A Study on Plasma Diffusion in a Uniform Magnetic Field

By

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Through gas plasmas of very low charge densities, electrons and ions diffuse freely because the space charge is negligible (free diffusion limit). On the contrary at high charge densities, the space-charge induced field saturates and causes the ambipolar diffusion which is a combination of diffusive and saturated mobility flow (ambipolar limit).

The transition from the ambipolar to the free diffusion is studied for the case of magnetized slab plasmas maintained in between two plane-walls through ionization by electron collision. The basic equation is nonilnear because of the mobility flow due to the space-charge field.

A simple analytic approximation is applied to the distribution of charged particles. The amplitude of charged particle distribution and the plasma thickness are determined by an iterative numerical procedure.

For typical values of the characteristic plasma parameters for magnetized hydrogen plasmas, variation of the charged particle density through the transition between both limits is investigated. Spatial distributions of electrons and ions inside the plasma slab are calculated.

1. Introduction

Plasma diffusion in the positive column with a longitudinal magnetic field is mostly treated by the modified Schottky theory with the assumption of ambipolar diffusion[1,2]. The ambipolar diffusion is based on the quasi-neutrality of plasma. Therefore, this treatment fails if deviation from the neutrality is significant. The deviation from plasma neutrality depends on the ratio of the Debye length to the overall dimension of the plasma in the directions of diffusion. This ratio increases with the applied magnetic field because of its effect to decrease plasma dimension.

As for the case where no magnetic field is applied, it has been shown by Allis and Rose[3] that the plasma diffusion becomes ambipolar in the limit of high charge density (the ambipolar limit), while it becomes free in the limit of low charge density (the free diffusion limit), and that the transition from free to ambipolar diffu-

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sion occurs in the region of intermediate charge densities between the two limits. In this case, however, the transition is expected to appear only at a high gas pressure with a very low plasma density. Since such an extremely weak ionization is difficult to maintain in a high pressure gas, the transition to free diffusion is hardly realizable.

When a uniform, longitudinal magnetic field is applied, the plasma dimension in the direction perpendicular to the applied field and parallel to the density gradient of the charged particles is determined by the ambipolar diffusion length in the ambipolar limit and by the electron diffusion length in the free diffusion limit. Since this dimension decreases as the magnetic field grows, the transitional or even the free diffusion can be attained at a medium charge density (up to $10^8/cm^3$) with a relatively low gas pressure, i.e. at a weak but realizable degree of ionization.

Investigation of the diffusion of charged particles through a magnetized positive column in its transitional state is of great interest because the diffusion is strongly affected by the electric field due to the space charge. The distribution of space charge is caused by the difference in the diffusion of electrons and ions, and the electric field due to the space charge brings about the mobility flows of charged particles. Accordingly, the diffusion in the transitional state is essentially non-linear. Because of this nonlinearity, we are forced either to an expensive machine solution of the diffusion equations or to simplified analytic approximations.

In Chapter 2, one dimensional diffusion equations for the transitional state in a magnetized plasma are derived. In Chapter 3, simple two-term analytic approximations for the density distributions of electrons and ions are presented. Distributions of charged particles are calculated in Chapter 4, for a variety of the Debye-length to diffusion-length and the ion-temperature to electron-temperature ratios.

2. Basic Equations for Magnetized Plasma Slab

2-1. Basic assumptions

We assume that an infinite slab of weakly ionized stationary plasma, consisting of electrons, one species of single-charged ions and neutral particles, is self-sustained in the space between two parallel infinite plane walls, where a uniform magnetic field is applied in parallel with the walls. In addition, the following conditions are assumed.

(1) Because of the applied magnetic field, the macroscopic diffusion velocity perpendicular to the field is small throughout the major part of the plasma slab, hence the inertial effect can be neglected.

(2) Neutral particles are ionized by collision with electrons. Consequently, the production rate of charged particles is proportional to the electron density. The charged particles recombine completely on their arrival at either of the walls.

(3) Since the ionization degree is low, collisions among charged particles is negligible as compared with that of charged particles with neutrals. The density of neutral particles is spatially constant.

