Heat Transfer to Laminar Flow of Pseudoplastic Fluids

By

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The temperature dependence of viscosity was taken into account in solving the equations of change for laminar flows of pseudoplastic fluids in circular tubes. The solutions were obtained for the entrance and the thermally well developed regions at the conditon of constant heat flux. Those analytical solutions were approximated with the equations of non dimensional moduli, which were in good agreement with the experimental results of the authors.

Furthermore, even if the temperature dependence of viscosity was taken into account, the plots of Nusselt numbers at the thermally fully developed region against u_{max}/u_m were shown to be just shifted on the same curve for those plots of nearly isothermal flow.

Introduction

In order to obtain the heat transfer coefficients of the laminar flow of non-Newtonian fluids in a circular tube, many investigators have solved analytically the equations of change with the various non-Newtonian models. Beek and Eggink[1] summarized these results for the following conditions, (1) entrance region —constant wall temperature, (2) entrance region—constant heat flux, (3) thermally fully developed region—constant wall temperature, (4) theromally fully developed region—constant hear flux. Those solutions were based on the assumption that all the transport properties of the fluid were independent of the temperature or that the flow was nearly isothermal.

In this paper, the authors will take into account the temperature dependence of viscosity in solving the equations of change for the entrance and the thermally well developed regions at the condition of constant wall heat flux. Those analytical solutions will be approximated with the equations of non diemnsional moduli, which are in good agreement with the experimental results. In the experiment, aqueous solutions of carboxy methyl cellulose (C.M.C.) were used and presumed

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to be pseudoplastic fluids, the flow characteristic of which was assumed to be represented by the Ostwald-de Waele's power law model.

Beek and Eggink[1] showed that the plot of Nusselt numbers in the thermally fully developed region against the ratios of the maximum to the average fluid velocity obtained analytically, for the nearly isothermal flow with use of any model of non-Newtonian fluids, could be correlated with a single curve. The authors will show that the plot of those obtained from taking into account the temperature dependence of viscosity can be correlated with the same curve as that given for the nearly isothermal flow.

Temperature Dependence of Viscosity

The flow characteristic of pseudoplastic fluids is assumed to be represented by the power law model

$$\tau = -m\left|\frac{du}{dr}\right|^{n} \tag{1}$$

As the experiment with the aqueous solutions of C.M.C. shows that the values of n are independent of the temperautre in the range of 15°C~60°C, while that of m changes considerably with temperature, it is assumed that this is the case common

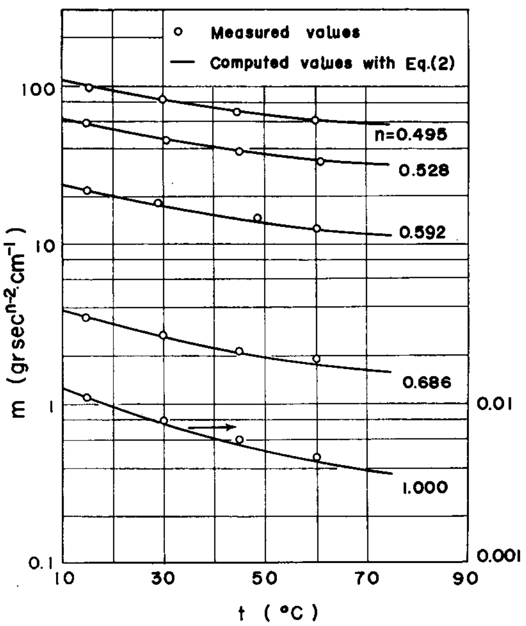

Fig. 1 Temperature dependence of m

to all pseudoplastic fluids. For the convenience of the following calculation, equation (2) is assumed to represent the temperature dependence of m.

$$m = \frac{m_0}{\left(1+\beta_c \frac{t-t_0}{t_0}\right)^n} \tag{2}$$

β_c is a characteristic constant of a material, and is determined by experiments as 0.5 for aqueous solutions of C.M.C.

As shown in Fig. 1, the results of calculation with equation (2) are in good agreement with those of measurement.

