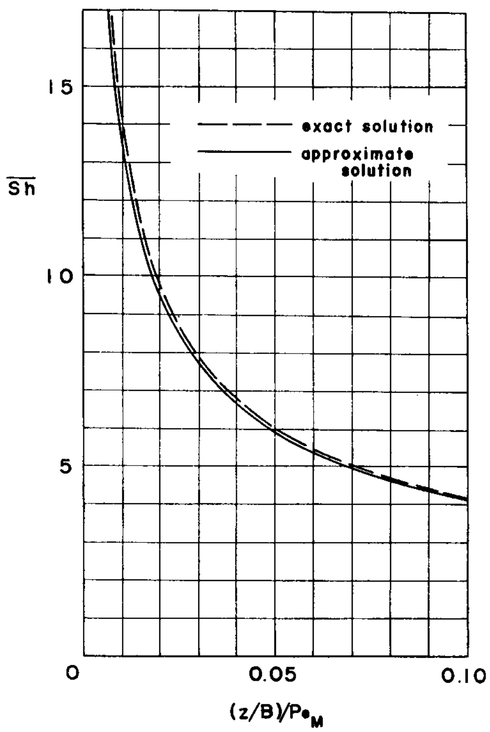

Fig. 6 Comparison of Mass Transfer Rate

and the initial condition is that

$$\text{at} \quad \tau = 0, \quad \zeta = 0 \tag{138}$$

Solution of this equation is

$$\tau = \frac{\zeta^2}{10} - \frac{2\zeta^3}{105} \tag{139}$$

$$\begin{aligned} \therefore \ \overline{Sh} &= \frac{1}{\tau}\int_0^1 Y\phi d\xi \\ &= \frac{1}{\tau}\int_0^\zeta \left(1-\frac{\xi}{\zeta}\right)^2 (1-\xi^2) d\xi \\ &= \frac{\zeta}{3\tau}\left(1-\frac{\zeta^2}{10}\right) \end{aligned} \tag{140}$$

The comparison of exact solution with this approximate solution is shown in Fig. 6.

Discussions and Conclusions

We have discussed the method of solving the several boundary value problems by use of our variational principle formulated in the previous work.

Though we treat the problems in which the exact solutions have been well known, the method described in this paper could apply to more complex problems which could not be solved exactly. The estimation of errors in the approximate calculations is always possible by use of the reciprocal formual with the maximum and minimum principle.

The general character of the approximate solution by the variational principle is that even though the grade of approximation for the profiles of velocities, temperatures and concentrations is not so good, we can obtain good approximate values for the quantities which are related to the average values in a space or the quantities connected with the functional (flow rate, Nusselt number, Sherwood number etc.).

It is necessary to select a proper admissible function intuitively. On the other hand, it could be said that the perturbation method is suitable to estimate the profiles of velocities, temperatures and concentrations near a wall. Recently, the digital computation techniques have been widely developed and the boundary value problems could be solved by use of a finite form of the partial differential equation, but some problems about stability and accuracy remain in solving the differential equation of parabolic type, which are encounterd in the problems of transport phenomena. If the partial differential equations are reduced to the ordinary differential equations and the ordinary differential equantions are reduced to the algebraical equations as shown in this paper, the labor of calculations could be decreased.

We have proposed the variational principle which is applicalbe to solve the problems of transport phenomena in the generalized formula, but the gernral transformation of the boundary problems to the equivalent variational problems is a unsolved subject of mathematics.

Further developement would be desirable.

Literature Cited

1) M. Hiraoka and K. Tanaka; Memoirs of the Faculty Eng. Kyoto University Vol. XXX, Part 3, pp. 235~263 (1968)
2) R.B. Bird, W.E. Stewart & E.N. Lightfoot: Transport Phenomena, p. 126, John Wiley & Sons (1960)
3) H. Schlichting: Boundary Layer Theory, p. 116, 4th Ed. Mc Graw-Hill (1960)
4) H. Schlichting: Ibid., p 238
5) H. Gröber, S. Erk & V. Grigull: Die Grundgesetze der Wärmeübertragung, 3rd Ed. Springer (1961)
6) R.B. Bird: Advances in Chemical Engineering, Vol. 1, p. 215, Academic Press (1956)

Nomencleature

B	:	thickness of liquid phase	[cm]
C_A	:	concentration of component A	[g/cm^3]
$C_A{}^*$	:	saturated concentration of component A	[g/cm^3]
C_{A0}	:	reference concentration of component A	[g/cm^3]
$\hat{C}_p$	:	heat capacity	[cal/g c]
D	:	dissipation function	
$\mathcal{D}$	:	diffusion coefficient	[cm^2/sec]
$D_0=2R$	:	diameter of circular duct	[cm]
f	:	friction coefficient	
$\boldsymbol{g}$	:	external force	[dyne/g]=[cm/sec^2]
H. I. J	:	functionals	
$\mathcal{H}$, $\mathcal{I}$	:	dimensionless functinals	
$H_0, I_0, \mathcal{H}_0, \mathcal{I}_0$	:	stationary values of functionals	
$H_0', I_0', \mathcal{H}_0', \mathcal{I}_0'$	:	approximate stationary values of functionals	
$\mathcal{K}$	:	quantity defined by Eq. (35)	
$L=T-U$	:	Lagrangean	[erg]
l	:	length	
p	:	pressure	[dyne/cm^2]
Q_H	:	rate of heat generation	[cal/cm^3 sec]
Q_i	:	generalized force ($i=1, 2, \cdots\cdots,$)	
Q_V	:	flow rate	
$\boldsymbol{q}$	:	heat flux	[cal/cm^2 sec]

q	:	generalized coordinates ($=1, 2, \cdots\cdots,$)	
R	:	radius of circular duct	[cm]
r	:	distance along radius	[cm]
S	:	surface of domain V	
T_0	:	reference temperature	[K]
T	:	absolute temperature	[K]
$\mathcal{T}_{\xi z}$	:	a component of dimensionless stress tensor	
t	:	time	
U	:	analogical potential	
V	:	domain	
$\boldsymbol{v}$	:	velocity vector	[cm/sec]
$\langle v_z \rangle$	:	average velocity of z-direction	[cm/sec]
Y	:	dimensionless concentration	
Nu	:	Nusselt number	
Sh	:	Sherwood number	
$P_{e_H}=R_e \cdot P_r$			
$P_{e_M}=R_e \cdot S_c$			
$\alpha=\lambda/\rho C_p$	:	thermal diffusivity	[cm²/sec]
$\Delta C=C^*-C_{A0}$			
$\Delta P=P_0-P_l$			[dyne/cm²]
$\Delta T=T-T_0$			[°K], [°C]
$\delta_B, \delta_p, \delta_V$	:	thickness of boundary layer	[cm]
ζ	:	dimensionless penetration depth	[—]
η	:	parameter used in Eqs. (56) and (83)	[—]
Θ	:	dimensionless temperature	[—]
$\theta=T-T_0$			
Λ	:	dimensionless flow rate	[—]
λ	:	thermal conductivity	[cal/cm·sec c]
μ	:	viscosity	[g/cm·sec]
$\nu=\mu/\rho$	:	kinematic viscosity	[cm²/sec]
$\xi=r/R, x/l, x/B$	:	dimensionless length	[—]
ρ	:	density	[g/cm³]
τ	:	dimensionless time	[—]
ϕ	:	dimensionless velocity	