(4) Both the electron and the ion temperature are constant throughout the plasma slab. In the gas discharge, charged particles gain energy from the applied electric field and lose the energy through collisions with neutral particles, therefore, the temperatures are determined by the condition of the energy balance.

(5) The density of the plasma is so low, that the magnetic field induced by the motion of charged particles is negligible.

(6) In a stationary state, all of the macroscopic variables depend only upon the coordinate X in the direction perpendicular to the walls. The origin is taken in the middle plane of the plasma slab between the walls.

2-2. Basic Equations

The component current densities by ions and electrons are equal everywhere, because ions and electrons are produced at the same rate. Under the above assumptions, the density Γ of number current of ions or electrons is given by

$$\Gamma = -D_{\pm}\frac{dN_{\pm}}{dX} \pm \mu_{\pm} E N_{\pm} \tag{2.1}$$

where $N_{\pm}$ is the particle density ($+$ for ions, $-$ for electrons), E is the electric field due to the space charge. $D_{\pm}$ and $\mu_{\pm}$ are the diffusion coefficient and the mobility respectively:

$$\mu_{\pm} = \frac{\nu_{\pm}}{\nu_{\pm}^2 + \omega_{\pm}^2} \cdot \frac{e}{m_{\pm}}, \quad D_{\pm} = \frac{T_{\pm}}{e}\mu_{\pm} \tag{2.2}$$

where $\nu_{\pm}$ is the frequency of collisions with neutrals, $\omega_{\pm}$ is the cyclotron frequency, and $T_{\pm}$ is the temperature. The equation of charge conservation is

$$\frac{d\Gamma}{dX} = \nu_i N_- \tag{2.3}$$

where ν_i is the frequency of ionization by electrons, hence $\nu_i N_-$ is the rate of production of charged particles. The electric field E is determined by Poisson's equation

$$\frac{dE}{dX} = 4\pi e(N_+ - N_-) \tag{2.4}$$

The four unknown variables $N_{\pm}$, Γ and E are to be determined by solving a set of four equations (2.1), (2.3) and (2.4) for given boundary conditions. The plasma slab is symmetric about the middle plane, hence the boundary conditions at the position of the middle plane

$$\Gamma = E = 0 \quad \text{at} \quad X = 0 \tag{2.5}$$

are required. The other boundary conditions are given on the walls. If the cyclotron radii of charged particles are small enough as compared with the width of the slab, it can be assumed that the density of charged particles on the wall is nearly equal to zero, that is

$$N_{\pm} = 0 \quad \text{at} \quad X = X_w \tag{2.6}$$

For the purpose of minimizing the number of parameters appearing in the plasma equations, we introduce the following dimensionless variables and parameters:

$$x = \frac{X}{L}, \quad n_{\pm} = \frac{N_{\pm}}{N_0}, \quad \varepsilon = \frac{LeE}{T_-}, \quad \gamma = \frac{\Gamma}{\nu_i L N_0}$$

$$\tau = \frac{T_+}{T_-}, \quad \sigma = \frac{\mu_-}{\mu_+}, \quad \alpha = \frac{\lambda_D}{L} \tag{2.7}$$

where, $L=(D_-/\nu_i)^{1/2}$ is the diffusion length of electrons, N_0 and λ_D are the electron density and the electron Debye length at $X=0$, respectively. Using these notations, the plasma equations (2.1), (2.3), (2.4) are rewritten as

$$\frac{dn_-}{dx} + \varepsilon n_- + \gamma = 0 \tag{2.8a}$$

$$\tau \frac{dn_+}{dx} - \varepsilon n_+ + \sigma\gamma = 0 \tag{2.8b}$$

$$\frac{d\gamma}{dx} = n_- \tag{2.8c}$$

$$\alpha^2 \frac{d\varepsilon}{dx} = n_+ - n_- \tag{2.8d}$$

The boundary conditions (2.5), (2.6) are converted into the following forms:

$$n_- = 1, \quad \gamma = \varepsilon = 0 \quad \text{at} \quad x = 0 \tag{2.9a}$$

and

$$n_+ = n_- = 0 \quad \text{at} \quad x = d \tag{2.9b}$$

where $d = X_w/L$ is half the width of the plasma divided by the electron diffusion length.

