Ultrasonic Wave Propagations in Non-Homogeneously and Dynamically Deformed Isotropic Elastic Materials

By

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The thermodynamical investigations of the wave propagation superposing on the non-homogeneously and dynamically deformed isotropic elastic material are discussed. The expansion forms of the free energy and of the wave deformation are determined by the characters of the isotropic tensors. The wave propagation equations are deduced for three superposed states: isothermal, isentropic and middle. The stress-acoustical law holds, that is, the phase difference of two polarized shear waves is proportional to the stress-difference and the wave frequency.

1. Introduction

With respect to the wave propagation in the deformed elastic materials, several theories have been published. Truesdell[1] and Truesdell and Noll[2] reported the complete properties of the propagation of the acceleration waves. The sinusoidal wave propagations were investigated by Hayes and Rivlin[3], Toupin and Bernstein[4], and Thurston[5]. Flavin and Green[6] and Green[7] investigated the wave propagations under isothermal or isentropic conditions in a thermoelastic body. But all of the above mentioned papers are restricted in a material subject to homogeneous deformation.

On the other-hand Benson and Raelson[8] proposed a new experimental nondestructive stress analysis called acoustoelasticity. Several experimental studies of this method have been reported[9-13]. The theoretical study of the acoustical birefringence was investigated by Tokuoka and Iwashimizu[14], who formulated the stress-acoustical law, that is, the acoustical birefringence is proportional to the difference of the secondary principal stresses.

In this paper, based on Tokuoka and Iwashmizu[14], the thermodynamical considerations are used for the deformation of an isotropic elastic material. The ultrasonic wave propagations superposing on the non-homogeneously and dynami-

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cally deforming state are analyzed by the method of quantitative order estimation. Then the stress-acoustical law is formulated.

2. Thermodynamical Considerations of Deformations of Isotropic Elastic Materials

The Thermodynamical treatments of the hyper-elastic material are reported by Landau and Lifshitz[15], England and Green[16], Green and Adkins[17] and Eringen[18] (chapter 5).

Consider an isotropic elastic material which is deformed from natural undeformed state I with constant temperature to state II. The coordinates, density, temperature and entropy are denoted by X_k ($k=1, 2, 3$), ρ_0, T_0 and S_0, and x_k, ρ, T and S in I and II respectively, where the coordinates are taken with respect to a rectangular Cartesian system and related with

$$x_k = X_k + u_k \tag{2.1}$$

by the displacement vector u_k, and the entropies are measured per unit volume of I.

The stress tensor in the state II is exactly expressed by[18]

$$t_{kl} = \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial E_{mn}} \frac{\partial x_k}{\partial X_m} \frac{\partial x_l}{\partial X_n}, \tag{2.2}$$

where Σ is the Helmholtz free energy per unit volume of I and assumed to be a function of E_{kl} and T, and

$$E_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_l} + \frac{\partial u_l}{\partial X_k} + \frac{\partial u_m}{\partial X_k}\frac{\partial u_m}{\partial X_l}\right) \tag{2.3}$$

is the Lagrangian strain tensor. The entropy is given by

$$S = -\frac{\partial \Sigma}{\partial T}. \tag{2.4}$$

We restrict the deformation within the proportional limit, where the strain and the displacement gradient may be less than 10^{-3} for iron and alminium and Hooke's law holds. In this case the non-linear terms of (2.3) may be omitted and we have the usual strain tensor

$$e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_l} + \frac{\partial u_l}{\partial X_k}\right). \tag{2.5}$$

