On the Noninteracting Control of Linear Time-Variant Multivariable Systems

By

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In this paper, we show two results with respect to the noninteracting control problem in multivariable linear systems.

In the first half, we show that the noninteracting control problem is a variational one. Necessary and sufficient conditions for the linear (time-variant and time-invariant) multivariable control system to be the noninteracting control system are shown with the aid of variational method used in the theory of invariance.

In the second half, it is shown that it is possbile to obtain the noninteracting control system by state variable feedback. Sufficient conditions for the noninteracting control system to be constructed by state variable feedback are obtained by using the concept of relative orders.

1. Introduction

The control of a multivariable plant is of considerable practical importance. Particular interest has been taken in so-called noninteracting controls, designed so that each system output was independently controlled by one of the inputs of the system.

The noninteracting control of time-invariant linear systems has been discussed by many authors.[1~4] In this paper, we will discuss the noninteracting control of the linear time-variant multivariable system. Since the system is time-variant, usual diagonalization techniques which are used in time-invariant systems cannot be applied. Therefore, we will use a variational method to otain the condition for the system to be a noninteracting control system.

Several definitions and formulations of the problem are shown in § 2. In § 3, a necessary and sufficient condition for th linear time-variant multivariable control system to be a noninteracting control system will be given. In § 6, we

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will discuss the design problem of the noninteracting control system. Sufficient conditions for the noninteracting control system to be constructed by state variable feedback will be given by using the concept of relative orders. Illustrative example will be shown in § 7.

2. Definitions

Let the linear ordinary differential equation

$$\begin{cases} \dot{\boldsymbol{x}} = A(t)\boldsymbol{x}+B(t)\boldsymbol{v}, \quad \boldsymbol{x}_0 = \boldsymbol{x}(0), \\ \boldsymbol{y} = C'(t)\boldsymbol{x} \end{cases} \tag{2.1}$$

represent the characteristics of the control system where $\boldsymbol{x}$ is an n-th state vector, $\boldsymbol{v}$ is an r-th input vector and $\boldsymbol{y}$ is an r-th output vector. $A(t)$, $B(t)$ and $C(t)$ are $n\times n$, $n\times r$ and $n\times r$ matrices, respectively. The transposed matrix $C(t)$ is expressed by $C'(t)$. We assume that the elements of $A(t)$, $B(t)$ and $C(t)$ are analytic functions and $n\geq r$.

Definition 1

If, for any $\boldsymbol{x}_0\in X^0$, for any $t\in T$ and for any inputs $v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_r$ where they are piece-wise continuous functions of t, the value of the i-th output y_i taking (2.2) into account is independent of $v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_r$ and is determined only by $\boldsymbol{x}_0$, t and v_i, then it is said that the output y_i is (X^0, T)- independent. Here X^0 is a set of possible initial values for the system (2.1) and the set T of times t is an interval $[0, \tau]\subset[0, \infty)$.

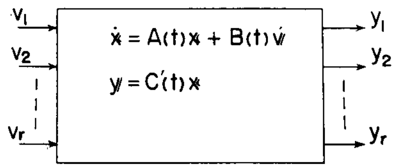

Fig. 1 The multivariable control system

Definition 2

If every output $y_i (i=1, \cdots r)$ is (X^0, T)- independent, then we say that the system is (X^0, T)- independent.

When the system is (X^0, T)- independent, each output y_i is never influenced by the inputs $v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_r$. However, the effect of the i-th input v_i which has influence on the i-th output y_i cannot be estimated from the notion of the (X^0, T)- independence. The noninteracting control means that the i-th output y_i should be independently controlled by the corresponding input v_i. Thus, we have the following defifinitions.

Definition 3

If the system (2.1) is (X^0, T)- independent and each output y_i is independently influenced by the corresponding input $v_i (i=1, \cdots, r)$, then we say that the system

(2.1) is the (X^0, T)- noninteracting control system.

Definition 4

If $X^0 \equiv R^n$ and $T \equiv [0, \infty)$, then it is said that the system (2.1) is the complete noninteracting control system. Here R^n is the whole space.

3. The (X^0, T)-Independence

Rozonoér has shown that the invariance problem is essentially a variational one. By using his method, the problem of the (X^0, T)- independence results in a similar variational problem.

Let us consider vectors $\bar{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}$ such that

$$\bar{\boldsymbol{v}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tilde{\boldsymbol{v}} = \begin{pmatrix} v_1 \\ \vdots \\ v_{i-1} \\ 0 \\ v_{i+1} \\ \vdots \\ v_r \end{pmatrix}, \quad \bar{\boldsymbol{v}} + \tilde{\boldsymbol{v}} = \boldsymbol{v}$$

and let the arbitrary increment of the vector function $\tilde{\boldsymbol{v}}(t)$ be $\Delta_i \boldsymbol{v}(t)$. Under this the functional $y_i(t)$ generally receives $\Delta_i y_i(t)$. Now, let the i-th column of $B(t)$ and $C(t)$ be $\boldsymbol{b}_i(t)$ and $\boldsymbol{c}_i(t) (i = 1, \cdots, r)$, respectively. Then, the following relation is obtained.[7,8] Here $(\boldsymbol{a}, \boldsymbol{b})$ is the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\begin{aligned} \Delta_i y_i(\tau) = (\boldsymbol{c}_i(\tau), \Delta_i \boldsymbol{x}(\tau)) = -\int_0^\tau [H(\boldsymbol{x}, \boldsymbol{p}, \bar{\boldsymbol{v}} + \tilde{\boldsymbol{v}} + \Delta_i \boldsymbol{v}) \\ -H(\boldsymbol{x}, \boldsymbol{p}, \bar{\boldsymbol{v}} + \tilde{\boldsymbol{v}})] dt = -\int_0^\tau \sum_{\substack{j=1 \\ j \neq i}}^{r} (\boldsymbol{p}(t), \boldsymbol{b}_j(t)) \Delta v_j(t) dt, \end{aligned} \tag{3.1}$$

where H is the Hamiltonian,

$$H = (\boldsymbol{p}, A(t)\boldsymbol{x} + B(t)\boldsymbol{v}) \tag{3.2}$$

and $\boldsymbol{p}(t)$ is a solution of the adjoint system:

$$\dot{\boldsymbol{p}} = -A'(t)\boldsymbol{p}, \ \boldsymbol{p}(\tau) = -\boldsymbol{c}_i(\tau). \tag{3.3}$$

The fulfillment of the (X^0, T)- independence's requirement always means that the increment of the functional $y_i(t)$ should disappear. Therefore, the criterion of the (X^0, T)- independence takes the form of the identity

$$\Delta_i y_i(t) \equiv 0, \text{ for all } t \in T, \tag{3.4}$$

which should be true for all $\bar{\boldsymbol{v}}(t)$, $\tilde{\boldsymbol{v}}(t)$ and $\Delta_i \boldsymbol{v}(t)$.