2-3. Ranges of the Characteristic Parameters

Three non-dimensional parameters τ, σ and α are involved in the equations

(2.8b) and (2.8d). In the case of low pressure discharge across a magnetic field, the parameter τ generally has a value of the order of magnitude $10^{-2}\sim 10^{-1}$. On the other hand, the value of σ varies with the magnitude of the ratio $\omega_\pm/\nu_\pm$. From (2.2) and (2.7) the following two limiting values are obtained.

$$\begin{aligned} \sigma &= \frac{\nu_+ m_+}{\nu_- m_-} \quad \text{for} \quad \omega_\pm \ll \nu_\pm \\ \sigma &= \frac{\nu_- m_-}{\nu_+ m_+} \quad \text{for} \quad \omega_\pm \gg \nu_\pm \end{aligned} \tag{2.10}$$

In general, σ takes a value between the above two limits. Provided that the collision cross sections for electrons and ions are inversely proportional to the velocity v of the incident charged particle, the collision frequencies $\nu_\pm$ are independent of the temperature $T_\pm$ of the charged particles and are proportional to the density of the neutral particles. Then the ratio ν_+/ν_-, accordingly the above limiting values of σ, depends only upon the species of ions.

By the equations (2.2) and (2.7) we have the relation

$$D_+/D_- = \tau/\sigma \tag{2.11}$$

Accordingly, if $\tau > \sigma$, ions diffuse faster than electrons, hence the induced electric field is directed inward. If $\tau < \sigma$, electrons diffuse out faster than ions. Consequently, the electric field induced by the space charge is directed outward. If τ is just equal to σ, the plasma is neutral everywhere and no electric field is induced.

From (2.7), we obtain the following expression for the coefficient α.

$$\alpha = [\nu_i(\nu_-^2 + \omega_-^2)/(\nu_- \omega_{p-}^2)]^{1/2} \tag{2.12}$$

where ω_{p-} is the electron plasma frequency at the origin ($x=0$) and is proportional to $n_0^{1/2}$. The ionization frequency ν_i is proportional to the density of neutral particles, and is sensitive to the electron temperature. Thus, by the relation (2.12), α depends upon the electron temperature, the electron density, the neutral density, and the magnetic field intensity. In the limit where $\alpha \ll 1$, the plasma is nearly quasi-neutral, and the diffusion is almost ambipolar. In another limit where $\alpha \gg 1$, the ratio n_+/n_- is, in the vicinity of the origin, nearly equal to its asymptotic value σ/τ. The diffusion is almost free in this case.

2–4. Hydrogen Plasma

As an example we will evaluate the characteristic parameters for the hydrogen plasma. From the data book[4] $\nu_-/\nu_+ \cong 20$. Since $m_+/m_- \cong 2000$, (2.10) gives $\sigma \cong 10^{-2}$ to 10^{+2} for $B=0$ to ∞. The values of α for the hydrogen plasma are shown in Fig. 1 for some typical values of the plasma parameters. From the later analysis,