Thermally Fully Develpoed Region

In order to take into account the temperature dependence of m, one has to solve the equation of motion, the equation of energy and equation (2) simultaneously. Since this calculation is tedious, the authors adopted the following approximation method of calculation. Equation of motion is

$$\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[rm\left(-\frac{\partial u}{\partial r}\right)^n\right] = 0 \tag{3}$$

B.C. (1) at $r = 0$ u = finite

(2) at $r = R$ $u = 0$ (4)

Equation of energy is

$$u\frac{\partial t}{\partial z} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) \tag{5}$$

B.C. (1) at $r = 0$ t = finite

(2) at $r = R$ $-\lambda\frac{\partial t}{\partial r} = q_w$ = constant

(3) at $z = 0$ $t = t_0$ (6)

1. The first approximation of velocity profile.
Substituting

$$X = \frac{r}{R} \tag{7}$$

into equations (3) and (4), one obtains

$$\frac{\partial}{\partial X}\left[mX\left(-\frac{\partial u}{\partial X}\right)^n\right] = -R^{n+1}\frac{dP}{dz}X \tag{8}$$

B.C. (1′) at $X = 0$ u = finite

(2′) at $X = 1$ $u = 0$ (9)

Solving these gives

$$u = \left(-\frac{R^{n+1}}{2}\frac{dP}{dz}\right)^{\frac{1}{n}}\int_X^1\left(\frac{X}{m}\right)^{\frac{1}{n}}dX \tag{10}$$

$$u_{\max} = \left(-\frac{R^{n+1}}{2}\frac{dP}{dz}\right)^{\frac{1}{n}}\int_0^1\left(\frac{X}{m}\right)^{\frac{1}{n}}dX \tag{11}$$

$$u_m = \left(-\frac{R^{n+1}}{2}\frac{dP}{dz}\right)^{\frac{1}{n}}\int_0^1\frac{X^{\frac{1}{n}+2}}{m^{\frac{1}{n}}}dX \tag{12}$$

$$\frac{u}{u_m} = \frac{\int_X^1\frac{X^{\frac{1}{n}}}{m^{\frac{1}{n}}}dX}{\int_0^1\frac{X^{\frac{1}{n}+2}}{m^{\frac{1}{n}}}dX} \tag{13}$$

$$\frac{u_{\max}}{u_m} = \frac{\int_0^1\frac{X^{\frac{1}{n}}}{m^{\frac{1}{n}}}dX}{\int_0^1\frac{X^{\frac{1}{n}+2}}{m^{\frac{1}{n}}}dX} \tag{14}$$

Assuming m and n are independent of the temperature, the first approximation of velocity distribution is calculated as

$$\frac{u}{u_m} = \frac{3n+1}{n+1}\left(1 - X^{\frac{1}{n}+1}\right) \tag{15}$$

$$\frac{u_{\max}}{u_m} = \frac{3n+1}{n+1} \tag{16}$$

2. The first approximation of temperature profile.

Substituting the first approximation of velocity distribution or equation (15) and

$$\theta = (t-t_0)\Big/\left(\frac{q_w R}{\lambda}\right) \tag{17}$$

$$Z = \alpha z/(R^2 u_m) \tag{18}$$

into equation (5),
one obtains

$$\frac{3n+1}{n+1}(1-X^{\frac{1}{n}+1})\frac{\partial\theta}{\partial z} = \frac{1}{X}\frac{\partial}{\partial X}\left(X\frac{\partial\theta}{\partial X}\right) \tag{19}$$

$$\text{B.C.}\quad \left.\begin{array}{llll} (1') & \text{at} & X=0 & \theta = \text{finite} \\ (2') & \text{at} & X=1 & -\partial\theta/\partial X = 1 \\ (3') & \text{at} & Z=0 & \theta = 0 \end{array}\right\} \tag{20}$$

Solving these equations gives the first approximation of temperature distribution as follows.

$$\theta = -\frac{2\Pi}{Gz} - \frac{(3n+1)}{2(n+1)}X^2 + \frac{2n^2}{(n+1)(3n+1)}X^{\frac{1}{n}+3} + \frac{19n^2+8n+1}{4(3n+1)(5n+1)} \tag{21}$$