[*Lemma 1*]

A necessary and sufficient condition for that

$$\Delta_i y_i(t) = 0\,, \quad t = \tau \tag{3.5}$$

is that, for each j ($j=1, \cdots, r\,; j \neq i$), the n vectors

$$\boldsymbol{b}_j(t),\ Q\boldsymbol{b}_j(t),\ \cdots,\ Q^{n-1}\boldsymbol{b}_j(t) \tag{3.6}$$

are linearly dependent on T and

$$\begin{aligned}&(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_j(t)) = 0\,,\\ &\quad k = 0,\ 1,\ \cdots,\ n-1\\ &\quad j = 1,\ \cdots,\ r;\ j \neq i\end{aligned} \tag{3.7}$$

hold at $t=\tau$, where Q is an operator such that

$$Q = D - A(t)\,;\ D = I\left(\frac{d}{dt}\right),\ I\text{: unit matrix.} \tag{3.8}$$

[*Lemma 2*]

A necessary and sufficient condition for that

$$\Delta_i y_i(t) \equiv 0 \text{ on } T \tag{3.9}$$

is the satisfaction of the relations

$$\begin{aligned}&(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_j(t)) = 0, \text{ on } T\\ &\quad k = 0,\ 1,\ \cdots,\ n-1\\ &\quad j = 1,\ \cdots,\ r;\ j \neq i\end{aligned} \tag{3.10}$$

Proof of Lemma 1.) From (3.1), it follows immediately that (3.5) is satisfied if and only if the relations $(\boldsymbol{p}(t), \boldsymbol{b}_j(t)) = 0$ for all $t \in T$ ($j=1, \cdots, i-1, i+1, \cdots, r$). We calculate $(n-1)$ derivatives of the function

$$z_j(t) \equiv (\boldsymbol{p}(t),\ \boldsymbol{b}_j(t)). \tag{3.11}$$

For the first derivative, (3.3) tells us

$$\begin{aligned}\dot{z}_j(t) &= (\dot{\boldsymbol{p}},\ \boldsymbol{b}_j) + (\boldsymbol{p},\ \dot{\boldsymbol{b}}_j) = (\boldsymbol{p},\ -A(t)\boldsymbol{b}_j) + (\boldsymbol{p},\ \dot{\boldsymbol{b}}_j)\\ &= (\boldsymbol{p},\ (D - A(t))\boldsymbol{b}_j) = (\boldsymbol{p},\ Q\boldsymbol{b}_j).\end{aligned} \tag{3.12}$$

Differentiating $(n-1)$ times and each time performing such an operation, we obtain

$$\begin{aligned}&z_j^{(k)}(t) = (\boldsymbol{p}(t),\ Q^k\boldsymbol{b}_j(t)),\\ &\quad k = 0,\ \cdots,\ n-1.\end{aligned} \tag{3.13}$$

By virture of $z_j(t)\equiv 0$ on T, all derivatives of the function $z_j(t)$ vaniish for all $t\in T$. Therefore

$$z_j^{(k)}(t)=(\boldsymbol{p}(t),\ Q^k\boldsymbol{b}_j(t))\equiv 0 \ \text{ on } \ T. \tag{3.14}$$

Since $\boldsymbol{p}(t)$ is a solution of (3.3), $\boldsymbol{p}(t)\neq 0$ for all $t\in T$. Hence from (3.14), $\boldsymbol{b}_j(t)$, $Q\boldsymbol{b}_j(t)$, $\cdots$, $Q^{n-1}\boldsymbol{b}_j(t)$ are linearly dependent for all $t\in T$. In particular, at $t=\tau$, taking into account the boundary condition $\boldsymbol{p}(\tau)=-\boldsymbol{c}_i(\tau)$, we have

$$z_j^{(k)}(\tau)=-(\boldsymbol{c}_i(\tau),\ Q^k\boldsymbol{b}_j(t)|_{t=\tau})=0. \tag{3.15}$$
$$k=0,\ 1,\ \cdots,\ n-1$$
$$j=1,\ \cdots,\ r;\ j\neq i$$

The sufficiency will be shown in the following. The vectors $\boldsymbol{b}_j(t)$, $Q\boldsymbol{b}_j(t)$, $\cdots$, $Q^{n-1}\boldsymbol{b}_j(t)$ are linearly dependent for all $t\in T$. Thus there exsist scalar functions $\lambda_0(t)$, $\lambda_1(t)$, $\cdots$, $\lambda_{n-1}(t)$ which do not vanish simulteneously for any $t\in T$ such that

$$\sum_0^{n-1}\lambda_s(t)Q^s\boldsymbol{b}_j(t)=0,\ \text{for } t\in T. \tag{3.16}$$

By multiplying $\boldsymbol{p}(t)$ scalarly, we obtain

$$\sum_0^{n-1}\lambda_s(t)(\boldsymbol{p}(t),\ Q^s\boldsymbol{b}_j(t))=0\,,\ t\in T.$$

Hence,

$$\sum_0^{n-1}\lambda_s(t)z_j^{(s)}(t)\equiv 0\,. \tag{3.17}$$

That is, $z_j(t)$ satisfies the homogeneous equation (3.17). Therefore according to the boundary condition (3.15), we have

$$z_j(t)=(\boldsymbol{p}(t),\ \boldsymbol{b}_j(t))\equiv 0\,,\ \text{on } \ T.$$

Thus, the lemma is proved.

The proof of Lemma 2 can be done without difficulty from the proof of Lemma 1.

[*Lemma 3*]

If $(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_j(t))\equiv 0$ on T, then $(\tilde{Q}^k\boldsymbol{c}_j(t),\ \boldsymbol{b}_j(t))\equiv 0$, on T, holds. The converce is also true. Here $\tilde{Q}\equiv D+A'(t)$. (The proof of this lemma is shown in reference [7].)

From these lemmas, we have the following theorem.

[*Theorem 1*]

A necessary and sufficient condition fot the system (2.1) to be $(X^0,\ T)$-

independent is the satisfaction of the set of relations

$$(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_j(t)) \equiv 0\,,\ t\in T. \qquad (3.18)$$
$$i=0,\ \cdots,\ r$$
$$k=0,\ 1,\ \cdots,\ n-1$$
$$j=1,\ \cdots,\ i-1,\ i+1,\ \cdots,\ r$$

Proof.) The necessity is apparent from Lemma 2. Thus we only need to show the sufficiency.

Now let the functions $F_{ik}(t)$ be

$$F_{ik}(t) \equiv (\tilde{Q}^k\boldsymbol{c}_i(t),\ \boldsymbol{x}(t)),\ k=0,\ \cdots,\ n-1. \qquad (3.19)$$

According to Lemma 3 and (3. 18), $\boldsymbol{c}_i(t)$, $\tilde{Q}\boldsymbol{c}_i(t)$, $\cdots$, $\tilde{Q}^{n-1}\boldsymbol{c}_i(t)$ are lineraly dependent for all $t\in T$. For simplicity, we assume that the vectors $\boldsymbol{c}_i(t)$, $\tilde{Q}\boldsymbol{c}_i(t)$, $\cdots$, $\tilde{Q}^{s-1}\boldsymbol{c}_i(t)$ are linearly independent on T where s is an integer such that $1\leq s\leq n-1$. It follows that there exist scalar functions $\lambda_0(t)$, $\cdots$, $\lambda_s(t)$ which do not vanish simultaneously for any t such that

$$\lambda_s(t)\tilde{Q}^s\boldsymbol{c}_i(t) = \sum_{j=0}^{s-1}\lambda_j(t)\tilde{Q}^j\boldsymbol{c}_i(t)\,. \qquad (3.20)$$

By multiplying $\boldsymbol{x}(t)$ scalarly to (3.20), we have

$$\lambda_s(t)F_{is}(t) = \sum_{j=0}^{s-1}\lambda_j(t)F_{ij}(t)\,. \qquad (3.21)$$

Differentiating (3.19) with respect to (2.1) and using the relations (3.18) and (3.21), we have the following system of differential equations;

$$\begin{cases} \dot{F}_{i_0} = F_{i_1}+(\boldsymbol{c}_i(t),\ \boldsymbol{b}_i(t))v_i(t) \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{F}_{i,s-1} = F_{i,s-1}+(\tilde{Q}^{s-2}\boldsymbol{c}_i(t),\ \boldsymbol{b}_i(t))v_i(t) \\ \lambda_s(t)\dot{F}_{i,s-1} = \lambda_s(t)\displaystyle\sum_{j=0}^{s-1}\lambda_j(t)F_{ij}(t)+\lambda_s(t)(\tilde{Q}^{s-1}\boldsymbol{c}_i(t),\ \boldsymbol{b}_i(t))v_i(t)\,. \end{cases} \qquad (3.22)$$

The solutions of these simulteneous differential equations only depend on the initial conditions $F_{ij}(0)=F_{ij}(\boldsymbol{x}_0)$ and the i-th input $v_i(t)$. That is, the i-th ouptut $y_i(t)=F_{i0}(t)$ does not depend on $\tilde{\boldsymbol{v}}(t)$. Thus the theorem is proved.