Now the temperature deviation $(T-T_0)$ associated with the deformation is assumed to be the same order of smallness with the strain, and the free energy Σ is expanded to the second order with respect to $(T-T_0)$ and e_{kl}. Then we have

$$\Sigma = \Sigma_0 - S_0(T-T_0) + A(T-T_0)^2 + A_{kl} e_{kl}(T-T_0) + A_{klmn} e_{kl} e_{mn} , \tag{2.6}$$

where A, A_{kl}, A_{klmn} are material constants and without loss of generality we may restrict them as

$$A_{kl} = A_{lk} ,$$
$$A_{klmn} = A_{lkmn} = A_{klnm} = A_{mnkl} . \tag{2.7}$$

The material considered is isotropic and then the material constants (2.7) must be isotropic tensors, i.e. tensor whose components have the same numerical value in any Cartesian coordinate system. Thus we may conclude according to Thomas[19] that

$$A_{kl} = -K\alpha\delta_{kl} ,$$
$$A_{klmn} = \frac{1}{2}\left[\lambda\delta_{kl}\delta_{mn} + \mu(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})\right], \tag{2.8}$$

where we may interpret physically that α is the thermal expansion coefficient, λ and μ are the isothermal Lamé elastic constants and $K \equiv \lambda + \frac{2}{3}\mu$ is the isothermal bulk modulus. Within the linear relations (2.2) and (2.4) give

$$t_{kl} = \lambda e\delta_{kl} + 2\mu e_{kl} - K\alpha(T-T_0)\delta_{kl} ,$$
$$S = S_0 - 2A(T-T_0) + K\alpha e , \tag{2.9}$$

where $e \equiv e_{mm}$ and δ_{kl} denotes the Kronecker delta, which equals 1 for $k=l$ and 0 for $k \neq l$.

In the isothermal deformation we have the usual Hooke's law

$$t_{kl} = \lambda e\delta_{kl} + 2\mu e_{kl} ,$$
$$S = S_0 + K\alpha e \tag{2.10}$$

and in the isentropic deformation, eliminating $(T-T_0)$ from $(2.9)_1$ by putting $S-S_0=0$ in $(2.9)_2$, we have

$$t_{kl} = \lambda^* e\delta_{kl} + 2\mu^* e_{kl} ,$$
$$T = T_0 + \frac{K\alpha}{2A} e , \tag{2.11}$$

where

$$\lambda^* \equiv \lambda - \frac{(K\alpha)^2}{2A} , \qquad \mu^* = \mu \tag{2.12}$$

are the isentropic Lamé elastic constants.

The well-known thermodynamic formula

$$\left(\frac{\partial V}{\partial p}\right)_S = \left(\frac{\partial V}{\partial p}\right)_T + \frac{T}{C_p}\left(\frac{\partial V}{\partial T}\right)_p^2 \tag{2.13}$$

gives the physical meaning of the material constant A, where V, p and C_p are volume, pressure and specific heat per unit volume at constant pressure respectively. If V is taken to be the volume occupied by material which occupied the unit volume in I,

$$\left(\frac{\partial V}{\partial T}\right)_p = \alpha\,, \quad \left(\frac{\partial V}{\partial p}\right)_S = -\frac{1}{K^*}\,, \quad \left(\frac{\partial V}{\partial p}\right)_T = -\frac{1}{K}\,, \tag{2.14}$$

where

$$K^* = \lambda^* + \frac{2}{3}\mu^* \tag{2.15}$$

is the isentropic bulk modulus. From (2.13)–(2.15) we have in the first approximation

$$\lambda^* = \lambda + \frac{T_0}{C_p}(K\alpha)^2 \tag{2.16}$$

and

$$A = -\frac{C_v}{2T_0} \tag{2.17}$$

by the comparison with $(2.12)_1$.

For the comparatively slow or quick deformation we may adopt $(2.10)_1$ or $(2.11)_1$ respectively as the stress-strain relation, but for the state of the middle deformation-rate $(2.9)_1$ must be correlated by the equation of heat conduction, which contains strain-rate. See e.g., Eringen[18] (chapter 8), Green and Adkins[17] and Boley and Weiner[20].

3. Ultrasonic Wave Propagations Superposing on Deformed Materials and Acoustical Birefringence

We consider the state III of propagation of ultrasonic waves superposing on the non-homogeneously and dynamically deformed materials. This state deviates from II by the displacement vector w_k, which is assumed to be infinitesimal and then in the following calculations the second and the higher order terms of w_k and of its space derivatives will be neglected. The quantities designated a prime indicate those in III.

