On the Resistance between the Two Disc Electrodes Applied to an Infinite Plate Conductor

By

Takeshi KIYONO* and Masaaki SHIMASAKI*

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The problem of disc electrodes is discussed on the basis of a Fredholm integral equation of the second kind. To compute the resistance between the two electrodes, numerical analysis is carried out utilizing the Laguerre-Gauss quadrature formula. The result is compared with that obtained from Riemann's solution which contains a physical contradiction.

A formula of infinite integral involving a Bessel function is also given.

1. Introduction

Though any problems of electro- or magnetostatic fields may be described as boundary value problems of the Laplace-type equations, rigorous solutions to these problems are in most cases extremely difficult. The Laplace-type equations may be solved numerically but sometimes *ad hoc* technique is used to obtain a solution of more convenient form easily. It is important however to verify whether the final solution exactly satisfies the boundary conditions when such a special mehtod is adopted.

The problem of disc electrodes treated in this paper is an example, if not typical, which shows how correct treatment of boundary conditions is important and how analytical considerations prior to numerical computation may be helpful.

As shown in 1), Riemann's solution to this problem does not satisfy exactly the boundary conditions and consequently leads to a physical contradiction. Even the method of images leads to the same fallacy. This problem may be reduced to a Fredholm integral equation of the second kind but it seems impossible to solve the integral equation analytically.

In this paper the integral equation is solved numerically to compute the resistance between the two electrodes. This approach seems to be easier than that

* Department of Electrical Engineering II.

of solving the Laplace equation directly. Though numerically, the relation between the resistance and the thickness of the plate is shown and the difference between Riemann's solution is discussed.

We also give a formula of infinite integral involving a Bessel function which was obtained as a by-product of computation of the resistance.

2. Statement of the Problem

As shown in Fig. 1, we consider an infinite plate of thickness $2h$ and of finite electrical conductivity σ. This plate is bounded by two parallel planes $z=\pm h$, and to these planes two identical perfectly conducting discs of radius a are applied as electrodes so that their centers lie in the z-axis of cylindrical coordinates as shown in Fig. 1. Between these electrodes we force a constant electric current I to flow. It is required to find the potential at any point and to compute the resistance between these electrodes.

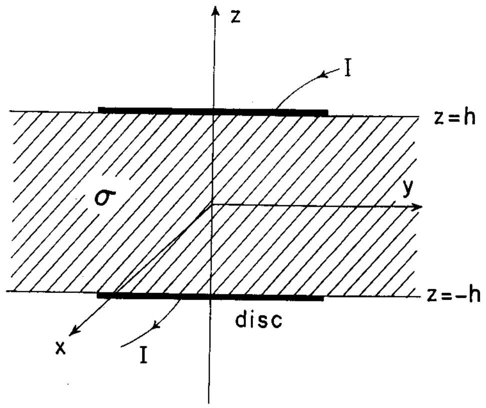

Fig. 1 Setup of the problem of two disc electrodes. Two electrodes are mounted coaxially on a conducting plate (shaded) of infinite extent.

The boundary conditions to be fulfilled at $z=\pm h$ are:

$$\frac{\partial V}{\partial z}=0, \qquad (r>a)\,; \qquad (1)$$

$$V=\pm V_0, \qquad (r<a). \qquad (2)$$

Here V_0, a constant, is the potential of the electrode at $z=h$, and $-V_0$ is that of the other electrode at $z=-h$.

Condition (1) shows that at the plate surface away from the disc there is no electric current flowing out into vacuum. Condition (2) is simply the requirement that the perfectly conducting electrode be an equipotential body.

3. Summary of the Previous Paper[1)]

In Riemann's solution quoted in the book of Gray and Mathews[2)], the condition (1) and the condition

$$\frac{\partial V}{\partial z}=\frac{I}{2\pi\sigma a}\cdot\frac{1}{\sqrt{a^2-r^2}}\qquad (z=\pm h,\quad r<a)\tag{3}$$

are adopted as boundary conditions in place of conditions (1) and (2). The above condition (3) results from the assumption that the normal component of the electric field at the electrode surface is identical with that of the case where a single disc electrode, providing a current source of $2I$, is immersed in an unlimited medium of conductivity σ. These conditions lead to the expression of the potential V in the plate:

$$V=\frac{I}{2\pi\sigma a}\int_0^\infty\frac{\sinh\lambda z}{\cosh\lambda h}\cdot\frac{\sin\lambda a}{\lambda}\cdot J_0(\lambda r)d\lambda.\tag{4}$$

As a matter of fact, this solution does not satisfy the condition (2) i. e., $V=$ constant at the electrode surface. Therefore, Riemann's solution (4) does not describe the actual physical situation of two disc electrodes facing each other. This can be made more definite by applying the method of images which leads to the same fallacy.

