Establishment of Linear Sequences

By

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This paper deals with the sequencing problem as an initial step to the study of production system. Sequence may be classified roughly into linear sequence and compound sequence. This paper descrives the systematical method of establishing linear sequences and the problem of minimum transition value as an example of determining an optimum linear sequence. The points to analyze the former are as follows: (1) The representative method of precedence relations. (2) The systematical method of establishing linear sequences. (3) The total number of feasible linear sequences. For these purposes, the fundamental matrix which makes precedence diagram into the form available to theoretical analysis, sequential product as the operational method by which precedence relations can be handled rationally, and then the linear product by which all of the feasible sequences can be established without overlapping have been introduced. Sequences are established easily, systematically and very mechanically by linear product. The technique to pick out the suitable sequences from tremendous feasible sequences is substantial to solve the latter. For this purpose, the concept of Lower Bound has been introduced. The algorithm can assure optimality. It can cope with the case of limitation in calculation time, and gives a suitable approximate solution.

1. Introduction

A sequencing problem arises to expect some effects by establishing a sequence. This problem is always encountered, when something should be done under consideration of time for a work.

Especially in machine industry, this problem is important because the productivity is one of the purposes in machine shops and it may significantly be influenced by sequence.

The characteristics of the problem are as follows:

a. This problem is essentially combinatorial and therefore the total number of feasible solutions is generally tremendous.

b. The solutions must satisfy various technological restrictions, especially precedence restrictions.

c. One or several solutions of the feasible solutions which will optimize a certain criterion must be picked out.

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The method of establishing sequences which satisfy various technological restrictions and how to pick out a suitable sequence of the tremendous feasible solutions are substantial to solve a sequencing problem.

At present, sequencing problems may be classified into the following categories; The "Traveling Salesman" Problem, The Assembly Line Balancing Problem and The Job Shop Scheduling Problem. Considerable efforts have been done to find solutions for the problems. However, sequencing problems, which may be encountered in practical production systems, do not necessarily belong to the above-mentioned categories. The study of the sequencing problem lies in the situation where some formulations have been made for the above three problems and various approaches are being made to them. In this way, the study of the sequencing problem may be at the starting point. Many difficult problems to solve analytically still remain unsettled and future research should be expected.

It is the purpose of this study to find the policy for control and optimization of the production system by picking up the problems systematically.

In machine industry there exist two characteristic types of production, classically identified as the straight-line type and the job shop type. The former is based on work pace rather than work content, which is taken in case of manufacturing a certain product in large quantities. Conversely, the latter is based on work content rather than work pace, which is taken in the case of turning out a wide variety of products in small quantities. These two characteristic types of production are essentially so distinct that we must clarify which type we are based in discussing sequence. First, this paper describes the production system of the straight-line type.

As the basic problem, it is necessary to establish feasible solutions which satisfy precedence restrictions. An effective method has not been established up to the present time.

By paying attention to working procedure, sequence may be classified roughly into linear sequence and compound sequence. In the most basic linear sequence of all sequences, plural work elements to be accomplished by the production system are done one by one. On the other hand, in compound sequence, a set of several work elements is done one at a time. It is the purpose of this paper to find a systematical method to establish linear sequences. The points to analyze it are as follows:

(1) The representative method of precedence restrictions.

(2) The systematical method of establishing linear sequences.

(3) The total number of feasible linear sequences.

We shall begin with definitions and notations we use in this paper. We shall then define the specific problem we are takling in Section 2. Next, we shall proceed to discuss (1) in Section 3, and then to make use of information we have thus gained we shall show how the problem (2) can be solved in Section 4 and Section 5. Then we shall consider (3) in Section 6. In Section 7, we shall deal with the problem of minimum transition value and the extension often encountered in the practical production system as as an example of determining an optimum linear sequence. Finally, we shall summarize our results in Section 8.

2. Definitions and Notations

The following definitions and notations are introduced.