4. The (X^0, T)-Noninteracting Control System

In this section, we will discuss the conditions for the system (2.1) to be a (X^0, T) -noninteracting control system.

[*Theorem 2*]

Let

$$M_i(t) \equiv [(\boldsymbol{c}_i(t),\ \boldsymbol{b}_i(t)),\ (\boldsymbol{c}_i(t),\ \tilde{Q}\boldsymbol{b}_i(t)),\ \cdots,\ (\boldsymbol{c}_i(t),\ Q^{n-1}\boldsymbol{b}_i(t))].$$
$$i = 1,\ \cdots,\ r$$

Then, a necessary and sufficient conditions for the system (2.1) to be the $(X^0,\ T)$-noninteracting control system are the satisfaction of relations

$$(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_j(t)) \equiv 0,\ t\in T, \qquad (4.1)$$
$$i = 1,\ \cdots,\ r$$
$$k = 0,\ \cdots,\ n-1$$
$$j = 1,\ \cdots,\ i-1,\ i+1,\ \cdots,\ r$$

and

$$\text{rank}\ M_i(t_i) = 1,\ \text{for some}\ t_i\in T. \qquad (4.2)$$
$$i = 1,\ \cdots,\ r$$

Proof.) Necessity. Suppose that the relations (4.2) do not hold. That is, we assume that

$$(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_i(t)) \equiv 0,\ \text{for all}\ t\in T. \qquad (4.3)$$
$$k = 0,\ \cdots,\ n-1$$

By Lemma 3, relations (4.3) are equivalent to the relations

$$(\tilde{Q}^k\boldsymbol{c}_i(t),\ \boldsymbol{b}_i(t)) \equiv 0,\ \text{for all}\ t\in T. \qquad (4.4)$$

Taking into account (4.1) and (4.4), (3.22) can be rewritten in the form:

$$\begin{cases} \dot{F}_{i_0} = F_{i_1} \\ \quad\vdots \\ \dot{F}_{i,s-2} = F_{i,s-1} \\ \lambda_s(t)\dot{F}_{i,s-1} = \lambda_s(t)\sum_{j=0}^{s-1}\lambda_j(t)F_{ij}(t) \end{cases} \qquad (4.5)$$

(4.5) shows that the i-th output $y_i(t)=F_{i_0}(t)$ has no relation to all inputs $v_1,\ \cdots,\ v_r$. Thus the conditions (4.1) and (4.2) are necessary.

Sufficiency. Since $\boldsymbol{b}_i(t)$ and $\boldsymbol{c}_i(t)$ are analytic functions, we can take a finite interval $(t_{i_1},\ t_{i_2})$ such that $(t_{i_1},\ t_{i_2})\subset T,\ t_i\in(t_{i_1},\ t_{i_2})$ and rank $M_i(t)\equiv 1$ for all $t\in(t_{i_1},\ t_{i_2})$. That is, if $(\boldsymbol{c}_i(t_i),\ Q^k\boldsymbol{b}_i(t_i)) \neq 0$ for some $k(0\leq k\leq n-1)$, then $(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_i(t)) \neq 0$ for $t\in(t_{i_1},\ t_{i_2})$. Hence, if $v_i(t) \neq 0$ on $(t_{i_1},\ t_{i_2})$,

$$(\tilde{Q}^k\boldsymbol{c}_i(t),\ \boldsymbol{b}_i(t))v_i(t) \neq 0\ \text{on}\ (t_{i_1},\ t_{i_2}). \qquad (4.6)$$

From this fact and (3.22), we can show that the i-th output y_i depends upon the corresponding input v_i on T. The theorem is thus proved.

Conditions given in Theorem 2 do not depend on the initial condition $\boldsymbol{x}_0$. It means that Theorem 2 also shows the condition for the system (2.1) to be the complete noninteracting control system.

5. (X^0, T)-Independence and (X^0, T)-Noninteracting Control in Time-Invariant System

In this section, we consider the linear time-invariant control system written in the matrix form

$$\begin{cases}\dot{\boldsymbol{x}} = A\boldsymbol{x}+B\boldsymbol{v} \\ \boldsymbol{y} = C'\boldsymbol{x}\end{cases} \tag{5.1}$$

where $\boldsymbol{x}$ is an n-th state vector, $\boldsymbol{y}$ is an r-th output vector, $\boldsymbol{v}$ is an r-th input vector. A, B and C are $n\times n$, $n\times r$ and $n\times r$ constant matrices, respectively. Then, the conditions for the system (5.1) to be the (X^0, T)-noninteracting control system are immediately obtained from the preceding results.

Let $\boldsymbol{b}_j$ and $\boldsymbol{c}_j$ be the i-th columns of B and C, respectively. Then we have the following theorems.

[*Theorem 3*]

A necessary and sufficient condition for the system (5.1) to be (X^0, T)-independent is the satisfaction of the relations

$$(\boldsymbol{c}_i, A^k\boldsymbol{b}_j) = 0\,. \tag{5.2}$$
$$i = 1, \cdots, r, \quad k = 0, \cdots, n-1, \quad j = 1, \cdots, i-1, i+1, \cdots, r$$

Proof.)

$$(\boldsymbol{c}_i(t), Q^k\boldsymbol{b}_j(t)) = (-1)^k(\boldsymbol{c}_i, A^k\boldsymbol{b}_j)$$

holds. Thus from Theorem 1, (5.2) is a necessary and sufficient condition.

[*Theorem 4*]

Let

$$M_i \equiv [(\boldsymbol{c}_i, \boldsymbol{b}_i), (\boldsymbol{c}_i, A\boldsymbol{b}_i), \cdots, (\boldsymbol{c}_i, A^{n-1}\boldsymbol{b}_i)]\,.$$
$$i = 1, \cdots, r$$

Then, necessary and sufficient conditions for the system (5.1) to be the (X^0, T)-noninteracting control system is the set of relations

$$\text{rank } M_i = 1, \quad i = 1, \cdots, r$$

and (5.2) hold.

Proof.)

$$(\boldsymbol{c}_i(t),\ Q^k\boldsymbol{b}_i(t)) = (-1)^k(\boldsymbol{c}_i,\ A^k\boldsymbol{b}_i)$$

holds. From this fact and Theorem 2 we can immediately prove the theorem.