The exact potential V is expressed in the form

$$V=V_0\int_0^\infty B(u)\cdot\sinh\zeta u\cdot J_0(\rho u)du,\tag{5}$$

where $z=a\zeta,\quad r=a\rho.$ (6)

The function $B(u)$ may be determined by a Fredholm integral equation of the second kind:

$$B(u)=f(u)+\int_0^\infty B(\lambda)K(\lambda,u)d\lambda,\tag{7}$$

where

$$f(u)=\frac{2}{\pi}\cdot\frac{\sin u}{u\cosh tu}\tag{8}$$

$$K(\lambda,u)=\frac{1}{\pi}\cdot\frac{e^{-t\lambda}}{\cosh tu}\left\{\frac{\sin(\lambda+u)}{\lambda+u}+\frac{\sin(\lambda-u)}{\lambda-u}\right\}\tag{9}$$

$$h=at.\tag{10}$$

The relation between I and V_0 is given by the equation

$$\begin{aligned}I&=\sigma\int_0^a(-E_z)_{z=h}2\pi r dr\\&=2\pi\sigma aV_0\int_0^\infty B(u)\cosh tu\,J_1(u)du.\end{aligned}\tag{11}$$

4. Numerical Solutions of the Integral Equation

Although the problem has been reduced to a Fredholm integral equation of the second kind, its analytical solution cannot be obtained easily. Therefore we have to find a way to solve it numerically.

To solve an integral equation, it is important to find a near approximation for the integral. From Eqs. (7), (8) and (9), we may write $B(u)\propto 1/\cosh tu$, and hence

$$B(\lambda)K(\lambda, u) \propto \exp(-2t\lambda).$$

Therefore by the Laguerre-Gauss quadrature formula, the integral on the right-hand side of Eq. (7) may be estimated quite efficiently. Putting $2t\lambda=x$, we have,

$$\int_0^\infty B(\lambda)K(\lambda, u)d\lambda=\int_0^\infty e^{-x}B\left(\frac{x}{2t}\right)G(x, u)dx$$

$$\cong\sum_{j=1}^{n} w_j B\left(\frac{x_j}{2t}\right)G(x_j, u), \tag{12}$$

where

$$G(x,u) = \frac{e^{x/2}}{2\pi t\cosh tu}\left\{\frac{\sin\left(\frac{x}{2t}+u\right)}{\frac{x}{2t}+u}+\frac{\sin\left(\frac{x}{2t}-u\right)}{\frac{x}{2t}-u}\right\}, \tag{13}$$

and x_j and w_j denote the nodes and weights of the Laguerre-Gauss n-point formula. Substituting Eq. (12) into Eq. (7), we have

$$B(u) = f(u)+\sum_{j=1}^{n} w_j B\left(\frac{x_j}{2t}\right)G(x_j, u). \tag{14}$$

Substituting $u=\frac{x_j}{2t}$ $(i=1, 2, \cdots, n)$ into Eq. (14), we have a system of linear simultaneous eqations of $B\left(\frac{x_j}{2t}\right)$'s:

$$B\left(\frac{x_i}{2t}\right) = f\left(\frac{x_i}{2t}\right)+\sum_{j=1}^{n} w_j B\left(\frac{x_j}{2t}\right)G\left(x_j, \frac{x_i}{2t}\right) \qquad (i=1, \cdots, n). \tag{15}$$

If we define an $n\times n$ matrix $C=(c_{ij})$ by

$$c_{ij} = w_j G\left(x_j, \frac{x_i}{2t}\right), \tag{16}$$

we have from Eqs. (15)

$$\begin{bmatrix} B\left(\frac{x_1}{2t}\right) \\ \vdots \\ B\left(\frac{x_n}{2t}\right) \end{bmatrix} = (I-C)^{-1} \begin{bmatrix} f\left(\frac{x_1}{2t}\right) \\ \vdots \\ f\left(\frac{x_n}{2t}\right) \end{bmatrix} . \tag{17}$$

Once $B\left(\frac{x_i}{2t}\right)$'s are determined, $B(u)$ may be computed by Eq. (14) for any value of u.

Now we describe the relation between this method and the method of successive approximation referred to in 1).

Expanding $B(u)$ as

$$B(u) = f(u) + B_1(u) + B_2(u) + \cdots\cdots, \tag{18}$$

we have

$$B_i(u) = \int_0^\infty B_{i-1}(\lambda) K(\lambda, u) d\lambda, \tag{19}$$

where $B_0(u) = f(u)$. If we estimate the right-hand side of Eq. (19) by the Laguerre-Gauss quadrature formula, we have

$$B_i(u) \cong \sum_{j=1}^{n} w_j B_{i-1}\left(\frac{x_j}{2t}\right) G(x_j, u). \tag{20}$$

Substitution of $u = \frac{x_i}{2t}$ into Eq. (20) gives

$$\begin{bmatrix} B_i\left(\frac{x_1}{2t}\right) \\ \vdots \\ B_i\left(\frac{x_n}{2t}\right) \end{bmatrix} = C \begin{bmatrix} B_{i-1}\left(\frac{x_1}{2t}\right) \\ \vdots \\ B_{i-1}\left(\frac{x_n}{2t}\right) \end{bmatrix} = C^i \begin{bmatrix} f\left(\frac{x_1}{2t}\right) \\ \vdots \\ f\left(\frac{x_n}{2t}\right) \end{bmatrix} . \tag{21}$$

From Eqs. (18), (21), we have

$$\begin{bmatrix} B\left(\frac{x_1}{2t}\right) \\ \vdots \\ B\left(\frac{x_n}{2t}\right) \end{bmatrix} = (I + C + C^2 + \cdots) \begin{bmatrix} f\left(\frac{x_1}{2t}\right) \\ \vdots \\ f\left(\frac{x_n}{2t}\right) \end{bmatrix} = (I-C)^{-1} \begin{bmatrix} f\left(\frac{x_1}{2t}\right) \\ \vdots \\ f\left(\frac{x_n}{2t}\right) \end{bmatrix} , \tag{22}$$

provided that the right-hand side of Eq. (22) converges.