$\boldsymbol{X}=\{x_1, x_2, \cdots, x_n\}$	: a set of n work elements which the production system must accomplish.
$\lvert\boldsymbol{X}\rvert$	: size of $\boldsymbol{X}$
$x_i<x_j$	: x_i precedes x_j indirectly.
$x_i\ll x_j$	: x_i precedes x_j directly.
$x_i\leqslant x_j$	: x_i precedes x_j indirectly or directly.
$G=G(\boldsymbol{X}, A)$	: precedence diagram which has only one starting node by arranging, using the ranking method. A see next.
A	: a set of arrows
α_{x_p,x_q}	: an arrow which directs from x_p to x_q.
π	: partial linear sequence.
Π	: complete linear sequence.
${}^m\pi_\nu(x_1, x_p)$	: partial linear sequence of cardinal number m which leads from starting node x_1 to node x_p. As there are generally several sequences which satisfy this condition, ν must be added for distinction.
$\vee$	: sequential union. The individuals represent independent sequences.
${}^m\pi(x_1, x_p)$	: $=\bigvee_\nu {}^m\pi_\nu(x_1, x_p)$
$S\{{}^m\pi_\nu(x_1, x_p)\}$	: a set of m nodes which are contained in sequence ${}^m\pi_\nu(x_1, x_p)$
$\sigma\{{}^m\pi_\nu(x_1, x_p)\}$	: a section of ${}^m\pi_\nu(x_1, x_p)$ by x_p $=\{x_i \mid x_i\leqslant x_p,\ x_i\in S\{{}^m\pi_\nu(x_1, x_p)\}\}$

A partial linear sequence of cardinal number m means a path which transits m nodes from the starting node without disturbing precedence relations. In case the cardinal number covers all of the work elements which the production system must accomplish, this path is called complete linear sequence.

The object of this paper is to find all of the feasible complete linear sequences. They are nothing but Hamiltonian paths in network diagram and may be arrived at by using multiplication-lathine of matrix. Unfortunately, this procedure leads to serious difficulties. First, we must rewrite the precedence diagram to the corresponding network diagram by adding necessary arrows, because a transition can occur only in existence of an arrow in network diagram. By this operation the precedence diagram may, however, become much more complicated and lose the merit of expressing conveniently the precedence relations between the work elements. Second, a large number of sequences which do not satisfy the precedence relations may be made out with proper sequences in seeking for feasible linear sequences from partial to complete step by step using multiplication-lathine. This is the fatal defect for establishing linear sequences systematically. It can be said that this trouble depends on the fact that network diagram has been easily used to represent precedence relations without taking the essential difference into consideration between the precedence diagram and the network diagram.

A precedence diagram has the following characteristics in comparison with a network diagram.

1. It is the convenient representation of precedence relations, and does not indicate a path along which work elements must be done. In other words, a path can be made by transition, if precedence relations are not disturbed, whether arrows exist or not.
2. All arrows are directed in one direction (generally from left to right).

In this paper we shall find a systematical method of establishing linear sequences by making use of these characteristics and taking the above-mentioned defects off. Detailed results are given below.

3. An analysis of precedence relations and making fundamental matrix.

To establish a sequence is to make a sequence from one of a low cardinal number to one of a higher cardinal number step by step without disturbing precedence relations. Therefore we must, first of all, clarify how we can get a sequence of high cardinal number from a sequence of lower cardinal number without disturbing precedence relations. We shall then consider how to establish all of the feasible sequences systematically without overlapping.

It is necessary to analyse a precedence diagram for the first purpose which has the following characteristics as mentioned above: A path can be made by transition, if precedence relations are not disturbed, whether arrows exist or not. This results in making the conditions clear under which a transition from one node to another may occur. We then introduce a matrix equivalent to the precedence diagram for this purpose because the precedence diagram itself can not be used for the theoretical analysis. At present vaiious matrices equivalent to the precedence diagram are made known, but these are not proper for establishing sequences systematically. As the reason, it can be said that these matrices explain only the conditions under which a transition can occur. For the above purpose, it must be more fully explained in what cases a transition may occur. From these points of view, we shall arrange the conditions under which a transition can occur from one node to another directly. They are summarized as follows:

I In case there exists an arrow α_{x_p,x_q} which connects from node x_p to x_q;

1. If there is only one arrow which directs to node x_q, *i.e.*, α_{x_p,x_q}, a transition can occur without any condition.
2. If there are several arrows which direct to node x_q, that is, arrows which direct from nodes $x_r, \cdots, x_s$ to x_q, it is necessary to pass through all of them beforehand, and so a conditional transition can occur.