3. The second approximation of velocity profile.

Equation (2) can be transformed into

$$m = \frac{m_0}{(1+\beta\theta)^n} \tag{22}$$

where

$$\beta = \beta_c \frac{Rq_w}{\lambda t_0} \tag{23}$$

Substituting equations (21) and (22) into equation (13) gives

$$\frac{u}{u_m} = \frac{3n+1}{n+1}\left[1 - A\frac{17n^2+8n+1}{30n^2+16n+2}\right]^{-1}$$
$$\times\left[(1-X^{\frac{1}{n}+1}) - \frac{A}{2}(1-X^{\frac{1}{n}+3}) + \frac{An^2}{(2n+1)(3n+1)}(1-X^{\frac{2}{n}+4})\right] \tag{24}$$

where

$$A = \left[\frac{1}{\beta} + \frac{19n^2+8n+1}{4(3n+1)(5n+1)} + \frac{2\Pi}{Gz}\right]^{-1} \tag{25}$$

4. Nusselt numbers.

Since it can be assumed that $\partial t/\partial z = \partial t_m/\partial z =$ constant for the constant heat flux in the thermally well developed region, Nusselt numbers can be calculated as

$$Nu_\infty = \left[2\int_0^1 X\frac{u}{u_m}dX\int_X^1 \frac{dX}{X}\int_0^X X\frac{u}{u_m}dX\right]^{-1} \tag{26}$$

By changing the order of integrals,

$$Nu_\infty = \left[2\int_0^1 \frac{dX}{X}\int_0^X X\frac{u}{u_m}dX\int_0^X X\frac{u}{u_m}dX\right]^{-1}$$
$$= \left[2\int_0^1 \frac{1}{X}\left\{\int_0^X X\frac{u}{u_m}dX\right\}^2 dX\right]^{-1} \tag{27}$$

Substituting equation (24) into equation (27), one obtains the approximate solution for Nusselt numbers as

$$Nu_\infty = 8\frac{15n^2+8n+1}{31n^2+12n+1}\left[1 - A\frac{17n^2+8n+1}{30n^2+16n+2}\right]^2 \times [1 - AS_1 + A^2S_2]^{-1} \tag{28}$$

where

$$S_1 = \frac{8114n^5+8751n^4+3720n^3+774n^2+78n+3}{3(3n+1)(4n+1)(7n+1)(31n^2+12n+1)} \tag{29}$$

$$S_2 = \frac{583663n^8+1047052n^7+819364n^6+364784n^5+100850n^4+17680n^3+1912n^3+116n+3}{4(3n+1)^2(5n+1)^2(7n+1)(11n+3)(31n^2+12n+1)} \tag{30}$$

Bird[2] and Grigull[3] have given the solution for nearly isothemral flow as

$$Nu_\infty = 8\frac{15n^2+8n+1}{31n^2+12n+1} \tag{31}$$

Comparing equation (28) with equation (31), it is noted that the term of $\left[1-A\frac{17n^2+8n+1}{30n^3+16n+2}\right]^2 \times [1-AS_1+A^2S_2]^{-1}$ is the correction term for temperature dependence of m.

Substituting equations (21) and (22) into equation (14), one obtains

$$\frac{u_{max}}{u_m} = \frac{3n+1}{n+1}\,\frac{1-A\dfrac{(n+1)(4n+1)}{2(2n+1)(3n+1)}}{1-A\dfrac{17n^2+8n+1}{2(3n+1)(5n+1)}} \tag{32}$$

Comparing equation (32) with equation (16), it is recognized that the term

$$\left[1-A\frac{(n+1)(4n+1)}{2(2n+1)(3n+1)}\right] \Big/ \left[1-A\frac{17n^2+8n+1}{2(3n+1)(5n+1)}\right]$$

is the correction term for temperature dependence of m.

In Fig. 2 the values of Nu_∞ computed with equation (28) are plotted against

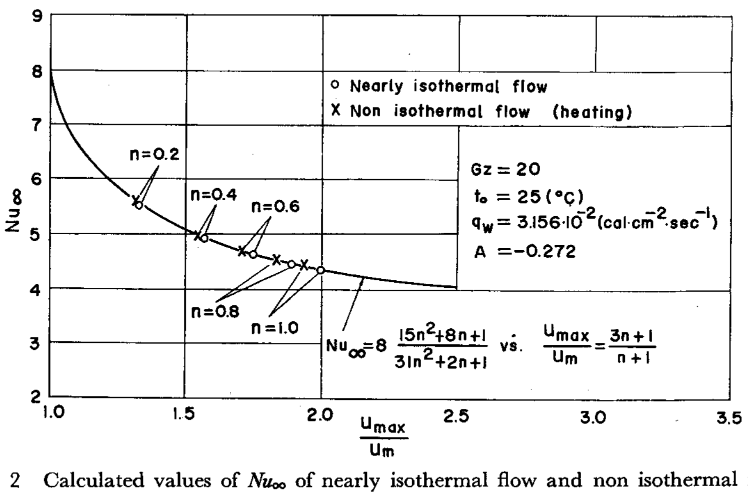

Fig. 2 Calculated values of Nu_∞ of nearly isothermal flow and non isothermal flow.

those of u_{max}/u_m computed with equation (32) and also those of Nu_∞ computed with equation (31) are plotted against those of u_{max}/u_m computed with equation (16). It is recognized that both plots can be correlated with a single curve.