Numerical solutions were obtained for several values of t from Eqs. (17), (14) utilizing the Laguerre-Gauss ten-point and twenty-point formulas. Results by ten- and twenty-point formulas were mutually in good agreement especially for

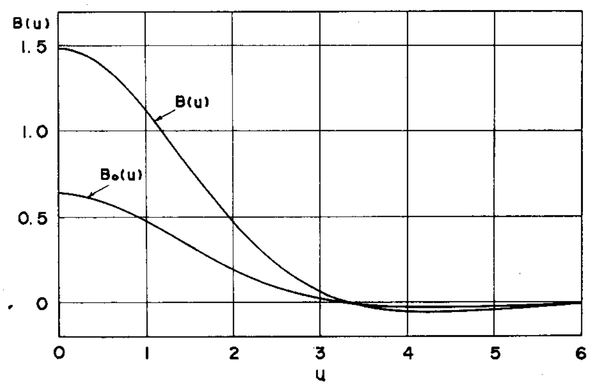

Fig. 2 Numerical solution of the integral equation (7) for t=0.5.

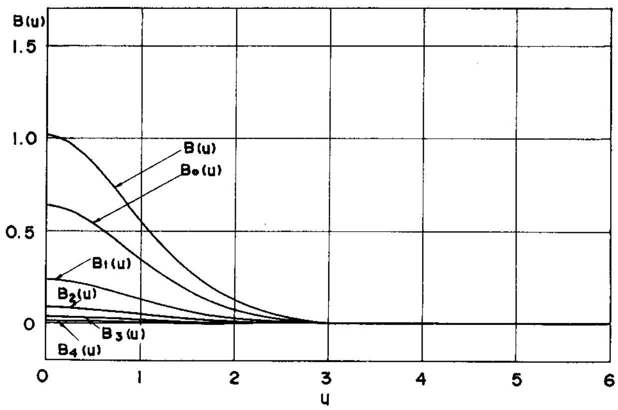

Fig. 3 Numerical solution of the integral equation (7) for t=1.0 and intermediate solutions by the method of successive approximation.

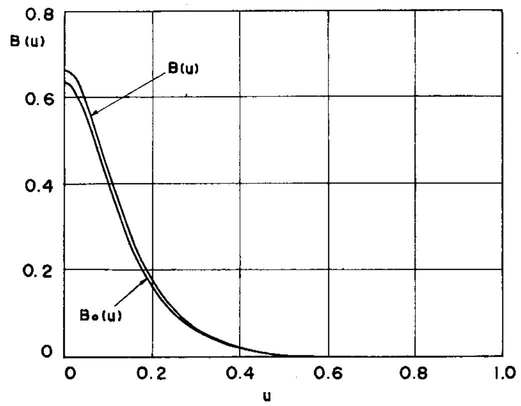

Fig. 4 Numerical solution of the integral equation (7) for t=10.

large values of t. Figs. 2, 3 and 4 show the typical results for $t=0.5$, 1.0 and 10.0 respectively.

To test the accuracy of this method, the result for $t=1.0$ was compared with results obtained by other methods. The methods compared are as follows:

1) A method to refine the approximate solution obtained by ten-point formula by substituting it into the integral equation (7). Transforming the integral part of Eq. (7) into

$$\int_0^\infty B(\lambda)K(\lambda, u)d\lambda = \sum_{k=1}^{m}\int_{(k-1)h}^{kh} B(\lambda)K(\lambda, u)d\lambda + \int_0^\infty B(\lambda+mh)K(\lambda+mh, u)d\lambda, \tag{23}$$

the first m terms of Eq. (23) were estimated by applying the Legendre-Gauss eight-point formula repeatedly and the last term was estimated by the Laguerre-Gauss twelve-point formula, where $h=0.5$, $m=12$.

2) The method of successive approximation. The fifth-order approximate solution was computed by the Simpson quadrature formula with step size $h=0.1$. Thus:

$$B(u) = \sum_{i=0}^{5} B_i(u).$$

Although the convergence was not necessarily sufficient, the values of $B_i(u)$'s $(i = 1, \cdots, 5)$ showed good agreement with those by the Laguerre-Gauss quadrature formula.

5. A Formula of Infinite Integral Involving a Bessel Function

We give a formula of infinite integral involving a Bessel function which plays an important role in computation of the resistance between the two electrodes. The formula is:

$$\int_0^\infty J_n(x)\frac{\sin(x+k)}{x+k}dx = \int_0^{\pi/2}\cos(n\phi+k\sin\phi)d\phi \tag{24}$$

$$= \begin{cases} \frac{\pi}{2}J_n(k)+2n(-)^{n/2}\sum_{m=0}^{\infty}(-)^m\frac{J_{2m+1}(k)}{n^2-(2m+1)^2} & (n:\text{even}) \\ -\frac{\pi}{2}J_n(k)+\frac{(-)^{(n-1)/2}}{n}J_0(k)+2n(-)^{(n-1)/2}\sum_{m=1}^{\infty}(-)^m\frac{J_{2m}(k)}{n^2-(2m)^2} & (n:\text{odd}) \end{cases} \tag{25}$$

where k is real.