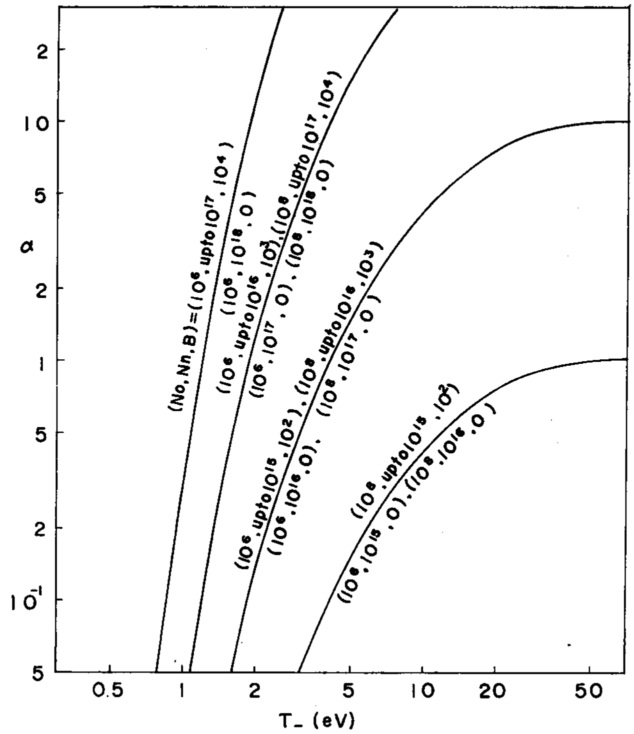

Fig. 1 α for the hydrogen plasma versus electron temperature T_-(eV). The first figure in the branket is central electron density $N_0(\mathrm{cm}^{-3})$, the second is neutral density $N_n(\mathrm{cm}^{-3})$, and the last represents magnetic field intensity B (Gauss).

one may see that the transition from the ambipolar diffusion takes place when the parameter α is the order of unity. Therefore, Fig. 1 shows that, when the applied magnetic field does not exist, the transition can appear only in a high pressure plasma of extremely low degree of ionization below 10^{-8}, while in a sufficiently strong magnetic field it may easily arise in a plasma of low neutral density. The transition may arise for the plasma of electron density higher than $10^8\,\mathrm{cm}^{-3}$. However, since the Debye length of such a plasma is less than 1 mm (for $T_- < 10\mathrm{eV}$), the width of the plasma slab in the transition region becomes too small to be of practical interest. In such a case, more exact investigations in which the inertial effect of the charged particles is taken into account and more realistic boundary conditions on the wall are required.

3. Approximate Solution

We are interested in the solutions which satisfy the following boundary conditions:

(1) The electron density n_- and the ion density n_+ are even functions of x, because of the symmetry about $x=0$.

(2) Both of them are zero on the walls at $x=\pm d$. On account of the condition (1), the current density γ and the electric field ε are odd functions of x and are zero at the origin $x=0$.

Now, as the approximate solutions satisfying the above boundary conditions, we assume the simplest analytic functions

$$n_- = (a_1 + a_2 \sin^2\theta)\cos\theta \tag{3.1a}$$

$$n_+ = (b_1 + b_2 \sin^2\theta)\cos\theta \tag{3.1b}$$

where

$$\theta = cx, \quad c = \pi/(2d) \tag{3.2}$$

From the condition (2.9a) it follows that $a_1 = n_-(x=0) = 1$. Using the above expressions in the basic equations (2.8c) and (2.8d), the following relations are obtained.

$$\gamma = \frac{1}{c}\left(a_1 \sin\theta + \frac{a^2}{3}\sin^3\theta\right) \tag{3.1c}$$

$$\varepsilon = \frac{1}{\alpha^2 c}\left[(b_1 - a_1)\sin\theta + \frac{b_2 - a_2}{3}\sin^3\theta\right] \tag{3.1d}$$

Substituting above four expressions (3.1a) thorugh (3.1d) into the basic equations (2.8a) and (2.8b), putting the coefficients of $\sin\theta$ and $\sin^3\theta$ equal to zero individually, and using the approximation

$$\cos\theta \cong 1 - \frac{1}{2}\sin^2\theta - \cdots\cdots \tag{3.3}$$

we obtain the four equations

$$1 - c^2 + 2c^2 a_2 + \frac{1}{\alpha^2}(b_1 - 1) = 0 \tag{3.4a}$$

$$(1 - 9c^2)a_2 + \frac{3}{\alpha^2}\left\{a_2 b_1 - \frac{4}{3}a_2 + \frac{b_2}{3} - \frac{b_1 - 1}{2}\right) = 0 \tag{3.4b}$$

$$\tau c^2(2b_2 - b_1) - \frac{1}{\alpha^2}b_1(b_1 - 1) + \sigma = 0 \tag{3.4c}$$

$$-9\tau c^2 b_2 - \frac{3}{\alpha^2}\left\{\frac{4}{3}b_1 b_2 - b_2 - \frac{1}{3}a_2 b_1 - \frac{1}{2}b_1(b_1 - 1)\right\} + \sigma a_2 = 0 \tag{3.4d}$$

It is difficult to solve the above set of equations algebraically and express a_2, b_1, b_2, and c^2 in terms of the characteristic parameters τ, σ, and α^2, with the exception of some special cases.