Entrance Region

At the entrance region, the heat penetrates only the thin layer near the wall. Bird[2] gave the solution of this case as

$$Nu_e = 0.650\left(\frac{pD_3}{\alpha z}\right)^{1/3} \tag{33}$$

where

$$p = -\frac{du}{dr}\bigg|_{r=R} \tag{34}$$

and for nearly isothermal flow

$$p = \frac{3n+1}{4n}\frac{8u_m}{D} \tag{35}$$

Consequently

$$Nu_e = 1.41\left(\frac{3n+1}{4n}\right)^{1/3} Gz^{1/3} \tag{36}$$

In order to take into account the temperature dependence of viscosity, the authors made the following assumptions (Refer to Fig. 3)

(1) The heat penetrates the thin layer of thickness y_c, namely until the plane of R_c.

(2) The temperature gradient in that layer is linear.

(3) The temperature of the core flow is t_0.

(4) Because the thickness y_c is very small, the average temperature of the flow is also t_0.

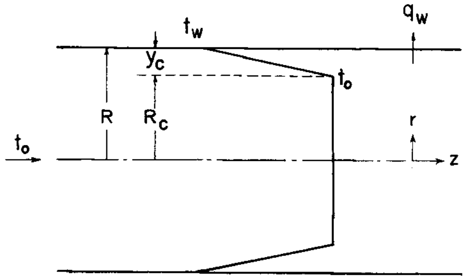

Fig. 3 Approximate temperature profile in themral entrance region.

Thus

$$q_w = \lambda \frac{t_0 - t_w}{y_c} = h(t_0 - t_w) \tag{37}$$

$$t_0 - t_w = \frac{q_w D}{\lambda} \Big/ Nu_e \tag{38}$$

and

$$y_c = R - R_c = D/Nu_e \tag{39}$$

The temperature phofile is,

for $R > R_c$

$$t = (t_w - t_0)\frac{r - R_c}{R - R_c} + t_0$$

Substituting equations (38) and (39) into this equation gives

$$t = -\left(\frac{r}{R} + \frac{2}{Nu_e} - 1\right)\frac{q_w R}{\lambda} + t_0$$

Accordingly

for $\quad 1 \geqq X > 1 - \dfrac{2}{Nu_e}$

$$\theta = -\left(X + \frac{2}{Nu_e} - 1\right) \tag{40}$$

And also one obtains

for $\quad 1 - \dfrac{2}{Nu_e} \geqq X \geqq 0$

$$\theta = 0 \tag{41}$$

Substituting equations (40), (41) and (22) into equation (13) gives

$$\frac{u_m}{u} = \frac{(3n+1)(4n+1)}{(n+1)(2n+1)} \times$$

$$\frac{(2n+1)\left\{1 + \beta\left(1 - \dfrac{2}{Nu_e}\right)\right\}(1 - X^{\frac{1}{n}+1}) - (n+1)\beta(1 - X^{\frac{1}{n}+2})}{(4n+1)\left\{1 + \beta\left(1 - \dfrac{2}{Nu_e}\right)\right\} - \beta(3n+1) - n\beta\left(1 - \dfrac{2}{Nu_e}\right)^{\frac{1}{n}+4}} \tag{42}$$

Consequently

$$p = -\frac{du}{dr}\Big|_{r=R} = -\frac{du}{RdX}\Big|_{X=1}$$

$$= \frac{3n+1}{4n}\cdot\frac{8u_m}{D}\left[1-\frac{1-\left(1-\dfrac{2}{Nu_e}\right)^{\frac{1}{n}+4}}{\left(\dfrac{1}{n}+4\right)\left(\dfrac{2}{Nu_e}-\dfrac{1}{\beta}\right)}\right]^{-1} \tag{43}$$

In equation (43) the bracketed term is a correction term. The values of this term can be calculated from the first approximation of Nu_e obtained from equation (36).