As a special case, we have for $n=1$,

$$\int_0^\infty J_1(x)\frac{\sin(x+k)}{x+k}dx = -\frac{\pi}{2}J_1(k)+\frac{\sin k}{k} \ . \tag{26}$$

We now give the proof.
Since

$$\frac{\sin(x+k)}{x+k} = \int_0^1 \cos t(x+k)dt \tag{27}$$

$$\int_0^\infty J_n(at)\sin btdt = \begin{cases} \dfrac{\sin\left(n\sin^{-1}\left(\frac{b}{a}\right)\right)}{\sqrt{a^2-b^2}} & a>b \\ \infty \quad \text{or} \quad 0 & a=b \\ \dfrac{a^n\cos\left(\frac{n\pi}{2}\right)}{\sqrt{b^2-a^2}\,(b+\sqrt{b^2-a^2}\,)^n} & a<b \end{cases} \tag{28}$$

$$\int_0^\infty J_n(at)\cos btdt = \begin{cases} \dfrac{\cos\left(n\sin^{-1}\left(\frac{b}{a}\right)\right)}{\sqrt{a^2-b^2}} & a>b \\ \infty \quad \text{or} \quad 0 & a=b \\ \dfrac{a^n\sin\left(\frac{n\pi}{2}\right)}{\sqrt{b^2-a^2}\,(b+\sqrt{b^2-a^2}\,)^n} & a<b, \end{cases} \tag{29}$$

we have

$$\begin{aligned} &\int_0^\infty J_n(x)\frac{\sin(x+k)}{x+k}\,dx \\ &= \int_0^1 dt\int_0^\infty J_n(x)\cos t(x+k)dx \\ &= \int_0^1\left\{\cos kt\,\frac{\cos(n\sin^{-1}t)}{\sqrt{1-t^2}} - \sin kt\,\frac{\sin(n\sin^{-1}t)}{\sqrt{1-t^2}}\right\}dt \\ &= \int_0^1\frac{\cos(kt+n\sin^{-1}t)}{\sqrt{1-t^2}}\,dt. \end{aligned} \tag{30}$$

If we write $t=\sin\phi$, we have Eq. (24):

$$\int_0^\infty J_n(x)\frac{\sin(x+k)}{x+k}dx = \int_0^{\pi/2}\cos(n\phi+k\sin\phi)d\phi.$$

Considering

$$J_n(z) = \begin{cases} \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\sin n\theta\sin(z\sin\theta)d\theta & n:\text{odd} \\ \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\cos n\theta\cos(z\sin\theta)d\theta & n:\text{even} \end{cases} \tag{31}$$

$$\cos(z\sin\theta) = J_0(z)+2\sum_{m=1}^{\infty} J_{2m}(z)\cos 2m\theta \tag{32}$$

$$\sin(z\sin\theta) = 2\sum_{m=0}^{\infty} J_{2m+1}(z)\sin(2m+1)\theta, \tag{33}$$

we arrive at the final results. That is, for even n, we have

$$\int_0^{\pi/2}\cos(n\phi+k\sin\phi)d\phi = \frac{\pi}{2}J_n(k)-2\sum_{m=0}^{\infty}J_{2m+1}(k)\int_0^{\pi/2}\sin n\phi\sin(2m+1)\phi d\phi$$

$$= \frac{\pi}{2}J_n(k)+2n(-)^{n/2}\sum_{m=0}^{\infty}(-)^m\frac{J_{2m+1}(k)}{n^2-(2m+1)^2}$$

and for odd n,

$$\int_0^{\pi/2}\cos(n\phi+k\sin\phi)d\phi = -\frac{\pi}{2}J_n(k)+J_0(k)\int_0^{\pi/2}\cos n\phi d\phi$$

$$+2\sum_{m=1}^{\infty}J_{2m}(k)\int_0^{\pi/2}\cos n\phi\cos 2m\phi d\phi$$

$$= -\frac{\pi}{2}J_n(k)+(-)^{(n-1)/2}\frac{J_0(k)}{n}+2n(-)^{(n-1)/2}\sum_{m=1}^{\infty}(-)^m\frac{J_{2m}(k)}{n^2-(2m)^2}.$$

In case of $n=1$, Eq. (25) may further be simplified to

$$\int_0^{\infty}J_1(x)\frac{\sin(x+k)}{x+k}dx = -\frac{\pi}{2}J_1(k)+\frac{\sin k}{k}$$

by the formulas such as

$$J_{\nu-1}(k)+J_{\nu+1}(k) = \frac{2\nu}{k}J_\nu(k) \tag{34}$$

$$\sin k = 2\sum_{m=0}^{\infty}(-)^m J_{2m+1}(k). \tag{35}$$

6. Resistance Between the Two Electrodes

From Eq. (11), the resistance R between the two electrodes may be written in the form

$$R = 2V_0/I = R_0/S, \tag{36}$$

where

$$R_0=\frac{1}{2\sigma a} \tag{37}$$

$$S=\frac{\pi}{2}\int_0^\infty B(u)\cdot \cosh tu\cdot J_1(u)du. \tag{38}$$

As t tends to infinity, $B(u)$ tends to $f(u)$ and therefore S to 1 and R to R_0 which is twice the resistance of a single electrode. This may be expected because for sufficiently large values of t, the two electrodes can be treated as two independent electrodes. The quantity S can be considered to express the dependence of the resistance on the thickness t of the plate.