II In case there exists no arrow which connects from node x_p to x_q;

3. If any nodes which precede x_p correspond to all of the nodes which direct to x_q, a transition can happen without any condition.
4. If any nodes which precede x_p do not correspond to all of the nodes which direct to x_q, it is necessary to pass through all of the nodes which direct to x_q beforehand, and so a conditional transition can occur.

In case none of 1, 2, 3, 4 above can occur, the direct transition from x_p to x_q can not occur.

By making use of this fact, we can define a matrix equivalent to the precedence diagram. The matrix is called fundamental matrix.

Fundamental matrix: $L_0=(a_{i,j})$

$$a_{i,j}=\begin{cases} x_j: & \text{in case the direct transition from } x_i \text{ to } x_j \text{ can occur.} \\ 0: & \text{in case the direct transition from } x_i \text{ to } x_j \text{ can not occur.} \end{cases}$$

Furthermore, we must indicate the following $c(i, j)$ to give the information on conditional transition.

$$c(i,j)=\begin{cases} \text{an empty set:} & \text{in case of unconditional transition.} \\ \text{a set of nodes which must be passed through beforehand} & \\ \quad : & \text{in case of conditional transition.} \end{cases}$$

We shall write $c(i, j)$ in the matrix making a cell at the northwestern corner of the box of each row and column. It depends on the second characteristics of the precedence diagram that the information on direct transition can be arranged so easily.

An arrangement of work elements in row and column is put in ranking order from up to down and from left to right.

For instance, the fundamental matrix for the precedence diagram shown in Fig. 1 is given in Fig. 2. In Fig. 2 the box of row 7 column 8 means that the direct transition can occur after passing through nodes 2, 5 and 6. Fig. 3 shows within which categories as mentioned above the ways of direct transition come in case direct transitions can occur in Fig. 2.

By introducing this matrix, we can use the information on precedence diagram in a concrete form.

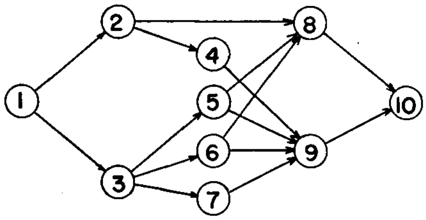


Fig. 1. Precedence diagram.

$L_0 =$

	1	2	3	4	5	6	7	8	9	10
1	0	2	3	0	0	0	0	0	0	0
2	0	0	3	4	(3) 5	(3) 6	(3) 7	(5,6) 8	0	0
3	0	2	0	(2) 4	5	6	7	0	0	0
4	0	0	3	0	(3) 5	(3) 6	(3) 7	(5,6) 8	(5,6,7) 9	0
5	0	2	0	(2) 4	0	6	7	(2,6) 8	(4,6,7) 9	0
6	0	2	0	(2) 4	5	0	7	(2,5) 8	(4,5,7) 9	0
7	0	2	0	(2) 4	5	6	0	(2,5,6) 8	(4,5,6) 9	0
8	0	0	0	4	0	0	7	0	(4,7) 9	(9) 10
9	0	0	0	0	0	0	0	8	0	(8) 10
10	0	0	0	0	0	0	0	0	0	0

Fig. 2. Fundamental matrix.

	1	2	3	4	5	6	7	8	9	10
1		1	1							
2			3	1	4	4	4	2		
3		3		4	1	1	1			
4			3		4	4	4	4	2	
5		3		4		3	3	2	2	
6		3		4	3		3	2	2	
7		3		4	3	3		4	2	
8				3			3		4	4
9								3		4
10										

Fig. 3.

4. Sequential product

The next problem to be discussed is to clarify the process with which sequences of high cardinal number can be made from a sequence of lower one. For this we must remember the difficulties that we encountered to adopt a network diagram for establishing sequences. That is, it is essential to consider the following for this problem:

(a) Not to disturb precedence relations

(b) Not to include cycle

These will be settled by making use of fundamental matrix above-mentioned for (a) because it makes the conditions of direct transition clear and by establishing the item which forbids cycle for (b). Taking these into consideration and arranging the process in case of establishing sequences from partial to complete successively with fundamental matrix, we get a flow chart as shown in Fig. 4. In the figure,