Equations of Non Dimensional Moduli

1. Thermally well developed region.

The analytical solution for nearly siothermal flow, equation (31) can be approximated

$$Nu_\infty = 4.36\left(\frac{3n+1}{4n}\right)^{1/3} \tag{44}$$

for the range of $n=0.2\sim3$, within the error of ±1 percent.

On the other hand, as shown in Fig. 4 the correction term for temperature dependence of m may be approximated with $(m/m_w)^{0.14/n^{0.7}}$. Though the values of A changes considerably with the value of Gz, the correction terms for $Gz=10$ (shown in Fig. 4) and 20 were shown to be approximated with the same modulus.

Consequently, the Nusselt numbers for constant heat flux and thermally well developed region are represented with

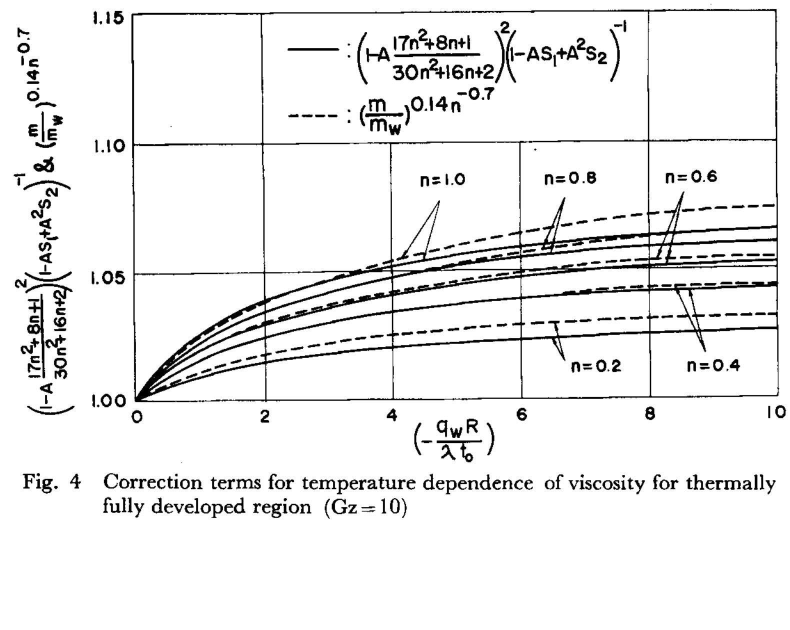

Fig. 4 Correction terms for temperature dependence of viscosity for thermally fully developed region (Gz = 10)

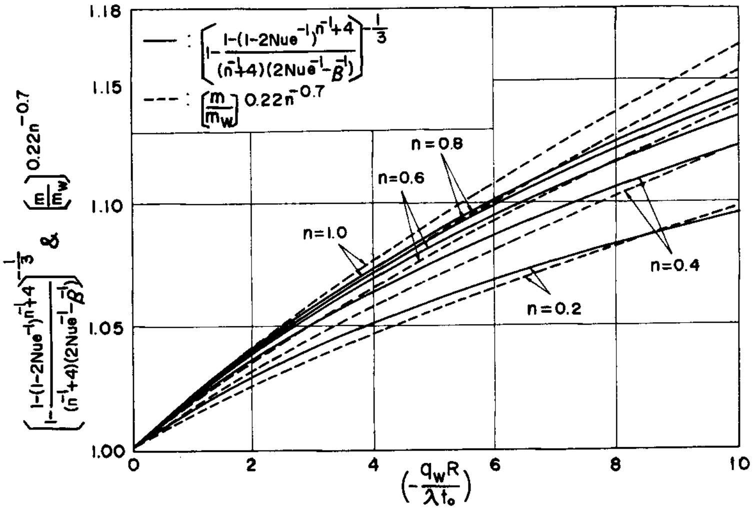

Fig. 5 Correction terms for temperature dependence of viscosity for thermally entrance region ($Nu_e = 10$)

$$Nu_\infty = 4.36\left(\frac{3n+1}{4n}\right)^{1/3}\left(\frac{m}{m_w}\right)^{0.14/n^{0.7}} \tag{45}$$