The integrand of Eq. (38) is proportional to $u^{-3/2}$ for a large value of u and is slow to converge. Utilizing Eqs. (7) and (26), Eq. (38) may be transformed into a more convenient form for numerical quadrature. We have

$$\begin{aligned} S &= \frac{\pi}{2}\int_0^\infty f(u)\cosh tu\, J_1(u)du+\frac{\pi}{2}\int_0^\infty\left\{\int_0^\infty B(\lambda)K(\lambda,u)d\lambda\right\}\cosh tu\, J_1(u)du \\ &= 1+\frac{1}{2}\int_0^\infty e^{-t\lambda}B(\lambda)d\lambda\int_0^\infty\left\{\frac{\sin(\lambda+u)}{\lambda+u}+\frac{\sin(\lambda-u)}{\lambda-u}\right\}J_1(u)du \\ &= 1+\int_0^\infty e^{-t\lambda}B(\lambda)\frac{\sin\lambda}{\lambda}\,d\lambda. \end{aligned} \tag{39}$$

Considering $B(\lambda)\propto \exp(-t\lambda)$ as $\lambda\to\infty$ and writing $2t\lambda=x$, we have by the Laguerre-Gauss quadrature formula,

$$\begin{aligned} S &= 1+\frac{1}{2t}\int_0^\infty e^{-x}\left\{e^{x/2}B\left(\frac{x}{2t}\right)\frac{\sin\left(\frac{x}{2t}\right)}{\frac{x}{2t}}\right\}dx \\ &\cong 1+\frac{1}{2t}\sum_{j=1}^{n} e^{x_j/2}B\left(\frac{x_j}{2t}\right)\frac{\sin\left(\frac{x_j}{2t}\right)}{\frac{x_j}{2t}}\,w_j, \end{aligned} \tag{40}$$

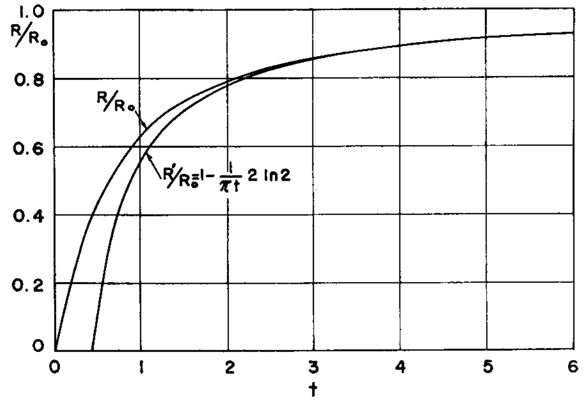

Fig. 5 The normalized resistance R/R_0 between the two electrodes. Approximation based on Riemann's solution $R'/R_0=1-2(\ln 2)/(\pi t)$ is also shown.

where x_j and w_j denote the nodes and weights of n-point formula respectively. We give the numerical results of R/R_0 by ten- and twenty-point forumlas in Fig. 5*.

7. Discussions

As mentioned in section 3, the electrode surface in Riemann's solution is not equipotential. Therefore, strictly speaking, we cannot define the resistance between the electrodes in Riemann's solution. But if we write $R=(2V/I)_{z=h}$, we have from Eq. (4).

$$R/R_0 = \frac{2}{\pi}\int_0^\infty \frac{\sinh\lambda h}{\cosh\lambda h} J_0(\lambda r)\frac{\sin\lambda a}{\lambda}\, d\lambda$$
$$= 1-\frac{2}{\pi}\int_0^\infty \frac{2e^{-\lambda h}}{e^{\lambda h}+e^{-\lambda h}} J_0(\lambda r)\frac{\sin\lambda a}{\lambda}\, d\lambda. \tag{41}$$

Expanding as

$$J_0(\lambda r)\frac{\sin\lambda a}{\lambda} = a\left(1-\frac{\lambda^2 r^2}{4}+\frac{\lambda^4 r^4}{64}-\cdots\cdots\right)\left(1-\frac{1}{3!}\lambda^2 a^2+\cdots\cdots\right)$$
$$= a\left\{1-\left(\frac{r^2}{4}+\frac{a^2}{6}\right)\lambda^2+\cdots\cdots\right\}, \tag{42}$$

and

$$\frac{e^{-\lambda h}}{e^{\lambda h}+e^{-\lambda h}} = \sum_{m=1}^{\infty}(-)^{m-1}e^{-2mh\lambda}, \tag{43}$$

we have

$$R/R_0 = 1-\frac{4a}{\pi}\sum_{m=1}^{\infty}(-)^{m-1}\int_0^\infty e^{-2mh\lambda}\left\{1-\left(\frac{r^2}{4}+\frac{a^2}{6}\right)\lambda^2+\cdots\cdots\right\}d\lambda$$
$$= 1-\frac{4a}{\pi}\sum_{m=1}^{\infty}(-)^{m-1}\left\{\frac{1}{2h}\frac{1}{m}-\left(\frac{r^2}{4}+\frac{a^2}{6}\right)\frac{2}{(2mh)^3}+\cdots\cdots\right\}$$
$$= 1-\frac{2a}{\pi h}\log_e 2+\frac{a^3}{6\pi h^3}\left(1+\frac{3}{2}\frac{r^2}{a^2}\right)\sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m^3}+O\left(\frac{r^5}{h^5},\frac{a^5}{h^5}\right)$$
$$= 1-\frac{2\log_e 2}{\pi t}+\frac{1}{6\pi t^3}\left(1+\frac{3}{2}\frac{r^2}{a^2}\right)\cdot\frac{1}{2}\int_0^\infty\frac{v^2}{e^v+1}dv+O\left(\frac{r^5}{h^5},\frac{a^5}{h^5}\right), \tag{44}$$

where

* The computer system KDC-II (HITAC 5020) at the Kyoto University Computation Center was used for this numerical calculation.