An example of the special cases is the case where τ is equal to σ. The condition

$\tau = \sigma$ is equivalent to the physical condition that electrons and ions have the same diffusion coefficient, i.e. $D_+ = D_-$. In this case the solution is

$$b_1 = 1, \quad a_2 = b_2 = 0, \quad c = 1$$

hence

$$n_- = n_+ = \cos x, \quad \gamma = \sin x, \quad \varepsilon = 0 \tag{3.5}$$

(3.5) is an exact solution to the equations (2.8). The plasma is neutral everywhere irrespective of the value of α^2. The distribution (3.5) is not altered throughout the whole range from the ambipolar to the free diffusion limit.

It can be seen directly from the basic equations (2.8) that the exact solution for the ambipolar limit where $\alpha^2 = 0$ is given by

$$n_- = n_+ = \cos \kappa x, \quad \kappa = [(1+\sigma)/(1+\tau)]^{1/2} \tag{3.6}$$

In the free diffusion limit where $\alpha^2 = \infty$, the solution is exactly

$$n_- = \cos x, \quad n_+ = \frac{\sigma}{\tau} \cos x. \tag{3.7}$$

Except those special cases, the solution to (3.4) has to be found by iterative numerical procedures. From (3.4) we obtain

$$c^2 = (1/\alpha^2)(\alpha^2 - 1 + b_1)/(1 - 2a_2) \tag{3.8a}$$

Substituting this into (3.4c), the following relation is obtained.

$$b_1^2 - pb_1 - q = 0; \tag{3.8b}$$

$$p = [1 + \tau(1-\alpha^2) - 2a_2 + 2\tau b_1]/(1 + \tau - 2a_2), \tag{3.8c}$$

$$q = [\alpha^2\sigma(1-2a_2) - 2\tau b_2(1-\alpha^2)]/(1+\tau-2a_2). \tag{3.8d}$$

Further, one obtains the following expression for b_2 by eliminating a_2 from (3.4b) and (3.4d).

$$b_2 = M\left[\sigma - \left(K - \frac{1}{\alpha^2}\right)b_1\right] \Big/ \left[\sigma + \frac{b_1}{\alpha^2} - K\left(9\tau c^2 + \frac{3+4b_1}{\alpha^2}\right)\right], \tag{3.8e}$$

$$M = \frac{3}{2}(b_1 - 1), \quad K = \alpha^2(9c^2 - 1) + 4 - 3b_1. \tag{3.8f}$$

From (3.4b) a_2 is expressed as follows:

$$a_2 = (b_2 - M)/K. \tag{3.8g}$$

A solution of the equations (3.4a) through (3.4d) can be found from the above relations by the following procedures.

(1) Initially, $a_2 = b_2 = 0$ is assumed.

(2) p and q are evaluated from (3.8c) and (3.8d), and then b_1 is determined by the relation

$$b_1 = \frac{1}{2}[p+\sqrt{(p^2+4q)}]. \tag{3.8b'}$$

(3) c^2 is evaluated from (3.8a).

(4) b_2 is evaluated from (3.8e).

(5) a_2 is evaluated from (3.8g).

The procedures (2) through (5) are repeated until they converge. If (3.8b) has no real roots, then it is advisable to restart the iteration by giving a new guess to b_1. The iteration procedure may be oscillatory especially for smaller values of b_1. By the authors' experience, under-relaxation with factor 1/2 is effective to suppress the oscillation and accelerate the convergence.