6. Design of the Noninteracting Control System by State Variable Feedback

Until now, we have assumed that the equation of the control system has already been given. That is, (2.1) and (5.1) were the equation of the control system. However, in most cases, equations which are given to us are not those of control systems but the equation of a plant. Thus, it is necessary for us to consider the design problem of the noninteracting control system. In this section, we will consider such a design problem. That is, for a given plant we construct a noninteracting contorl system by state variable feedback. For this purpose, results obtained in the preceding sections are very useful. In the following, we will discuss the conditions which are imposed on the feedback when we use the linear state variable feedback control to realize the (X^0, T)-noninteracting control system.

6.1 Time-Variant System

Let

$$\begin{cases} \dot{\boldsymbol{x}} = F(t)\boldsymbol{x} + G(t)\boldsymbol{u} \\ \boldsymbol{y} = H'(t)\boldsymbol{x} \end{cases} \tag{6.1}$$

be the equation of a plant written in the matrix form. Here, $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{u}$ are an n-th state vector, r-th output vector and r-th control vector, respectively. $F(t)$, $G(t)$ $H(t)$ are $n\times n$, $n\times r$ and $n\times r$ matrices, respectively. We assume that they are analytic functions and $n\geq r$.

If, $P(t)$ is an $n\times r$ matrix and $S(t)$ is an nonsingular $r\times r$ matrix, then the substitution of

$$\boldsymbol{u} = P'(t)\boldsymbol{x} + S(t)\boldsymbol{v}\,,$$

where $\boldsymbol{v}$ represents the new r-th input vector, into (6.1) shall be called the linear state variable feedback[2,3]. Then, the equation of the control system is given as follows (Fig. 2):

$$\begin{cases} \dot{\boldsymbol{x}} = (F(t) + G(t)P'(t))\boldsymbol{x} + G(t)S(t)\boldsymbol{v}\,, \\ \boldsymbol{y} = H'(t)\boldsymbol{x}\,. \end{cases} \tag{6.2}$$

Now, according to Theorem 2 and Lemma 3, necessary and sufficient conditions for the system to be the (X^0, T)-noninteracting control system is as follows:

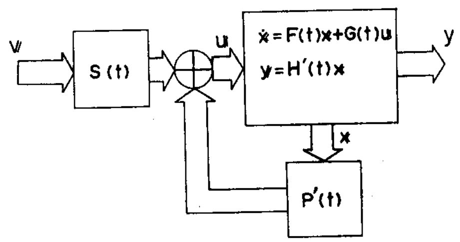

Fig. 2 The noninteracting control system by state variable feedback

$$(\tilde{R}^k \boldsymbol{h}_i(t), \underline{G(t)S(t)_j}) \equiv 0, \ t \in T, \quad (6.3)$$
$$i = 1, \cdots, r$$
$$k = 0, \cdots, n-1$$
$$j = 1, \cdots, i-1, i+1, \cdots, r$$

$$\text{rank } M_i(t) = \text{rank } [(\boldsymbol{h}_i(t), \underline{G(t)S(t)_i}), \cdots, (\tilde{R}^{n-1}\boldsymbol{h}_i(t), \underline{G(t)S(t)_i})] = 1, \quad (6.4)$$
for some $t_i \in T$, $i = 1, \cdots, r$

where $\boldsymbol{h}_i(t)$ is the i-th column of $H(t)$, $\underline{G(t)S(t)}_j$ is the j-th column of $G(t)S(t)$ and $\tilde{R} \equiv D + F'(t) + P(t)G'(t)$. Thus the relaization fo the noninteracting control system results in the determination of the feedback pair fo matrices $P(t)$ and $S(t)$ which satisfy the relations (6.3) and (6.4). Therefore it should be necessary to consider the existence of such feedback pair of matrices $P(t)$ and $S(t)$.

Concerning this problem, the concepts of relative orders of the multivariable system play an important role[3~5]. Let define positive integers ρ_i $(i=1, \cdots, r)$ such that

$$\boldsymbol{h}_i'(t)G(t) = \boldsymbol{h}_i'(t)\{QG(t)\} = \cdots = \boldsymbol{h}_i'(t)\{Q^{\rho_i - 1}G(t)\} \equiv \boldsymbol{0}$$
$$\boldsymbol{h}_i'(t)\{Q^{\rho_i}G(t)\} \not\equiv \boldsymbol{0}, \quad t \in T \quad (6.5)$$

where the prerator $Q \equiv D - F(t)$. Note that, according to Lemma 3, the relations of (6.5) are rewritten in the form:

$$\boldsymbol{h}_i'(t)G(t) = \{\tilde{Q}\boldsymbol{h}_i(t)\}'G(t) = \cdots = \{\tilde{Q}^{\rho_i - 1}\boldsymbol{h}_i(t)\}'G(t) = \boldsymbol{0}$$
$$\{\tilde{Q}^{\rho_i}\boldsymbol{h}_i(t)\}'G(t) \not\equiv \boldsymbol{0}, \quad t \in T \quad (6.5)'$$

where $\tilde{Q} \equiv D + F'(t)$. Then we will have the following existence theoem.

[*Theorem 5*]

Let

$$K(t) = \begin{bmatrix} \{\tilde{Q}^{\rho_1}\boldsymbol{h}_1(t)\}'G(t) \\ \cdots\cdots\cdots \\ \{\tilde{Q}^{\rho_r}\boldsymbol{h}_r(t)\}'G(t) \end{bmatrix}$$

Then, if rank $K(t) = r$ for all $t \in T$, there exists a feedback pair of matrices $P(t)$ and

$S(t)$ which satisfy the relations (6.3) and (6.4).

Proof.) From the definition of relative orders, the following relations

$$\begin{cases} \tilde{R}\boldsymbol{h}_j(t) = (\tilde{Q}+P(t)G'(t))\boldsymbol{h}_j(t) = \tilde{Q}\boldsymbol{h}_j(t), \\ \tilde{R}^2\boldsymbol{h}_j(t) = \tilde{R}^2\boldsymbol{h}_j(t), \\ \tilde{R}^{\rho_j}\boldsymbol{h}_j(t) = \tilde{Q}^{\rho_j}\boldsymbol{h}_j(t) \qquad (j=1,\cdots,r) \end{cases} \tag{6.6}$$

hold. Thus,

$$\begin{cases} \{\tilde{R}\boldsymbol{h}_j(t)\}'G(t) \equiv \boldsymbol{0}, \\ \cdots\cdots\cdots\cdots \\ \{\tilde{R}^{\rho_j-1}\boldsymbol{h}_j(t)\}'G(t) \equiv \boldsymbol{0} \end{cases} \tag{6.7}$$

hold. Now let us choose a feedback pair $P(t)$ and $S(t)$ such that

$$\begin{cases} P^*(t) = -N^*(t)\{K'(t)\}^{-1} \\ S^*(t) = K(t)^{-1} \end{cases} \tag{6.8}$$

where

$$N^*(t) = [\tilde{Q}^{\rho_1+1}\boldsymbol{h}_1(t), \cdots, \tilde{Q}^{\rho_r+1}\boldsymbol{h}_r(t)]\,.$$