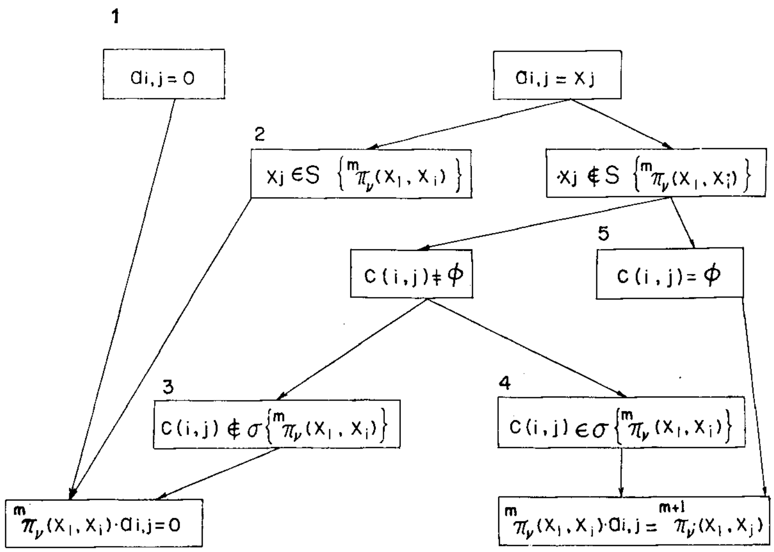


Fig. 4. Sequential product.

(1) means that there is no partial linear sequence in case the direct transition from the ending node x_i of ${}^m\pi_\nu(x_1, x_i)$ to node x_j can not occur. (2) is established to forbid cycle. (3) excludes the case which violates the precedence relations. Finally, we can get a new sequence in case (4) or (5) condition is satisfied. The suffix ν' shows the corresponding relation to ν. We call this process sequential product. We can get sequences of high cardinal number from sequences of lower one by using this process successively.

The basis of the method of establishing sequences we discuss in this paper lies in the very sequential product. We can handle precedence restrictions in the most rational form by introducing sequential product, and clarify the process of establishing sequences, which satisfy precedence restrictions, only with sequential product. By arranging precedence restrictions with the concept of direct transition, it can be made much easier to handle precedence restrictions analytically. Furthermore, adding some consideration for cycle to the idea and bringing the process of establishing sequences into the form of sequential product, unproper sequences can never be made out. Sequential product makes it possible to exclude all the defects.

5. Linear product

We have settled the problems on the basic operations we encounter in establishing sequences by introducing fundamental matrix and sequential product. Then the problem finally to be discussed is to settle a establishing method of linear sequences by using them. For this purpose it may be suitable to introduce the operational method of multiplication of matrix. As a result this will be arranged in the following form, called linear product: We assume that the product of individual elements obey sequential pıoduct on making the product of $(1, n)$ matrix by L_0.

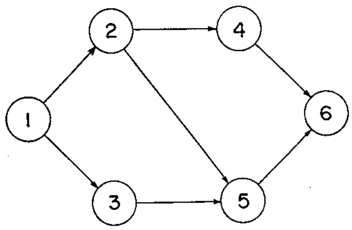
Then

linear product: $[M]^{m+1}=[M]^{m}\cdot L_0$. (*)

where, supposing that

$$\begin{matrix} & & x_1 & x_2 & \cdots & x_i & \cdots & x_n \\ [M]^m & = & x_1[0 & \pi^m(x_1, x_2) & \cdots & {}^m\pi(x_1, x_i) & \cdots & {}^m\pi(x_1, x_n)] \end{matrix}$$

(i, j)-element, *i.e.*, ${}^{m+1}\pi(x_1, x_j)$ of matrix $[M]^{m+1}$ is,



(a) Precedence diagram.

	1	2	3	4	5	6
1	0	2	3	0	0	0
2	0	0	3	4	3/5	0
3	0	2	0	2/4	2/5	0
4	0	0	3	0	3/5	5/6
5	0	0	0	4	0	4/6
6	0	0	0	0	0	0

(b) Fundamental matrix.