4. Transition from the Ambipolar to the Free Diffusion of Magnetized Hydrogen Plasma

By the method described in the previous chapter the amplitudes b_1, a_2, and b_2 of electron and ion distributions with the half thickness c of plasma slab were calculated for the various values of τ and α which are typical for magnetized hydrogen plasma. The results are shown in Table 1. The higher mode amplitudes a_2 and b_2 vanish both in the ambipolar limit and in the free diffusion limit. Thus there exists only the fundamental mode which is proportional to $\cos\theta$. The higher mode of the form $\sin^2\theta\cdot\cos\theta$ occurs in the transition region. The higher mode amplitude is relatively small for τ greater than σ, even midway in the transition.

The half thickness of the plasma slab, d, is $(\pi/2)((1+\tau)/(1+\sigma))^{1/2}$ in the ambipolar limit and is $\pi/2$ in the free diffusion limit. In the present sample calculations for magnetized hydrogen plasma where $\sigma=0.01$ and $\tau\leqq 1$, the half width d does not show large variation throughout the transition between both limits. In the case where no magnetic field is applied to plasma slab, the width is much larger in the free diffusion limit than in the ambipolar limit. This is due to the large value of the electron-mobility to ion-mobility ratio σ.

The amplitude b_1, the ion density in the middle of the plasma slab, varies monotonically from unity to $\sigma/\tau(=D_-/D_+)$ by transition from the ambipolar to the free diffusion limit, as clearly shown in Fig. 2. It can be concluded from the $b_1-\alpha^2$ curves of Fig. 2 that the transition from the ambipolar limit begins rather gradually by increasing α^2, while it ends rather abruptly. The turning point is shifted to larger values of α^2 as the ion- to electron-temperature ratio τ becomes smaller.

Typical density distribution of charged particles in the transition region are

Table 1. Amplitude a of electron distribution, b of ion distribution, and half width d of plasma slab for $\sigma=0.01$.

τ	α^2	a_1	a_2	b_1	b_2	d
—	0	1.000 0	0.000 0	1.000 0	0.000 0	*
	∞	1.000 0	0.000 0	σ/τ	0.000 0	1.570 8
0	1	1.000 0	−0.001 2	1.009 9	0.004 0	1.564 9
	10	1.000 0	0.000 7	1.091 6	0.203 9	1.563 1
0.001	1	1.000 0	−0.001 3	1.009 0	0.001 7	1.565 8
	10	1.000 0	0.000 9	1.082 6	0.180 9	1.562 9
	50	1.000 0	0.021 4	1.822 5	10.319	1.524 5
	100	1.000 0	0.243 1	15.063	482.5	1.043 7
0.01	—	1.000 0	0.000 0	1.000 0	0.000 0	1.570 8
0.1	1	1.000 0	0.013 2	0.914 2	−0.014 1	1.621 7
	3	1.000 0	0.015 8	0.667 7	−0.121 0	1.639 3
	5	1.000 0	0.303 7	0.144 3	−0.121 9	1.672 2
1	0.1	1.000 0	0.054 5	0.948 0	0.006 1	2.136
	0.3	1.000 0	0.091 6	0.836 8	0.001 8	2.102
	1.0	1.000 0	0.130 7	0.398 3	−0.032 4	2.139
	1.5	1.000 0	1.142 3	0.080 1	−0.014 7	2.134
	1.6	1.000 0	0.134 9	0.061 6	−0.009 9	2.081
	10.0	1.000 0	0.018 9	0.011 6	−0.000 2	1.623 2