Then, from (6.5)′ and (6.8),

$$\begin{aligned} \tilde{R}^{\rho_j+1}\boldsymbol{h}_j(t) &= (\tilde{Q}+P^*(t)G'(t))\{\tilde{Q}^{\rho_j}\boldsymbol{h}_j(t)\} \\ &= \tilde{Q}^{\rho_j+1}\boldsymbol{h}_j(t)+P^*(t)G'(t)\{\tilde{Q}^{\rho_j}\boldsymbol{h}_j(t)\} \\ &= \tilde{Q}^{\rho_j+1}\boldsymbol{h}_j(t)-N^*(t)\{K'(t)\}^{-1}[G'(t)\{\tilde{Q}^{\rho_j}\boldsymbol{h}_j(t)\}] \\ &= \tilde{Q}^{\rho_j+1}\boldsymbol{h}_j(t)-N^*(t)\{K'(t)\}^{-1}\underline{K'(t)}_j \\ &= \tilde{Q}^{\rho_j+1}(t)-N^*(t)\boldsymbol{e}_j \\ &= \tilde{Q}^{\rho_j+1}\boldsymbol{h}_j(t)-\tilde{Q}^{\rho_j+1}\boldsymbol{h}_j(t) \equiv \boldsymbol{0} \end{aligned}$$

holds where $\underline{K'(t)}_j$ is the j-th colunm of $K'(t)$ and $\boldsymbol{e}_j$ is the j-th unit vector. Hence,

$$\tilde{R}^{\rho_j+k}\boldsymbol{h}_j(t) \equiv \boldsymbol{0} \text{ for all } t\in T,\ (k=1,2,\cdots) \quad j=1,\cdots,r \tag{6.9}$$

holds. In addition to this fact.

$$\begin{aligned} \{\tilde{R}^{\rho_j}\boldsymbol{h}_j(t)\}'\,G(t)S^*(t) &= [\{\tilde{Q}^{\rho_j}\boldsymbol{h}_j(t)\}'\,G(t)]K(t)^{-1} \\ &= \boldsymbol{e}_j' = (0,\cdots 0,\underset{\substack{\frown \\ j}}{1},0,\cdots,0), \\ (j=1,\cdots,r) \end{aligned} \tag{6.10}$$

holds. From (6.9) and (6.10), we can immediately obtain the relations for all $t\in T$;

$$\begin{cases} (\tilde{R}^k \boldsymbol{h}_i(t),\ \underline{G(t)S(t)}_j) \equiv 0, \\ \quad i = 1, \cdots, r, \quad j = 1, \cdots, r, \quad k = 0, \cdots, n-1;\ k \neq \rho_i \\ (\tilde{R}^{\rho_i} \boldsymbol{h}_i(t),\ \underline{G(t)S(t)}_j) \equiv 0, \quad j = 1, \cdots, r;\ j \neq i \\ (\tilde{R}^{\rho_i} \boldsymbol{h}_i(t),\ \underline{G(t)S(t)}_i) \equiv 1\,. \end{cases}$$

These relations satisfy the conditions of the noninteracting control (6.3) and (6.4). Thus the theorem is proved.

The theorem just proved shows the sufficient condition for the (X^0, T)-noninteracting control system (or complete noninteracting control system) to be realizable with linear state variable feedback.

6.2 Time Invariant System

Let

$$\begin{cases} \dot{\boldsymbol{x}} = F\boldsymbol{x} + G\boldsymbol{u} \\ \boldsymbol{y} = H'\boldsymbol{x} \end{cases} \tag{6.11}$$

be the equation of a plant. In this case, the matrices F, G and H are contsant. Orders of vectros and matrices are the same as (6.1).

Let the feedback be

$$\boldsymbol{u} = P'\boldsymbol{x} + S\boldsymbol{v} \tag{6.12}$$

where $|S| \neq 0$ and $\boldsymbol{v}$ is an new r-th input vector. By substituting (6.12) into (6.11), we can obtain the equation of the feedback control system;

$$\begin{cases} \dot{\boldsymbol{x}} = (F + GP')\boldsymbol{x} + GS\boldsymbol{v}, \\ \boldsymbol{y} = H'\boldsymbol{x}\,. \end{cases} \tag{6.13}$$

Hence from Theorem 4 the conditions for the control system (6.13) to be the (X^0, T)-noninteracting control system (or complete noninteracting control system) are given as follows;

$$\begin{aligned} (\boldsymbol{h}_i, \underline{GS}_j) &= (\boldsymbol{h}_i, (F+GP')\underline{GS}_j) = \cdots \\ &= (\boldsymbol{h}_i, (F+GP')^{n-1}\underline{GS}_j) = 0, \\ & i = 1, \cdots, r, \quad j = 1, \cdots, r;\ j \neq i \end{aligned} \tag{6.14}$$

$$\text{rank } [(\boldsymbol{h}_i, \underline{GS}_i), \cdots, (\boldsymbol{h}_i, (F+GP')^{n-1}\underline{GS}_i)] = 1, \quad i = 1, \cdots, r \tag{6.15}$$

where $\boldsymbol{h}_i$ is the i-th column of H and $\underline{GS}_j$ is the j-th column of GS.

The relative orders of the system (6.11) are defined by the relations

$$\begin{cases} \boldsymbol{h}_i'G = \boldsymbol{h}_i'FG = \cdots = \boldsymbol{h}_i'F^{\rho_i-1}G = \mathbf{0}, \\ \boldsymbol{h}_i'F^{\rho_i}G \neq \mathbf{0}. \end{cases}$$

Thus, the existence of the feedback pair of matrices P and S which satisfy the relations (6.14) and (6.15) can be considered in the same manner used in the preceding section.

[*Theorem 6*]

Let

$$K = \begin{bmatrix} \boldsymbol{h}_1'F^{\rho_1}G \\ \cdots\cdots \\ \boldsymbol{h}_r'F^{\rho_r}G \end{bmatrix}.$$

Then a necessary and sufficient condition fot rhe relations (6.14) and (6.15) to be satisfied by P and $S(|S| \neq 0)$ is that

$$\text{rank } K = r.$$

(See Appendix 1).

The theorem just proved shows that the concept of relative orders plays a very important role in the problem of the noninteracting contorl. This fact has already been indicated by P. L. Falb and W. A. Wolovich[3], and authors[4]. Falb and Wolovich has introduced the concept of "decoupling" to the multivariable system. "Decoupling" and the (X^0, T)-noninteracting control defined here are similar concepts. For example, the conditions for the time invariant system (6.11) with a state variable feedback (6.12) to be a (X^0, T)-noninteracting control system are consistent with the conditions for which the control system (6.13) is the decoupling system. However, "decoupling" is only defined in the time invariant control system represented by the system (6.11) with feedback (6.12). Hence, it seems to be difficult to consider the general expantion of the concept of "decoupling". On the other hand, the (X^0, T)-nonintreacting control defined in this paper can be applicable to the general continuous linear systems. And even if the system under consideration is nonlinear, the (X^0, T)-noninteracting control may be realizable[6]. Hence, we may conclude that the concept of (X^0, T)-noninteracting control is a more general concept than that of "decoupling".

7. Selection of $P(t)$ and $S(t)$

If, $K(t)$ is nonsingular for all $t \in T$, then, by choosing $P(t)$ and $S(t)$ such as (6.8), the system can be decoupled. However, the matrices $P(t)$ and $S(t)$ which satisfy the conditions (6.3) and (6.4) are not unique. It is possible to exploit this nonuniqueness to obtain various desired closed loop pole configurations.