$[M]^1=$	0	2	3	0	0	0
$[M]^2=$	0	1·2	1·3	0	0	0
$[M]^3=$	0	1·3·2	1·2·3	1·2·4	0	0
$[M]^4=$	0	0	1·2·4·3	1·3·2·4 1·2·3·4	1·3·2·5 1·2·3·5	0
$[M]^5=$	0	0	0	1·3·2·5·4 1·2·3·5·4	1·2·4·3·5 1·2·3·4·5 1·3·2·4·5	0
$[M]^6=$	0	0	0	0	0	1·3·2·5·4·6 1·2·3·5·4·6 1·2·4·3·5·6 1·3·2·4·5·6 1·2·3·4·5·6

(c)

Fig. 5.

$$^{m+1}\pi(x_1, x_j) = \bigvee_{\nu'} {}^{m+1}\pi_{\nu'}(x_1, x_j) = \bigvee_{i=1}^{n} (\bigvee_{\nu} {}^{m}\pi_{\nu}(x_1, x_i) \cdot a_{i,j})$$

We adopt the row of starting node as $[M]^1$ and the matrix, where the element x_1 corresponding to the starting node is written in the form of sequential product ahead of the elements which are not zero in $[M]^1$, as $[M]^2$.

We can get sequences of higher cardinal number successively by the equation (*) and finally complete linear sequences by $[M]^n$. The precedence diagram is shown in Fig. 5 (a) as an example and the fundamental matrix is given in Fig. 5 (b). By the steps in Fig. 5 (c), we finally get the five complete linear sequences. In fact there are no more feasible sequences other than these five sequences. In the next section it will be explained that there are no more sequences in this case except these five sequences, but it may be obvious from the origin of linear product.

The characteristics of linear product is that sequences of high cardinal number can easily be established successively without uselessness by sequential product and also systematically by operational method of multiplication of matrix.

6. The total number of distinct feasible linear sequences

We will discuss how many feasible sequences there are, in relation to the problem of establishing linear sequences which satisfy precedence relations. This is not necessarily the fundamental problem on the settlement of the sequencing problem, but this total number can give various multipliers on the occasion of calculation. Klein and others are concerned about the same kind of problem, but did not find an effective method. The basic way of thinking in this paper is as follows: To seek the total number of feasible sequences is to apply the combinatorial theory to this problem by grouping precedence relations to several groups and rearranging it to the form easier to find them. From this point of view, a basic method can be introduced and called the box method.

6.1 The box method

This is a method that seeks the total number of linear sequences by aligning the boxes whose number is equal to that of work elements the production system must accomplish, and by considering the positions of the work elements in the boxes without disturbing precedence relations. Elementary Lemmas applicable to this method are as follows:

Lemma 1. In a set of work elements $X=\{x_1, x_2, \cdots, x_n\}$, assumed that the following precedence relations are given:

$$\left.\begin{array}{l} P: x_{h_1} \leqslant x_{h_2} \cdots \leqslant x_{h_s} \\ Q: x_{i_1} \leqslant x_{i_2} \cdots \leqslant x_{i_t} \\ \quad \cdots\cdots \\ R: x_{j_1} \leqslant x_{j_2} \cdots \leqslant x_{j_u} \end{array}\right\} \quad \text{where, } s+t+\cdots+u \leqq n$$

If the precedence relations P, Q, $\cdots$, R are independent of one another, that is, there is no common element at all, the total number of feasible sequences is

$$N = n!/s!t! \cdots u!$$

Lemma 2. Divide all of the work elements in X into the work elements restricted in position by their location of linear sequence (fixed work elements), and the work elements whose positions are not restricted (variable work elements) by precedence relations:

$$\left.\begin{array}{ll}\text{Fixed work elements} & : x_{k_1}, x_{k_2}, \cdots, x_{k_v} \\ \text{Variable work elements} & : x_{l_1}, x_{l_2}, \cdots, x_{l_w}\end{array}\right\} \text{ where, } v+w=n$$

Then the total number of linear sequences is

$N=nCv\cdot$(the possible number of combinations in the fixed work elements)$\cdot$ $(n-v)!$

Lemma 1 is self-evident. The proof of Lemma 2 is as follows: In $X=\{x_1, x_2, \cdots, x_n\}$, there are ways of nCv kinds in taking the positions of the boxes for fixed work elements, and of $(n-v)!$ kinds for variable work elements because the order of the set are quite unrestricted. Also some possible combinations can exist among the fixed work elements. So we get the above equation.