The stress in III is given by

$$t'_{kl} = \frac{\rho'}{\rho_0}\left(\frac{\partial \Sigma}{\partial e_{mn}}\right)' \frac{\partial x_k'}{\partial X_m}\frac{\partial x_l'}{\partial X_n}\,, \tag{3.1}$$

where

$$x_k' = x_k + w_k \tag{3.2}$$

and

$$\left(\frac{\partial \Sigma}{\partial e_{mn}}\right)' = \frac{\partial \Sigma}{\partial e_{mn}} + \frac{\partial^2 \Sigma}{\partial e_{mn} \partial T}(T'-T) + \frac{\partial^2 \Sigma}{\partial e_{mn} \partial e_{pq}}(e_{pq}'-e_{pq}) \, . \tag{3.3}$$

The entropy in III is

$$S' = S - \frac{\partial^2 \Sigma}{\partial T^2}(T'-T) - \frac{\partial^2 \Sigma}{\partial T \partial e_{pq}}(e_{pq}'-e_{pq}) \, . \tag{3.4}$$

When the wave frequency is comparatively high and less than 10^9 cycle/sec at ordinary temperature, the wave deformation is regarded to be isentropic[5] (pp. 76 and 62). Putting $S'=S$ in (3.4) and eliminating $(T'-T)$ from (3.3) we have

$$\left(\frac{\partial \Sigma}{\partial e_{mn}}\right)' = \frac{\partial \Sigma}{\partial e_{mn}} + \left[\frac{\partial^2 \Sigma}{\partial e_{mn} \partial e_{pq}} - \frac{\dfrac{\partial^2 \Sigma}{\partial e_{mn} \partial T} \dfrac{\partial^2 \Sigma}{\partial T \partial e_{pq}}}{\dfrac{\partial^2 \Sigma}{\partial T^2}}\right](e_{pq}'-e_{pq}) \, . \tag{3.5}$$

Substituting (2.6), (2.8), (2.17), (3.3) and (3.5) into (3.1), we have

$$t_{kl}' = \left(1 - \frac{\partial w_m}{\partial x_m}\right) t_{kl} + t_{km} \frac{\partial w_l}{\partial x_m} + t_{ml} \frac{\partial w_k}{\partial x_m} + \tau_{kl} \, , \tag{3.6}$$

where

$$\begin{aligned} \tau_{kl} \equiv\ & (1-e)\left[\mu^*\left(\frac{\partial w_k}{\partial x_l} + \frac{\partial w_l}{\partial x_k}\right) + \lambda^* \frac{\partial w_m}{\partial x_m} \delta_{kl}\right] \\ & + 2\mu^* \left[e_{km}\left(\frac{\partial w_m}{\partial x_l} + \frac{\partial w_l}{\partial x_m}\right) + e_{ml}\left(\frac{\partial w_k}{\partial x_m} + \frac{\partial w_m}{\partial x_k}\right)\right] \\ & + 2\lambda^* \left(e_{kl} \frac{\partial w_m}{\partial x_m} + e_{mn} \frac{\partial w_m}{\partial x_n} \delta_{kl}\right) . \end{aligned} \tag{3.7}$$

The equations of motion in II and III are

$$\begin{aligned} & \frac{\partial t_{kl}}{\partial x_l} + \rho f_k = \rho \ddot{u}_k \, , \\ & \frac{\partial t_{kl}'}{\partial x_l'} + \rho' f_k = \rho'(\ddot{u}_k + \ddot{w}_k) \, , \end{aligned} \tag{3.8}$$

where f_k is the body force per unit mass. Substituting (3.6) into $(3.8)_2$ and referring $(3.8)_1$ and $\partial/\partial x_l' = \partial/\partial x_l - (\partial w_m/\partial x_l)\partial/\partial x_m$, we have

$$t_{ml} \frac{\partial^2 w_k}{\partial x_m \partial x_l} + \frac{\partial \tau_{kl}}{\partial x_l} + \frac{\partial t_{ml}}{\partial x_l} \frac{\partial w_k}{\partial x_m} = \rho \ddot{w}_k \, . \tag{3.9}$$