Now, let us choose matrices $P(t)$ and $S(t)$ such that

$$P^{**}(t) = -(N^*(t)+N_1(t))\{K'(t)\}^{-1} \tag{7.1}$$

$$S^{**}(t) = K(t)^{-1} \tag{7.2}$$

where

$$N_1(t) = \left[\sum_{j=0}^{\rho 1}\lambda_{1j}\tilde{Q}^j\boldsymbol{h}_1(t), \cdots, \sum_{j=0}^{\rho r}\lambda_{rj}\tilde{Q}^j\boldsymbol{h}_r(t)\right]$$

and λ_{ij} are arbitrary constant. We will show that matrices $P^{**}(t)$ and $S^{**}(t)$ satisfy the noninteracting conditions (6.3) and (6.4). According to the definition of relative orders,

$$\begin{aligned}
\tilde{R}\boldsymbol{h}_j(t) &= \tilde{Q}\boldsymbol{h}_j(t)\\
&\vdots\\
\tilde{R}^{\rho j}\boldsymbol{h}_j(t) &= \tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\\
\tilde{R}^{\rho j+1}\boldsymbol{h}_j(t) &= \{\tilde{Q}+P^{**}(t)G'(t)\}\{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\}\\
&= \tilde{Q}^{\rho j+1}\boldsymbol{h}_j(t)+P^{**}(t)[G'(t)\{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\}]\\
&= \tilde{Q}^{\rho j+1}\boldsymbol{h}_j(t)-N^*(t)\{K'(t)^{-1}[G'(t)\{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\}]\\
&\qquad -N_1(t)K'(t)^{-1}[G'(t)\{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)]\\
&= \tilde{Q}^{\rho j+1}\boldsymbol{h}_j(t)-\tilde{Q}^{\rho j+1}\boldsymbol{h}_j(t)-\sum_{k=0}^{\rho j}\lambda_{jk}\tilde{Q}^k\boldsymbol{h}_j(t)\\
&= -\sum_{k=0}^{\rho j}\lambda_{jk}\tilde{Q}^k\boldsymbol{h}_j(t)\\
\tilde{R}^{\rho j+2}\boldsymbol{h}_j(t) &= \{\tilde{Q}+P^{**}(t)G'(t)\}\left\{-\sum_{k=0}^{\rho j}\lambda_{jk}\tilde{Q}^k\boldsymbol{h}_j(t)\right\}\\
&= -\sum_{k=0}^{\rho j}\lambda_{jk}\tilde{Q}^{k+1}\boldsymbol{h}_j(t)\\
&\qquad -\sum_{k=0}^{\rho j}\lambda_{jk}P^{**}(t)[G'(t)\{\tilde{Q}^k\boldsymbol{h}_j(t)\}]\\
&= -\sum_{k=0}^{\rho j}\lambda_{jk}\tilde{Q}^{k+1}\boldsymbol{h}_j(t)\\
&\qquad +\sum_{k=0}^{\rho j}\lambda_{jk}N^*(t)\{K'(t)\}^{-1}[G'(t)\{\tilde{Q}^k\boldsymbol{h}_j(t)\}]
\end{aligned}$$

$$+\sum_{k=0}^{\rho j}\lambda_{jk}N_1(t)\{K'(t)\}^{-1}[G'(t)\{\tilde{Q}^k\boldsymbol{h}_j(t)\}]$$

$$= -\sum_{k=0}^{\rho j}\lambda_{jk}\tilde{Q}^{k+1}\boldsymbol{h}_j(t)$$

$$+\lambda_{j\rho j}N^*(t)\{K'(t)\}^{-1}[G'(t)\{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\}]$$

$$+\lambda_{j\rho j}N_1(t)\{K'(t)\}^{-1}[G'(t)\{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\}]$$

$$= \left\{-\sum_{k=0}^{\rho j-1}\lambda_{jk}\tilde{Q}^{k+1}\boldsymbol{h}_j(t) - \lambda_{j\rho j}\tilde{Q}^{\rho j+1}\boldsymbol{h}_j(t)\right\}$$

$$+ \lambda_{j\rho j}\tilde{Q}^{\rho j+1}\boldsymbol{h}_j(t) + \lambda_{j\rho j}\sum_{k=0}^{\rho j}\lambda_{jk}\{\tilde{Q}^k\boldsymbol{h}_j(t)\}$$

$$= \sum_{k=0}^{\rho j}\{\lambda_{j\rho j}\lambda_{jk} - \lambda_{j,\,k-1}\}\,\tilde{Q}^k\boldsymbol{h}_j(t)$$

$$= \sum_{k=0}^{\rho j}\lambda'_{jk}\tilde{Q}\boldsymbol{h}_j(t)$$

....

hold. Here $\lambda_{j\rho j}\lambda_{jk} - \lambda_{j,k-1} = \lambda'_{jk},\ \lambda_{j,-1} = 0.$

Hence $\tilde{R}^{\rho j+k}\boldsymbol{h}_j(t)$ $(k = 1, 2, \cdots)$ can be expressed by a linear combination of $\boldsymbol{h}_j(t)$, $\tilde{Q}\boldsymbol{h}_j(t), \cdots, \tilde{Q}^{\rho j}\boldsymbol{h}_j(t)$. Thus,

$$\{\tilde{R}^k\boldsymbol{h}_j(t)\}'G(t)S^{**}(t) \equiv \mathbf{0}, \quad k = 0, 1, \cdots, \rho_j - 1,$$

$$\{\tilde{R}^{\rho j}\boldsymbol{h}_j(t)\}'G(t)S^{**}(t) = \{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\}'G(t)K(t)^{-1} = (0, \cdots, 0, \underset{\widehat{j}}{1}, 0, \cdots, 0)$$

$$\{\tilde{R}^{\rho j+1}\boldsymbol{h}_j(t)\}'G(t)S^{**}(t) = -\sum_{k=0}^{\rho j}\lambda_{jk}\{\tilde{Q}^k\boldsymbol{h}_j(t)\}'G(t)K(t)^{-1} = -\lambda_{j\rho j}\{\tilde{Q}^{\rho j}\boldsymbol{h}_j(t)\}'G(t)K(t)^{-1} = (0, \cdots, 0, \underset{\widehat{j}}{-\lambda_{j\rho j}}, 0, \cdots, 0)$$

$$\{\tilde{R}^{\rho j+2}\boldsymbol{h}_j(t)\}'G(t)S^{**}(t) = (0, \cdots, 0, \underset{\widehat{j}}{\lambda'_{j\rho j}}, 0, \cdots, 0)$$

....

hold. From these, we have the following relations. That is, for $s \neq j$,

$$(\tilde{R}^k\boldsymbol{h}_j(t),\ \underline{G(t)S^{**}(t)_s}) \equiv 0 \tag{7.3}$$

$$k = 0, 1, \cdots, n-1$$
$$s = 1, \cdots, r$$
$$j = 1, \cdots, r$$

and for $s = j$,

$$\begin{cases} (\boldsymbol{h}_j(t),\ \underline{G(t)S^{**}(t)_j}) \equiv 0 \\ \cdots\cdots\cdots\cdots \\ (\tilde{R}^{\rho_j-1}\boldsymbol{h}_j(t),\ \underline{G(t)S^{**}(t)_j}) \equiv 0 \\ (\tilde{R}^{\rho_j}\boldsymbol{h}_j(t),\ \underline{G(t)S^{**}(t)_j}) = 1 \\ (\tilde{R}^{\rho_j+1}\boldsymbol{h}_j(t),\ \underline{G(t)S^{**}(t)_j}) = \lambda_{j\rho_j}^{(0)} \\ \cdots\cdots\cdots\cdots \\ (\tilde{R}^{n-1}\boldsymbol{h}_j(t),\ \underline{G(t)S(t)_j}) = \lambda_{j\rho_j}^{(n-\rho_j-2)} \end{cases} \tag{7.4}$$

hold, where $\lambda_{j\rho_j}^{(0)}, \cdots, \lambda_{j\rho_j}^{(n-\rho_j-2)}$ are constants determined from $\lambda_{j0}, \cdots, \lambda_{j\rho_j}$. (7.3) and (7.4) satisfy the conditions (6.3) and (6.4).