This method is applicable to the case where precedence relations can be divided into some independent ones by proper contribance, but some supplementary methods mentioned below should be used in case precedence relations are too complicated to divide properly.

6.2 The fixing method

This is a method which cuts the complicated precedence relations into the simpler ones by grouping and fixing a certain work element to the specific position of the boxes. As the fixing work element, we select the one that makes the precedence relations simpler by fixing, or that has a smaller number of possible positions in coming out in complete linear sequence.

For example, we shall seek the total number of feasible linear sequences for Fig. 1 by this method. Following the box method, we make ten boxes. Then work element 1 enters the first box and 10 the last box. If we fix our attention upon the work element 9, this work element enters box 8 or 9.

We divide the precedence relations into two cases by the fixing method. There are 90 kinds in which case 9 enters the box 8, and 156 kinds in which case 9 enters the box 9. They are independent of each other, so we understand that there are 246 kinds of complete linear sequences. This method is effective however complicated the precedence relations are. We never fail to seek the total number of feasible linear sequences by this method.

6.3 The inverse arrow method

We describe this method as an example of the case where we can get the total number more easily by the combinatorial theory. In the method, we abbreviate the arrow for a while which prevents the application of the box method, for instance, α_{x_p,x_q} and seek the number of complete linear sequences $N(S)$ in that case by using Lemmas. We denote the number of linear sequences which satisfy the precedence relation $x_p \leqslant x_q$ by $N(S_{p,q})$, and the number of linear sequences which satisfy the precedence relation $x_q \leqslant x_p$ by $N(S_{q,p})$. Then,

$$N(S) = N(S_{p,q}) + N(S_{q,p})$$

Therefore,

$$N(S_{p,q}) = N(S) - N(S_{q,p})$$

In some cases of the precedence diagram, $N(S_{q,p})$ can be arrived at more easily in comparison with $N(S_{p,q})$. This method is effective in this case. Even if we abbreviate two or more arrows, the same system is applicable. For instance, in case of abbreviating arrows, α_{x_p,x_q}, α_{x_r,x_s},

$$N(S_{p,q} \cdot S_{r,s}) = N(S) - N(S_{q,p}) - N(S_{s,r}) + N(S_{q,p} \cdot S_{s,r})$$

For the precedence diagram shown in Fig. 6, we abbreviate arrows $\alpha_{3,4}$ and $\alpha_{2,5}$. From Lemma 1, $N(S)=6$

$$N(S_{4,3}) = N(S_{5,2}) = 1 . \quad N(S_{4,3} \cdot S_{5,2}) = 0$$

and therefore

$$N(S_{3,4} \cdot S_{2,5}) = 6-1-1+0 = 4$$

This is the total number of feasible linear sequences for the precedence diagram shown in Fig. 6.

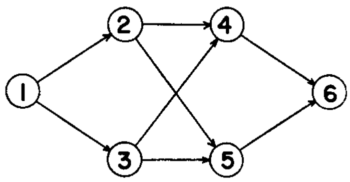


Fig. 6. Precedence diagram.

6.4 The grouping method

In the large scale of the production system, the method is often chosen which considers the system by dividing it into several subgroups from the point of analysis and various practical restrictions. We describe the grouping method applicable to such a method. This is a method which makes precedence relations simple by dividing plural work elements into several subgroups and seeks the number of complete linear sequences on that case and then counts the complete linear sequences lost by grouping. In this case it is desirable for the lost complete linear sequences to be counted easily for the way of grouping. We assume that, after all of the work elements of one subgroup are done, a transition may occur to another with precedence relations between subgroups. Under this assumption, the following situation may occur: In case a certain node x_α of an arbitrary subgroup is connected to other node x_β of the same subgroup by an arrow, a direct transition can not occur from node x_α to any node of other subgroups. Moreover, in case a certain node x_γ of an arbitrary subgroup is connected to another node x_δ of the same subgroup by an arrow, a direct transition can not occur from any node of other subgroups to node x_δ. These facts can be used to count the number of complete linear sequences lost by grouping. Further, we must pay attention to the following. Under the above-mentioned assumption no complete linear sequence may come to exist by way of grouping. We can judge it by intuition in simple cases, while we utilize the following fact in complex cases. We assume that a set of work elements X are divided into subgroups $W_1, W_2, \cdots, W_m$ and make out (m, m) matrix whose elements are $W_1, W_2, \cdots, W_m$. As the $(i.j)$-element of the matrix we shall give 1 in case there is at least one arrow which directs from a node of W_i to a node of W_j and, 0 otherwise. We shall then rank subgroups $W_1, W_2, \cdots, W_m$ by this matrix. If there is no cycle in this case, complete linear sequences do exist.