As the deformation is non-homogeneous, the complete plane wave does not propagate and then the wave displacement is expressed as

$$\boldsymbol{w} = \boldsymbol{W} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \,, \tag{3.10}$$

were the amplitude $\boldsymbol{W}$ and the wave vector $\boldsymbol{k}$ may be functions of e_{kl} and T of II. Expanding W_k to the first order with respect to e_{kl} and $(T-T_0)$, we obtain

$$W_k = W_{0k} + a_k(T-T_0) + b_k e + c_l e_{kl} + a_{klm} e_{lm} \,, \tag{3.11}$$

where W_{0k} is the constant amplitude in I.

By the same.reason mentioned in Section 2, a_k, b_k and c_k must be isotropic vectors and a_{klm} is an isotropic tensor of the third order. By similar analysis described in Thomas[19] we can conclude that

$$a_k = b_k = c_k = 0 \,,$$
$$a_{klm} = a \gamma_{klm} \,, \tag{3.12}$$

where a is any constant and γ_{klm} is the alternating tensor, which is antisymmetric with respect to any pair of suffixes[20]. Thus, by the same argument with $\boldsymbol{k}$, the amplitude and the wave vector may be constant within the first order of the strain and the temperature deviation. With respect to the frequency ω, it may constant along an acoustical path. Therefore the gradient of the displacement w_k may be replaced by $\boldsymbol{k} w_k$.

The propagation velocity in iron and alminium is about $3 \sim 7 \times 10^5$ cm/sec, then, when the ultrasonic frequency is adopted as larger than 5×10^6 cycle/sec, the magnitude of wave vector $k = \omega/v$ is larger than 40/cm in longitudinal wave and 90/cm in transverse wave. If the maximum strain gradient is assumed to be 10^{-3} /cm, the ratio of the strain gradient to the wave displacement gradient is less than 1/40 or 1/90. By this order estimation we may neglect the terms of the derivative of strain of the second and third terms of the left hand side of (3.9).

For simplicity the wave propagates along a principal axis of stress, and this direction is taken as x_3-axis. By the simple calculations we obtain the propagation equation

$$\mathbf{A} \boldsymbol{w} = \rho_0 v^2 \ddot{\boldsymbol{w}} \,, \tag{3.13}$$

where

$$v = \frac{\omega}{k} \tag{3.14}$$

is the wave velocity and

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_{;} + \mathbf{A}_{.} + \mathbf{A} \tag{3.15}$$

is called the acoustical tensor, where

$$\mathbf{A}_0 \equiv \begin{bmatrix} \mu^* & & 0 \\ & \mu^* & \\ 0 & & \lambda^*+2\mu^* \end{bmatrix},$$

$$\mathbf{A}_1 \equiv \left\{\frac{\mu+\mu^*}{\mu^*}t_3-\frac{2\lambda}{3K}t+\frac{4}{3}\mu\alpha(T-T_0)\right\}\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix},$$

$$\mathbf{A}_2 \equiv \frac{\lambda^*+\mu^*}{\mu^*}\left\{2t_3-\frac{2\lambda}{3K}t+\frac{4}{3}\mu\alpha(T-T_0)\right\}\begin{bmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 1 \end{bmatrix}, \tag{3.16}$$

$$\mathbf{A}_3 \equiv \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_3 \end{bmatrix}.$$

In this derivation we use the relation

$$e_{kl} = \frac{1}{2\mu}\left(t_{kl}-\frac{\lambda}{3K}t\delta_{kl}\right)+\frac{\alpha}{3}(T-T_0)\delta_{kl},$$
$$e = \frac{t}{3K}+\alpha(T-T_0), \tag{3.17}$$
$$t \equiv t_{mm}$$

deduced from $(2.9)_1$. When the deformation from I to II is isothermal, we must put $T-T_0=0$ in (3.16) and (3.17), and these coincide with Tokuoka and Iwashimizu[14]. When the deformation is isentropic, we have

$$\mathbf{A}_1+\mathbf{A}_2 = \left(2t_3-\frac{2\lambda^*}{3K^*}t\right)\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \frac{\lambda^*+2\mu^*}{\mu^*} \end{bmatrix}, \tag{3.18}$$

which is obtained from $(2.11)_1$ instead of $(2.9)_1$.