$$\frac{1}{2}\int_0^\infty \frac{v^2}{e^v+1}\,dv \cong 1. \tag{45}$$

Thus the resistance obtained by Riemann's method is not constant but depends on r especially for small values of t. As Gray and Mathews adopted only the first approximation $R/R_0=1-2(\ln 2)/(\pi t)$, they seem to have overlooked this physical contradiction.

In case the thickness t is very large, say, greater than 10 where the mutual interaction of two electrodes is very small, their approximation $R/R_0=1-2(\ln 2)/(\pi t)$ shows a good agreement with the exact result. But for small values of t, such is not the case. This follows from the fact that in Riemann's solution, the mutual interaction between the two electrodes is not taken into consiedration.

On the other hand, Fig. 5 shows that the exact resistance R for small values of t is given by $R=R_0t$, which seems quite reasonable.

References

1) T. Kiyono and T. Tsuda: Some Contributions to the Theory of Electrostatic Field, THIS MEMOIRS, Vol. XXX, Part 3 (Sept. 1968) pp. 315-326.
2) A. Gray and G.B. Mathews: "A Treatise on Bessel Functions and their Applications to Physics," MacMillan and Co., London, pp. 128-131 (1895).

Appendix

Numerical results are given in Tables 1, 2, 3, 4, and 5.

Table 1. Numerical solution of the integral equation (7) for $t=0.5$ by the Laguerre-Gauss N-point formula.

u	$B(u)$	
	$N=10$	$N=20$
0.000	0.1483813 E + 01	0.1483765 E + 01
0.200	0.1467262 E + 01	0.1467215 E + 01
0.400	0.1418724 E + 01	0.1418680 E + 01
0.600	0.1341397 E + 01	0.1341357 E + 01
0.800	0.1240164 E + 01	0.1240131 E + 01
1.000	0.1121016 E + 01	0.1120989 E + 01
1.200	0.9904149 E + 00	0.9903949 E + 00
1.400	0.8546983 E + 00	0.8546855 E + 00
1.600	0.7196010 E + 00	0.7195951 E + 00
1.800	0.5899328 E + 00	0.5899329 E + 00
2.000	0.4694142 E + 00	0.4694194 E + 00
2.200	0.3606487 E + 00	0.3606581 E + 00
2.400	0.2651957 E + 00	0.2652081 E + 00
2.600	0.1837048 E + 00	0.1837192 E + 00
2.800	0.1160851 E + 00	0.1161006 E + 00
3.000	0.6167849 E − 01	0.6169415 E − 01
3.200	0.1942480 E − 01	0.1943994 E − 01
3.400	−0.1199270 E − 01	−0.1197863 E − 01
3.600	−0.3402539 E − 01	−0.3401281 E − 01
3.800	−0.4816193 E − 01	−0.4815115 E − 01
4.000	−0.5585422 E − 01	−0.5584535 E − 01
4.200	−0.5846380 E − 01	−0.5845691 E − 01
4.400	−0.5722619 E − 01	−0.5722119 E − 01
4.600	−0.5322959 E − 01	−0.5322638 E − 01
4.800	−0.4740545 E − 01	−0.4740387 E − 01
5.000	−0.4052791 E − 01	−0.4052773 E − 01
5.200	−0.3321994 E − 01	−0.3322093 E − 01
5.400	−0.2596434 E − 01	−0.2596625 E − 01
5.600	−0.1911776 E − 01	−0.1912035 E − 01
5.800	−0.1292656 E − 01	−0.1292962 E − 01
6.000	−0.7543372 E − 02	−0.7546701 E − 02
6.200	−0.3043568 E − 02	−0.3046987 E − 02
6.400	0.5590634 E − 03	0.5557021 E − 03
6.600	0.3297765 E − 02	0.3294580 E − 02
6.800	0.5239786 E − 02	0.5236868 E − 02
7.000	0.6475007 E − 02	0.6472420 E − 02

Table 2. Numerical solution of the integral equation (7) for $t=1.0$ by the Laguerre-Gauss N-point formula.