Example. Let

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sum_{j=1}^{4} f_{2j}(t)x_j + \sum_{j=1}^{2} g_{2j}(t)u_j \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \sum_{j=1}^{4} f_{4j}(t)x_j + \sum_{j=1}^{2} g_{4j}(t)u_j \end{cases} \tag{7.5}$$

$$\begin{cases} y_1 = h_{11}x_1 + h_{12}x_3 \\ y_2 = h_{21}x_1 + h_{22}x_3 \end{cases} \tag{7.6}$$

be a system of differential equations of a multivariable plant. For this plant, we will design a noninteracting control system. In the following, we assume that

$$\delta \neq 0,\ \Delta(t) \neq 0, \text{ for all } t \in T \tag{7.7}$$

where

$$\delta = h_{11}h_{22} - h_{12}h_{21}, \quad \Delta(t) = g_{21}(t)g_{42}(t) - g_{22}(t)g_{41}(t).$$

(i) Relative orders

$$\boldsymbol{h}_1'G(t) = (0,0)$$
$$\{\tilde{Q}\boldsymbol{h}_1\}'G(t) = (h_{11}g_{21}(t) + h_{12}g_{41}(t),\ h_{11}g_{22}(t) + h_{12}g_{42}(t))$$
$$\boldsymbol{h}_2'G(t) = (0,0)$$
$$\{\tilde{Q}\boldsymbol{h}_2\}'G(t) = (h_{21}g_{21}(t) + h_{22}g_{41}(t),\ h_{21}g_{22}(t) + h_{22}g_{42}(t))$$

where

$$\boldsymbol{h}_1' = (h_{11}, 0, h_{12}, 0),$$

$$\boldsymbol{h}_2' = (h_{21}, 0, h_{22}, 0),$$

$$G(t) = \begin{pmatrix} 0 & 0 \\ g_{21}(t), & g_{22}(t) \\ 0 & 0 \\ g_{41}(t), & g_{42}(t) \end{pmatrix};$$

Hence, $\rho_1 = \rho_2 = 1$.

(ii) Feedback pair $P(t)$ and $G(t)$

By assumption (7.7), $|K(t)| = \delta\Delta(t) \neq 0$ for all $t \in T$ where

$$K(t) = \begin{bmatrix} \{\tilde{Q}\boldsymbol{h}_1\}'G(t) \\ \{\tilde{Q}\boldsymbol{h}_2\}'G(t) \end{bmatrix}$$

$$= \begin{bmatrix} h_{11}g_{21}(t) + h_{12}g_{41}(t), & h_{11}g_{22}(t) + h_{12}g_{42}(t) \\ h_{21}g_{21}(t) + h_{22}g_{41}(t), & h_{21}g_{22}(t) + h_{22}g_{42}(t) \end{bmatrix}.$$

Therefore we can take a state variable feedback $\boldsymbol{u}$ as follows;

$$\boldsymbol{u} = \{P^{**}(t)\}'\boldsymbol{x} + S^{**}(t)\boldsymbol{v} \tag{7.8}$$

where

$$P^{**}(t) = -\{N^*(t) + N_1(t)\}\{K'(t)\}^{-1},$$

$$S^{**}(t) = K(t)^{-1},$$

$$N^*(t) = [\tilde{Q}^2\boldsymbol{h}_1, \tilde{Q}\boldsymbol{h}_2]$$

$$= \begin{pmatrix} h_{11}f_{21}(t) + h_{12}f_{41}(t), & h_{21}f_{21}(t) + h_{22}f_{41}(t) \\ h_{11}f_{22}(t) + h_{12}f_{42}(t), & h_{21}f_{22}(t) + h_{22}f_{42}(t) \\ h_{11}f_{23}(t) + h_{12}f_{43}(t), & h_{21}f_{23}(t) + h_{22}f_{43}(t) \\ h_{11}f_{24}(t) + h_{12}f_{44}(t), & h_{21}f_{24}(t) + h_{22}f_{44}(t) \end{pmatrix},$$

$$N_1(t) = [\lambda_{10}\boldsymbol{h}_1 + \lambda_{11}\tilde{Q}\boldsymbol{h}_1, \quad \lambda_{20}\boldsymbol{h}_2 + \lambda_{21}\tilde{Q}\boldsymbol{h}_2]$$

$$= \begin{pmatrix} \lambda_{10}h_{11}, & \lambda_{20}h_{21} \\ \lambda_{11}h_{11}, & \lambda_{21}h_{21} \\ \lambda_{10}h_{21}, & \lambda_{20}h_{22} \\ \lambda_{11}h_{12}, & \lambda_{21}h_{22} \end{pmatrix}.$$

(iii) The equation of the noninteracting control system

By substituting (7.8) into (7.5), we can obtain the equation of the closed loop control system:

$$\left\{\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{\delta}\{(\lambda_{10}h_{11}h_{22}-\lambda_{20}h_{12}h_{21})x_1 \\
&\quad +(\lambda_{11}h_{11}h_{22}-\lambda_{21}h_{12}h_{21})x_2 \\
&\quad +(\lambda_{11}-\lambda_{20})h_{12}h_{22}x_3 \\
&\quad +(\lambda_{11}-\lambda_{21})h_{12}h_{22}x_4\} \\
&\quad +\frac{h_{22}}{\delta}v_1-\frac{h_{12}}{\delta}v_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{\delta}\{(\lambda_{20}-\lambda_{10})h_{11}h_{21}x_1+(\lambda_{21}-\lambda_{11})h_{11}h_{21}x_2 \\
&\quad +(\lambda_{20}h_{11}h_{22}-\lambda_{10}h_{12}h_{21})x_3 \\
&\quad +(\lambda_{21}h_{11}h_{22}-\lambda_{11}h_{12}h_{21})x_4\} \\
&\quad -\frac{h_{21}}{\delta}v_1+\frac{h_{11}}{\delta}v_2
\end{aligned}\right. \tag{7.9}$$

$$\left\{\begin{aligned}
y_1 &= h_{11}x_1+h_{12}x_3 \\
y_2 &= h_{21}x_1+h_{22}x_3 .
\end{aligned}\right. \tag{7.10}$$

By differentiating (7.10) with respect to (7.9), we have

$$\left\{\begin{aligned}
\ddot{y}_1 &= \lambda_{10}y_1+\lambda_{11}\dot{y}_1+v_1 , \\
\ddot{y}_2 &= \lambda_{20}y_2+\lambda_{21}\dot{y}_2+v_2 .
\end{aligned}\right. \tag{7.11}$$

Thus, the multivariable control system (7.9) and (7.10) is a noninteracting control system.

(iv) Stability

A necessary and sufficient condition for the system (7.11) to be a stable system is apparent. That is,

$$\lambda_{i0}<0, \quad \lambda_{i1}<0, \quad i=1,2. \tag{7.12}$$

This condition can be obtained directly from (7.9) by applying the Routh-Hurwitz's condition. Taking into account the fact that λ_{ij} are arbitrary constants, it is possible to stabilize the noninteracting control system (7.9).

8. Conclusions

In this paper, the noninteracting control of linear (time-variant and time-invariant) multivariable control system is studied.

We showed first that the problem of the noninteracting control is essentially a variational one. Then, we discussed the design problem of the noninteracting control system. It is shown that it is possbile to obtain the noninteracting control system by state variable feedback. Sufficient conditions for the noninteracting control system to be constructed by state variable feedback are given by using the concept of relative orders.