2. Entrance region

Substituting equation (43) into equation (33), one obtains

$$Nu_e = 1.41\left(\frac{3n+1}{4n}\right)^{1/3} Gz^{1/3}\left[1-\frac{1-\left(1-\frac{2}{Nu_e}\right)^{\frac{1}{n}+4}}{\left(\frac{1}{n}+4\right)\left(\frac{2}{Nu_e}-\frac{1}{\beta}\right)}\right]^{-1/3} \tag{46}$$

For $Nu_e = 10$, the correction term may be approximated with $\left(\frac{m}{m_w}\right)^{0.22/n^{0.7}}$ as shown in Fig. 5. The same procedure gives $\left(\frac{m}{m_w}\right)^{0.17/n^{0.7}}$ for $Nu_e = 6.54$ and $\left(\frac{m}{m_w}\right)^{0.14/n^{0.7}}$ for $Nu_e = 5$.

Consequently, the Nusselt numbers for constant heat flux and entrance region can be represented by

$$Nu_e = 1.41\left(\frac{n+1}{4n}\right)^{1/3} Gz^{1/3}\left(\frac{m}{m_w}\right)^{c/n^{0.7}} \tag{47}$$

where $c = 0.045 Nu_e^{0.7}$ (48)

Substituting Nu_e obtained from equation (36) into equation (48), the values of c is computed.

Experimental Results

The test section is consisted of a brass tube of 17.3 mm O.D., 12.7 mm I.D. and 4000 mm long, wound with electrically heating nichrome wires of 0.8 mm O.D., and preceded by a fore-flow section of 1000 mm long. The heating section is covered with a thermal insulator, on which heating wire is wound again as a compensator.

The wall temperature and the temperature prfile of the flow were measured with copper- constantan thermocouples near the exit of the test section. The average temperature of the fluid was computed from its inlet temperature and the amount of Watt applied to the heater. The heat flux was computed from the heat added and the heat transfer surface area. Thus, the heat transfer coefficients at the measuring point were calculated.

The values of the rheological properties, m and n were changed by varying concentration of C.M.C. of the solution, and they were measured with a Shimazu Universal Rheometer UR-IM (coaxial cylinders type). Those properties changed with time by the effect of heating so much that it was necessary to measure them

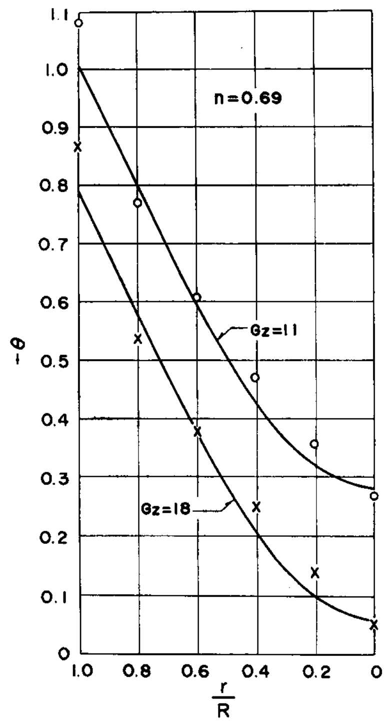

Fig. 6 Measurement and first appromimation profiles of temperature.

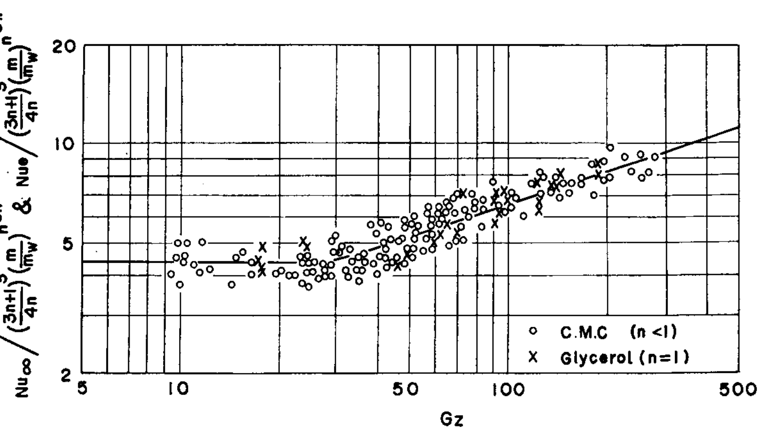

Fig. 7 Experimental results of Nusselt number.