u	$B(u)$	
	$N=10$	$N=20$
0.000	0.1015904 E + 01	0.1015906 E + 01
0.200	0.9894333 E + 00	0.9894353 E + 00
0.400	0.9153848 E + 00	0.9153866 E + 00
0.600	0.8075296 E + 00	0.8075314 E + 00
0.800	0.6827646 E + 00	0.6827661 E + 00
1.000	0.5561601 E + 00	0.5561612 E + 00
1.200	0.4383606 E + 00	0.4383615 E + 00
1.400	0.3353044 E + 00	0.3353051 E + 00
1.600	0.2492049 E + 00	0.2492054 E + 00
1.800	0.1798356 E + 00	0.1798359 E + 00
2.000	0.1256212 E + 00	0.1256214 E + 00
2.200	0.8439842 E − 01	0.8439855 E − 01
2.400	0.5388195 E − 01	0.5388203 E − 01
2.600	0.3191979 E − 01	0.3191981 E − 01
2.800	0.1661363 E − 01	0.1661364 E − 01
3.000	0.6359641 E − 02	0.6359628 E − 02
3.200	−0.1546241 E − 03	−0.1546469 E − 03
3.400	−0.3975325 E − 02	−0.3975351 E − 02
3.600	−0.5916903 E − 02	−0.5916931 E − 02
3.800	−0.6598966 E − 02	−0.6598992 E − 02
4.000	−0.6481655 E − 02	−0.6481678 E − 02
4.200	−0.5897617 E − 02	−0.5897637 E − 02
4.400	−0.5079797 E − 02	−0.5079812 E − 02
4.600	−0.4184884 E − 02	−0.4184896 E − 02
4.800	−0.3312596 E − 02	−0.3312606 E − 02
5.000	−0.2521229 E − 02	−0.2521235 E − 02
5.200	−0.1839892 E − 02	−0.1839897 E − 02
5.400	−0.1277947 E − 02	−0.1277950 E − 02
5.600	−0.8321042 E − 03	−0.8321061 E − 03
5.800	−0.4916283 E − 03	−0.4916291 E − 03
6.000	−0.2420356 E − 03	−0.2420359 E − 03
6.200	−0.6761427 E − 04	−0.6761412 E − 04
6.400	0.4694616 E − 04	0.4694657 E − 04
6.600	0.1155835 E − 03	0.1155840 E − 03
6.800	0.1503649 E − 03	0.1503654 E − 03
7.000	0.1613144 E − 03	0.1613151 E − 03

Table 3. Numerical solution of the integral equation (7) for $t=10.0$ by the Laguerre-Gauss N-point formula.

u	$B(u)$	
	$N=10$	$N=20$
0.000	0.6659423 E + 00	0.6659425 E + 00
0.020	0.6527985 E + 00	0.6527984 E + 00
0.040	0.6158376 E + 00	0.6158376 E + 00
0.060	0.5614195 E + 00	0.5614192 E + 00
0.080	0.4973940 E + 00	0.4973942 E + 00
0.100	0.4308479 E + 00	0.4308479 E + 00
0.120	0.3669087 E + 00	0.3669087 E + 00
0.140	0.3086008 E + 00	0.3086009 E + 00
0.160	0.2572702 E + 00	0.2572702 E + 00
0.180	0.2131481 E + 00	0.2131482 E + 00
0.200	0.1758313 E + 00	0.1758313 E + 00
0.220	0.1446140 E + 00	0.1446140 E + 00
0.240	0.1186925 E + 00	0.1186925 E + 00
0.260	0.9727648 E − 01	0.9727649 E − 01
0.280	0.7964340 E − 01	0.7964338 E − 01
0.300	0.6515898 E − 01	0.6515898 E − 01
0.320	0.5328023 E − 01	0.5328026 E − 01
0.340	0.4354942 E − 01	0.4354943 E − 01
0.360	0.3558452 E − 01	0.3558452 E − 01
0.380	0.2906885 E − 01	0.2906886 E − 01
0.400	0.2374105 E − 01	0.2374106 E − 01
0.420	0.1938601 E − 01	0.1938602 E − 01
0.440	0.1582710 E − 01	0.1582710 E − 01
0.460	0.1291942 E − 01	0.1291942 E − 01
0.480	0.1054429 E − 01	0.1054429 E − 01
0.500	0.8604512 E − 02	0.8604512 E − 02
0.520	0.7020552 E − 02	0.7020552 E − 02
0.540	0.5727337 E − 02	0.5727337 E − 02
0.560	0.4671659 E − 02	0.4671658 E − 02
0.580	0.3810019 E − 02	0.3810020 E − 02
0.600	0.3106851 E − 02	0.3106851 E − 02
0.620	0.2533092 E − 02	0.2533093 E − 02
0.640	0.2064992 E − 02	0.2064992 E − 02
0.660	0.1683150 E − 02	0.1683150 E − 02
0.680	0.1371715 E − 02	0.1371715 E − 02
0.700	0.1117741 E − 02	0.1117740 E − 02

Table 4. Numerical solution of the integral equation (7) for $t=1.0$ by the Laguerre-Gauss 10-point formula and the refined solution.