$* = (\pi/2)[(1+\tau)/(1+\sigma)]^{1/2}$

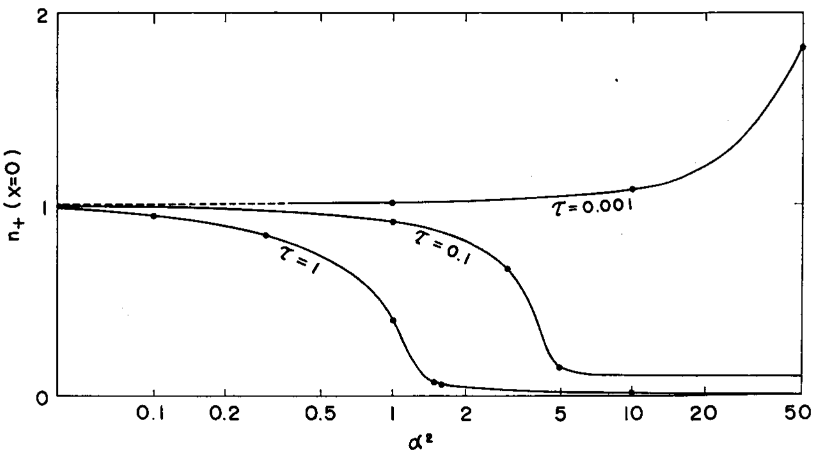

Fig. 2 Ion density in the middle of a plasma slab versus α^2. The dots on the curves are calculated values,

shown in Fig. 3 (a), (b) and (c) with the distributions in both limits as references. In the transition region, the electron distribution shows a peculiar behavior, namely it does not change monotonically from the abmipolar to the free distribution but first moves in the opposite direction and then turns back to the free distribution. This peculiar behavior is also observed in the results of more accurate machine integration of the basic equations.

Typical examples of the space charge-induced electric field are shown in Fig. 4. It is well known that the distribution of the electric field in the ambipolar limit is of the form tan θ. The approximate expression for the electric field (3.1d) fails near

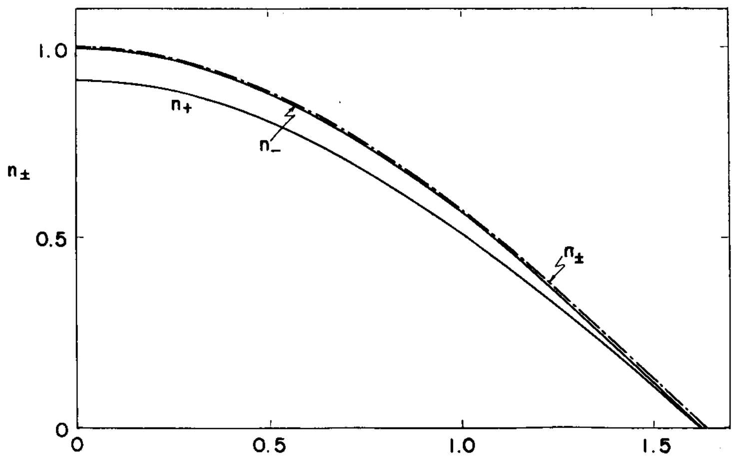

Fig. 3 (a)

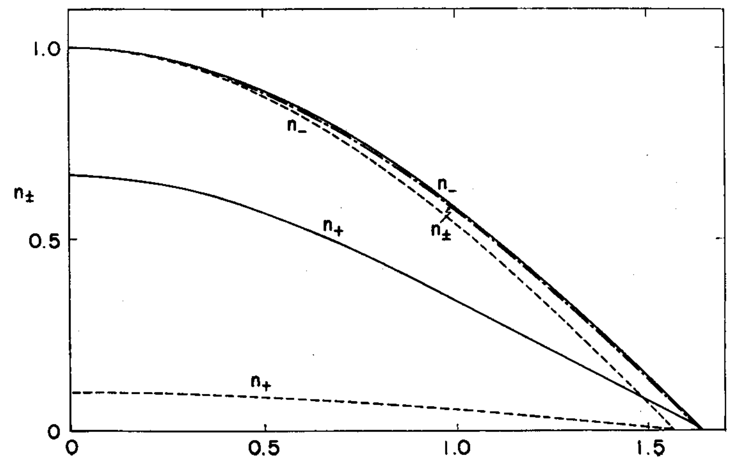

Fig. 3 (b)