We apply this method to the precedence diagram shown in Fig. 1, paying attention to this fact. We cut the diagram in Fig. 1 to two subgroups W_1, W_2 as shown in Fig. 7. In this case there exists the precedence relation $W_1 \leqslant W_2$ between the two subgroups. So complete linear sequences do exist apparently.
The number of them N is

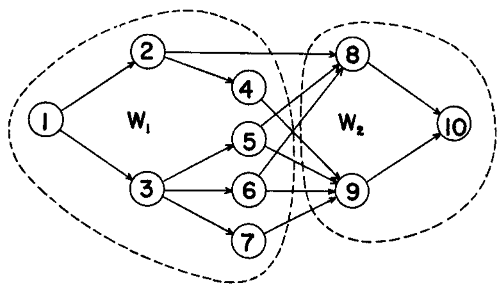


Fig. 7. Precedence diagram.

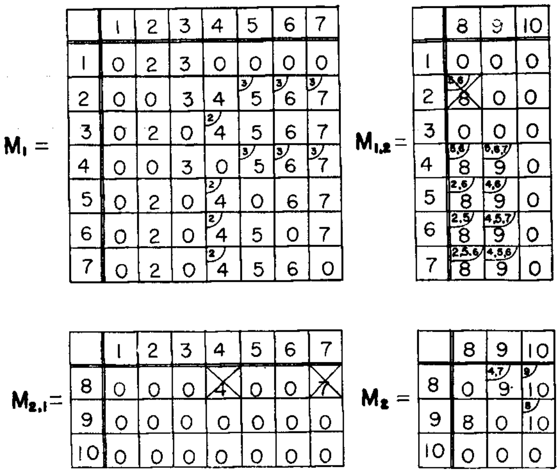


Fig. 8. Fundamental matrix.

$$N = 90 \times 2 = 180$$

The fundamental matrix of this case is shown in Fig. 8, where

M_1 : the fundamental matrix of subgroup W_1

M_2 : " " W_2

$M_{1,2}$: the transition matrix from W_1 to W_2

$M_{2,1}$: " " from W_2 to W_1.

The transition matrix $M_{2,1}$ loses its meaning from the above assumption, and moreover the arrow $\alpha_{2,8}$ in $M_{1,2}$ gets lost by grouping. Hereupon we must count the number without overlapping. We denote the number of complete linear sequences which include transition $\alpha_{2,8}$, $\alpha_{8,4}$, and $\alpha_{8,7}$ by N_2, N_3, and N_4 respectively, and transitions $\alpha_{2,8}$ and $\alpha_{8,4}$, $\alpha_{2,8}$ and $\alpha_{8,7}$, $\alpha_{8,4}$, and $\alpha_{8,7}$, and $\alpha_{2,8}$, $\alpha_{8,4}$ and $\alpha_{8,7}$ by $N_{2,3}$, $N_{2,4}$, $N_{3,4}$ and $N_{2,3,4}$ respectively. Then the total number of complete linear sequences is

$$\begin{aligned} N &= N_1+N_2+N_3+N_4-N_{2,3}-N_{2,4}-N_{3,4}+N_{2,3,4} \\ &= 130+10+38+28-8-2-0+0 = 246 \end{aligned}$$

This number corresponds with the one seeked previously. The lost number of sequences is 27% of all sequences. If we apply the methods developed above to the more practical problem shown in Fig. 9, we get about 1.8×10^{12} sequences (accurately 1,762,551,451,584) as a result. The number of sequences in case there is no precedence relation is $29! \fallingdotseq 8.8 \times 10^{30}$. In this case the feasible linear sequences by the precedence relations are about 2.0×10^{-17}% of all sequences. This fact indicates that it is very important to establish sequences systematically.