u	$B(u)$	
	$N=10$	Refined Solution
0.000	0.1015904 E + 01	0.1015905 E + 01
0.100	0.1009197 E + 01	0.1009198 E + 01
0.200	0.9894333 E + 00	0.9894345 E + 00
0.300	0.9576372 E + 00	0.9576384 E + 00
0.400	0.9153848 E + 00	0.9153862 E + 00
0.500	0.8646332 E + 00	0.8646345 E + 00
0.600	0.8075296 E + 00	0.8075308 E + 00
0.700	0.7462324 E + 00	0.7462335 E + 00
0.800	0.6827646 E + 00	0.6827657 E + 00
0.900	0.6189110 E + 00	0.6189119 E + 00
1.000	0.5561601 E + 00	0.5561608 E + 00
1.100	0.4956864 E + 00	0.4956870 E + 00
1.200	0.4383606 E + 00	0.4383611 E + 00
1.300	0.3847792 E + 00	0.3847797 E + 00
1.400	0.3353044 E + 00	0.3353049 E + 00
1.500	0.2901064 E + 00	0.2901068 E + 00
1.600	0.2492049 E + 00	0.2492052 E + 00
1.700	0.2125064 E + 00	0.2125066 E + 00
1.800	0.1798356 E + 00	0.1798358 E + 00
1.900	0.1509621 E + 00	0.1509623 E + 00
2.000	0.1256212 E + 00	0.1256213 E + 00
2.100	0.1035296 E + 00	0.1035297 E + 00
2.200	0.8439842 E − 01	0.8439849 E − 01
2.300	0.6794144 E − 01	0.6794147 E − 01
2.400	0.5388195 E − 01	0.5388197 E − 01
2.500	0.4195693 E − 01	0.4195694 E − 01
2.600	0.3191979 E − 01	0.3191978 E − 01
2.700	0.2354196 E − 01	0.2354195 E − 01
2.900	0.1094375 E − 01	0.1094373 E − 01
3.100	0.2706288 E − 02	0.2706257 E − 02
3.300	−0.2345770 E − 02	−0.2345802 E − 02
3.500	−0.5138140 E − 02	−0.5138173 E − 02
3.700	−0.6383272 E − 02	−0.6383304 E − 02
3.900	−0.6616801 E − 02	−0.6616828 E − 02
4.100	−0.6231376 E − 02	−0.6231400 E − 02
4.300	−0.5506597 E − 02	−0.5506615 E − 02
4.500	−0.4634610 E − 02	−0.4634626 E − 02
4.700	−0.3741456 E − 02	−0.3741469 E − 02
4.900	−0.2904417 E − 02	−0.2904427 E − 02
5.100	−0.2165852 E − 02	−0.2165860 E − 02
5.300	−0.1543974 E − 02	−0.1543980 E − 02
5.500	−0.1041059 E − 02	−0.1041063 E − 02
5.700	−0.6495471 E − 03	−0.6495500 E − 03
5.900	−0.3564475 E − 03	−0.3564495 E − 03
6.100	−0.1464090 E − 03	−0.1464102 E − 03
6.300	−0.3762463 E − 05	−0.3763194 E − 05
6.500	0.8620387 E − 04	0.8620349 E − 04
6.700	0.1365315 E − 03	0.1365313 E − 03
6.900	0.1582719 E − 03	0.1582718 E − 03
7.100	0.1604332 E − 03	0.1604333 E − 03
7.300	0.1500940 E − 03	0.1500942 E − 03
7.500	0.1326129 E − 03	0.1326130 E − 03
7.700	0.1118828 E − 03	0.1118829 E − 03
7.900	0.9059533 E − 04	0.9059544 E − 04

Table 5. The normalized resistance R/R_0

t	R/R_0		$R'/R_0 = 1 - 2\cdot(\ln 2)/(\pi t)$
	$N=10$	$N=20$	
0.100	−0.2441338 E + 00	0.8208976 E − 01	−0.3412712 E + 01
0.200	0.2018067 E + 00	0.2134419 E + 00	−0.1206356 E + 01
0.300	0.2998587 E + 00	0.2981674 E + 00	−0.4709040 E + 00
0.400	0.3689218 E + 00	0.3688071 E + 00	−0.1031780 E + 00
0.500	0.4290431 E + 00	0.4290570 E + 00	0.1174576 E + 00
0.600	0.4807292 E + 00	0.4807199 E + 00	0.2645480 E + 00
0.700	0.5252381 E + 00	0.5252339 E + 00	0.3696126 E + 00
0.800	0.5637791 E + 00	0.5637768 E + 00	0.4484110 E + 00
0.900	0.5973188 E + 00	0.5973170 E + 00	0.5096987 E + 00
1.000	0.6266537 E + 00	0.6266523 E + 00	0.5587288 E + 00
1.200	0.6752263 E + 00	0.6752257 E + 00	0.6322740 E + 00
1.400	0.7135043 E + 00	0.7135037 E + 00	0.6848063 E + 00
1.600	0.7442391 E + 00	0.7442384 E + 00	0.7242055 E + 00
1.800	0.7693429 E + 00	0.7693429 E + 00	0.7548493 E + 00
2.000	0.7901655 E + 00	0.7901648 E + 00	0.7793644 E + 00
2.500	0.8292189 E + 00	0.8292189 E + 00	0.8234915 E + 00
3.000	0.8562905 E + 00	0.8562899 E + 00	0.8529096 E + 00
4.000	0.8911378 E + 00	0.8911372 E + 00	0.8896822 E + 00
6.000	0.9268923 E + 00	0.9268923 E + 00	0.9264548 E + 00
8.000	0.9450270 E + 00	0.9450264 E + 00	0.9448411 E + 00
10.000	0.9559680 E + 00	0.9559680 E + 00	0.9558729 E + 00
15.000	0.9706102 E + 00	0.9706102 E + 00	0.9705819 E + 00
20.000	0.9779486 E + 00	0.9779486 E + 00	0.9779364 E + 00