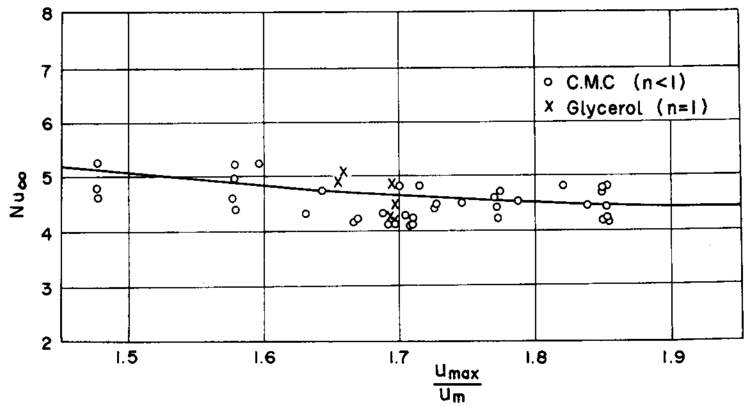

Fig. 8 Experimental values of Nu_∞ *vs.* u_{max}/u_m computed for non isothermal flow.

during every run. The range of the values of n of the solution used was 1.00~0.33.

In Fig. 6 the measured temperature profiles and equation (21) of the first approximation of them are compared. It is recognized that both of them are in fairly good agreement with each other.

The experimental results of Nusselt number are plotted against Graetz number in Fig. 7. Equations (45) and (47) correlate the data well. It should be noted that $Gz=30$ is the critical point between the entrance and thermally fully developed region.

In Fig. 8, the experimental results of Nusselt number in the thermally fully developed region *vs.* u_{max}/u_m computed from equation (32) are plotted. The theoretical curve as same as that in Fig. 2 correlates the data fairly well.

In Fig. 7 and 8, the data of aqueous solutions of gylcerol are included.

Conclusion

This paper dealt with the heat transfer to laminar flow of pseudoplastic fluids in a circular tube at the condition of constant wall heat flux.

The temperature dependence of viscosity was taken into account in solving the equations of change to obtain the Nusselt number at the entrance and thermaly fully developed region.

Those analytical solutions were approximated with the equations of non-dimensional moduli, which were shown to be in good agreement with the experimental results obtained by the authors.

Nomenclature

Nomenclature

A	:	Equation (25)	[—]
c	:	Equation (48)	[—]
D	:	Diameter of tube	[cm]
Gz	:	Graetz number $= \dfrac{\frac{\Pi}{4} D^2 u_m}{\alpha_z}$	[—]
h	:	Heat transfer coefficient	[cal/cm²sec·deg·C]
m	:	Constant in power law model	[gr·sec^{n-2}/cm]
m_0, m_w	:	m at t_0 and t_w respectively	[gr·sec^{n-2}/cm]
u	:	Power in power law model	[—]
Nu_e	:	Nusselt number at entrance region	[—]
Nu_∞	:	Nusselt number at fully developed region	[—]
P	:	Pressure	[gr/sec^2·cm]
p	:	Velocity gradient at wall	[1/sec]
q_w	:	Heat flux at wall	[cal/cm^{2}sec]
R	:	Radius of tube	[cm]
R_c	:	Fig. 3	[cm]
r	:	Radius	[cm]
S_1, S_2	:	Equation (29) and (30) respectively	[—]
t	:	Temperature	[°C]
t_0	:	Inlet temperature of fluid	[°C]
t_w	:	Temperature of wall	[°C]

u	:	Velocity of fluid	[cm/sec]
u_m	:	Average velocity in tube	[cm/sec]
u_{max}	:	Velocity at center	[cm/sec]
X	:	Non-dimensional radius, equation (7)	[—]
y_c	:	Fig. 3	[cm]
z	:	Distance along axis of tube	[cm]
Z	:	Non-dimensional distance along axis of tube, equation (18)	[—]
α	:	Temperature conductivity of fluid	[cm²/sec]
β	:	Coefficient of temperature dependence of m, equations (22) and (23)	[—]
β_c	:	Coefficient of temperature dependence of m, equation (2)	[—]
θ	:	Non-dimensional temperature	[—]
λ	:	Thermal conductivity of fluid	[cal/cm·sec·deg·°C]
τ	:	Shear stress	[gr/cm·sec²]

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