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Appendix 1

1. Necessity. Suppose that there exists a feedback pair of matrices P and $S(|S|\neq 0)$ which satisfy the relations (6.14) and (6.15). Now, from the definition of relative orders, we have

$$\begin{cases} \boldsymbol{h}_i'(F+GP')^j = \boldsymbol{h}_i'F^j \ : j=0,\cdots,\rho_i \\ \boldsymbol{h}_i'(F+GP')^j = \boldsymbol{h}_i'F^{\rho_i}(F+GP')^{j-\rho_i} \ : \ j=\rho_i+1,\cdots,n-1 \end{cases} \tag{A.1}$$

Hence, for j $(0\leq j\leq \rho_i-1)$,

$$\begin{cases} \boldsymbol{h}_i'GS\equiv \boldsymbol{0} \\ \boldsymbol{h}_i'(F+GP')GS=\boldsymbol{h}_i'FGS\equiv \boldsymbol{0} \\ \boldsymbol{h}_i'(F+GP')^{\rho_i-1}GS=\boldsymbol{h}_i'F^{\rho_i-1}GS\equiv \boldsymbol{0} \quad i=1,\cdots,r \end{cases} \tag{A.2}$$

hold. When $j=\rho_i$,

$$\boldsymbol{h}_i'(F+GP')^{\rho_i}G=\boldsymbol{h}_i'F^{\rho_i}G\neq \boldsymbol{0}\,.$$

Thus, according to the nonsingularity of S,

$$\boldsymbol{h}_i'(F+GP')^{\rho_i}GS = \boldsymbol{h}_i F^{\rho_i}GS \neq \boldsymbol{0} \tag{A.3}$$

hold. (Here, $\rho_i < n-1$. If $\rho_i = n-1$, then (6.12) never hold.) Now from (6.14), we have

$$\begin{aligned}\boldsymbol{h}_i'F^{\rho_i}GS &= [\boldsymbol{h}_i'F^{\rho_i}\underline{GS}_1, \cdots, \boldsymbol{h}_i'F^{\rho_i}\underline{GS}_r] \\ &= [0, \cdots, 0, \boldsymbol{h}_i'F^{\rho_i}\underline{GS}_i, 0, \cdots 0].\end{aligned}$$

Hence, by (A.3),

$$\boldsymbol{h}_i'F^{\rho_i}\underline{GS}_i = \alpha_i \neq 0, \ i = 1, \cdots, r$$

should hold where α_i are nonzero scalars. Thus we can obtain the following relations. That is,

$$\begin{bmatrix}\boldsymbol{h}_1'F^{\rho_1}GS \\ \vdots \\ \boldsymbol{h}_r'F^{\rho_r}GS\end{bmatrix} = \begin{bmatrix}\boldsymbol{h}_1'F^{\rho_1}G \\ \vdots \\ \boldsymbol{h}_r'F^{\rho_r}G\end{bmatrix} S = KS = \begin{bmatrix}\alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_r\end{bmatrix}.$$

Taking into account the nonsingularity of S, we have

$$|K| = |S^{-1}|\sum_{j=1}^{r}\alpha_j \neq 0 .$$

2. Sufficiency. Let us choose the feedabck pair of matrices such that

$$P = -N(K')^{-1}, \quad S = K^{-1}$$

where

$$N' = \begin{bmatrix}\boldsymbol{h}_1'F^{\rho_1+1} \\ \vdots \\ \boldsymbol{h}_r'F^{\rho_r+1}\end{bmatrix}.$$

Then the sufficiency can be proved without difficulty by using the same manner used in the proof of Theoem 6.

Appendix 2 (($\boldsymbol{X}$, $\boldsymbol{T}$)-noninteracting control and diagonalization method)

The multivariable control system (6. 13) can be written in the transfer function matrix from:

$$\boldsymbol{Y}(s) = H'(sI-F-GP')^{-1}GS\boldsymbol{V}(s) \tag{A.4}$$

where $\boldsymbol{Y}(s) = \mathcal{L}\{\boldsymbol{y}(t)\}$, $\boldsymbol{V}(s) = \mathcal{L}\{\boldsymbol{v}(t)\}$. If, ($X^0$, T)-noninteracting control holds, that is, if Theorem 5 holds, (A.6) will be diagonalized with respect to $\boldsymbol{Y}(s)$ and $\boldsymbol{V}(s)$.

This fact will be shown in the following. By using the Laplace's theorem, (A.4) can be expressed in the following form:

$$\boldsymbol{y}(t) = \int_0^t H'\Phi(t, \tau)GS\boldsymbol{v}(\tau)d\tau \tag{A.5}$$

where

$$\Phi(t, \tau) = \exp\{(F+GP')(t-\tau)\}.$$

From the definition of exponential function, we have

$$H'\Phi(t, \tau)GS$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} H'(F+GP')^k GS(t-\tau)^k.$$

Hence,

$$y_i(t) = \int_0^t \left\{\sum_0^{\infty} \frac{1}{k!} \boldsymbol{h}_i'(F+GP')^k G(S)(t-\tau)^k\right\}\boldsymbol{v}(\tau)d\tau. \tag{A.6}$$

Let us consider the integrand in (A.6). Let us choose matrices P and S such that

$$P' = -K^{-1}N', \quad S = K^{-1}$$

where

$$N' = \begin{bmatrix} \boldsymbol{h}_1'F^{\rho_1+1} \\ \cdots \\ \boldsymbol{h}_r'F^{\rho_r+1} \end{bmatrix}.$$

Then,

$$\left\{\begin{aligned} &\boldsymbol{h}_i'GS = \boldsymbol{0} \\ &\cdots\cdots \\ &\boldsymbol{h}_i'(F+GP')^{\rho_i-1}GS = \boldsymbol{0} \\ &\boldsymbol{h}_i'(F+GP')^{\rho_i}GS = \boldsymbol{h}_i'F^{\rho_i}GS = (0, \cdots 0, \overset{i}{\check{1}}, 0, \cdots 0) \\ &\boldsymbol{h}_i'(F+GP')^{\rho_i+1}GS = (\boldsymbol{h}_i'F^{\rho_i+1}+\boldsymbol{h}_i'F^{\rho_i}GP')GS \\ &\qquad = (\boldsymbol{h}_i'F^{\rho_i+1}-\boldsymbol{h}_i'F^{\rho_i}FK^{-1}N')GS \\ &\qquad = (\boldsymbol{h}_i'F^{\rho_i+1}-\boldsymbol{h}_i'F^{\rho_i+1})GS \\ &\qquad \equiv \boldsymbol{0} \\ &\boldsymbol{h}_i'(F+GP)^{\rho_i+2}GS \equiv \boldsymbol{0} \\ &\cdots\cdots \end{aligned}\right. \tag{A.7}$$

hold. By using (A.7) to (A.6), we have

$$y_i(t) = \int_0^t \frac{1}{\rho_i!}(t-\tau)^{\rho_i} v_i(\tau) d\tau .$$

Hence,

$$\mathcal{L}\left\{y_i(t)\right\} = \mathcal{L}\left\{\frac{t^{\rho_i}}{\rho_i!}\right\}\mathcal{L}\left\{v_i(t)\right\} .$$

holds. Thus, we have

$$\boldsymbol{Y}(s) = \begin{bmatrix} \frac{1}{s^{\rho_1+1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s^{\rho_r+1}} \end{bmatrix} \boldsymbol{V}(s) .$$