Relations (3.13)–(3.16) show that the wave separates three parts one is the longitudinal wave whose velocity is

$$v_{\parallel} = v_{0\parallel}\left[1+\frac{1}{\mu^*}\left\{2t_3-\frac{2\lambda}{3K}t+\frac{4}{3}\mu\alpha(T-T_0)\right\}+\frac{t_3}{\lambda^*+2\mu^*}\right]^{1/2}, \tag{3.19}$$

and the other two are shear waves polarized along the secondary principal axes of stress in the wave-front whose velocities are

$$v_{\perp\alpha} = v_{0\perp}\left[1+\frac{1}{\mu^*}\left\{2t_3-\frac{2\lambda}{3K}t+\frac{4}{3}\mu\alpha(T-T_0)+t_\alpha\right\}\right]^{1/2} \quad (\alpha=1,\ 2) \tag{3.20}$$

where

$$v_{0\parallel} \equiv \left(\frac{\lambda^*+2\mu^*}{\rho_0}\right)^{1/2}, \quad v_{0\perp} \equiv \left(\frac{\mu^*}{\rho_0}\right)^{1/2} \tag{3.21}$$

are longitudinal and transverse wave velocities respectively in the natural state I and t_α is the secondary principal stress.

Relation (3.20) shows that perpendicularly polarized two shear waves propagate, in general, in different velocities and we have

$$\frac{\Delta v_\perp}{v_\perp} \equiv \frac{v_{\perp1}-v_{\perp2}}{v_{\perp\,\mathrm{mean}}} = \frac{t_1-t_2}{2\mu} = e_1-e_2 \tag{3.22}$$

and

$$\delta = \omega h\left(\frac{1}{v_{\perp2}}-\frac{1}{v_{\perp1}}\right) = \alpha h(t_1-t_2)\ , \tag{3.23}$$

where we use $(2.12)_2$, and δ is the phase difference between the thickness of a specimen h and

$$\alpha \equiv \frac{\omega}{2\rho\, v_{0\perp}^{\ 3}} \tag{3.24}$$

means the acoustoelastic sensitivity.

The velocity deviation ratio (3.22) is independent of the used wave frequency and its maximum value in the proportional limit takes about 0.1~0.2 percent. The relation (3.23) shows the stress-acoustical low[14] and the acoustoelastic sensitivity (3.24) has the same value for any thermodynamical state of II e.g., isothermal or isentropic.

4. Conclusions

1. The generalization and the precise analysis from Tokuoka and Iwashimizu [14] are investigated with respect to the wave propagation in the non-homogeneously and dynamically deformed isotropic elastic material.

2. The free energy, the wave amplitude and the wave vector are expanded with respect to the strain and the temperature deviation. From the general properties of isotropic tensors, the physical interpretations of the expansion coefficients are determined and the amplitude and the wave vector may be constant within the first order approximation.

3. For the deformation from the natural state to the superposed state, three case are investigated, i.e., isothermal, isentropic and middle state between them.

With respect to the infinitesimal isentropic wave deformation the propagation equation (3.13) is obtained from the restriction: strain $< 10^{-3}$, strain gradient $< 10^{-3}$/cm, and wave frequency $> 5 \times 10^{6}$ cycle/sec.

4. When the wave vector coincides with a principal axis of stress, the wave separates into three parts: one longitudinal and two transverse waves which are polarized along the secondary principal stresses.

5. The stress-acoustical law holds, that is, the phase difference is proportional to stress-difference, wave frequency and length of acoustical path.

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