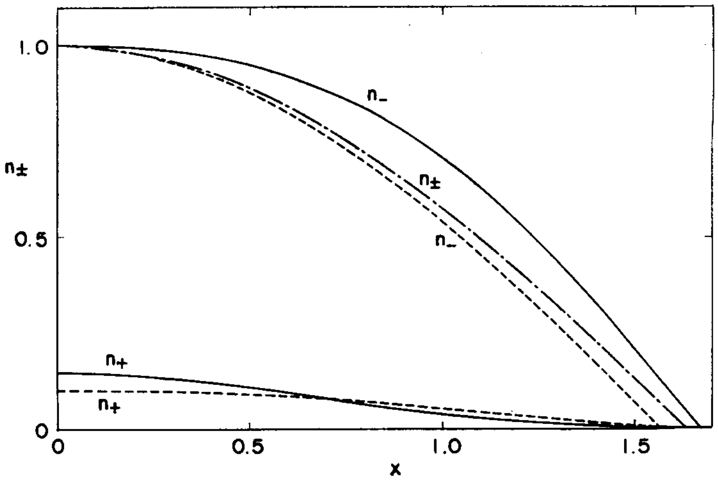

Fig. 3 (c)

Fig. 3 Distribution of electron density n_- and ion density n_+ in plasma slab for $\tau=0.1$; (a) for $\alpha^2=1$, (b) for $\alpha^2=3$, (c) for $\alpha^2=5$. The chain curves show the ambipolar limit for $\alpha^2=0$. The dashed curves represent the free diffusion limit for $\alpha^2=\infty$.

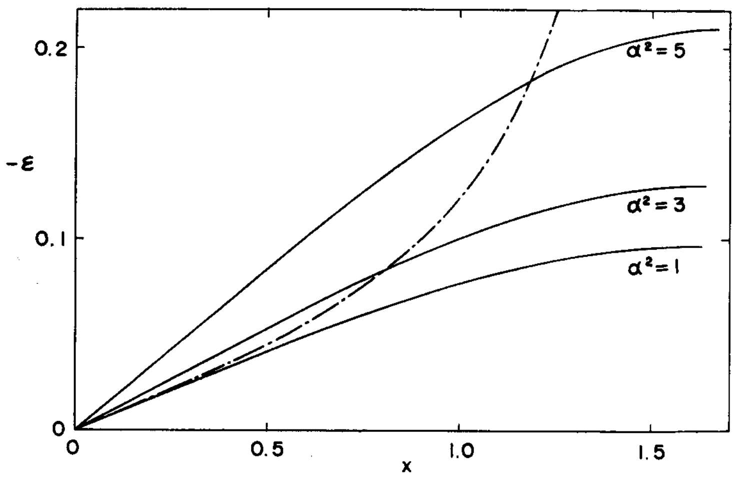

Fig. 4 Distribution of electric field due to space charge in plasma slab for $\tau=0.1$; $\alpha^2=1,3$ and 5. The chain curve shows the ambipolar limit at $\alpha^2=0$.

this limit. On the contrary, near the free diffusion limit, the distribution of the electric field can well be fit by the expression (3.1d). In the transition from the ambipolar limit, the electric field increases at first and then decreases to its free limit zero. Such a behavior has also been observed in the machine solution of the basic equation with the exception of the sheath region where the electric field monotonically decreases by transition from the ambipolar limit. Inside the sheath region in the vicinity of the wall surface, the expression (3.1) will fail to represent the detail of the charged particle distribution. Therefore, it is desirable to treat the sheath region separately as in the works on the probe theory in the continuum media[5,6].

If only the density of charged particles is decreased keeping the other parameters constant, that is, if α^2 is increased keeping τ and σ constant, then transition from the ambipolar to the free diffusion occurs with small variation of the plasma width. On the other hand, if the magnetic field is increased keeping the other parameters constant, α^2 is increased with a decrease of the plasma width. It has been shown by Lehnert's experiment[2] that, in the positive column of dc discharge, the electron temperature decreases, as the magnetic field increases, in a compensative way to keep the radius of positive column constant. In this case τ increases as the magnetic field increases, since the ion temperature keeps an approximately constant value nearly equal to the temperature of the gas. According to the relation $D_+/D_- = \tau/\sigma$, the variation of τ causes a variation in the ratio of diffusion coefficients D_+/D_-. Thus, if τ is initially less than σ, hence D_+ is less than D_-, then in the course of increasing the magnetic field D_+ becomes larger than D_- hence the direction of induced electric field, may be reversed.

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