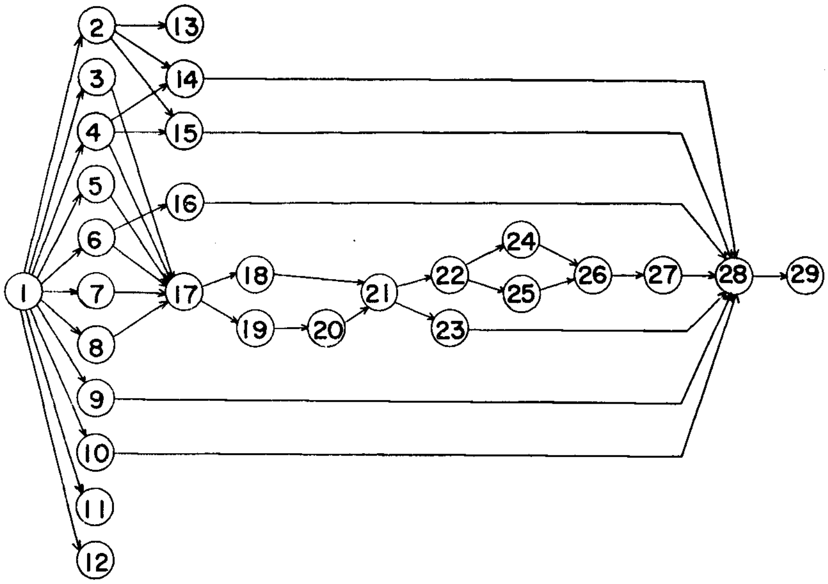


Fig. 9. Precedence diagram.

As linear product never counts the sequences which violate precedence relations, it may be well known how effective and useful linear product is in establishing linear sequences.

7. Determination of an optimum linear sequence

There is the following basic problem of establishing a linear sequence: 'There are the work elements which the production system must accomplish. It needs time to transit from an arbitrary work element x_i to another work element x_j, and the time differs with pairs of work elements. When precedence relations are imposed on the work elements, what sequence shall we make out to complete all of the work elements in the smallest time without disturbing the precedence relations?' For the problem, present knowledge is not enough to give an effective method. The reason is why precedence relations make the problem very complicated. The purpose of this section is to find out an effective approach to this problem by the establishing method of linear sequences. One more technique is necessary for this purpose, that is, the technique on the operation to pick out suitable sequences from tremendous feasible sequences. There are several methods for that. In this paper we use the concept of Lower Bound, which is the basis of the Branch & Bound Algorithm used by Little et al for the 'Traveling Salesman' problem. This has the following merit: It makes up for the lack of the criterion selecting work elements on including work elements in sequence. But the Branch & Bound Algorithm is originated from the assignment problem, so it can not be used as it is in existence of precedence relations. That is, several new problems come to exist. By settling these problems and introducing linear product, this paper develops the algorithm effective to solve the basic problem.

7.1 The Formulation of The Problem.

Time, which is necessary to shift from a work element x_i to another work element x_j, is called transition value by restating it generally and denoted by $S(i, j)$. That is, $S(i, j)$ does not always mean time, but in some cases distance, cost, value of other criterion, or weighted mean value of other criteria. When precedence relations are imposed on X, a finite value is given to $S(i.j)$ for $a_{i,j} \neq 0$ and a infinite value for $a_{i,j}=0$ in the fundamental matrix. These transition values form a matrix, but we use the form of L_0 for convenience'sake by making cell at the southeastern corner of each box in L_0. For the precedence diagram shown in Fig. 1, for example, the fundamental matrix L_0 with transition values is given in Fig. 10.

$L_0 =$

	1	2	3	4	5	6	7	8	9	10
1	0	2_{11}	3_{88}	0	0	0	0	0	0	0
2	0	0	3_{53}	4_{6}	$^{3}5_{9}$	$^{3}6_{11}$	$^{3}7_{5}$	$^{5,6}8_{92}$	0	0
3	0	2_{6}	0	$^{2}4_{97}$	5_{33}	6_{94}	7_{24}	0	0	0
4	0	0	3_{20}	0	$^{3}5_{28}$	$^{3}6_{24}$	$^{3}7_{89}$	$^{5,6}8_{74}$	$^{5,6,7}9_{75}$	0
5	0	2_{61}	0	$^{2}4_{18}$	0	6_{19}	7_{74}	$^{2,6}8_{67}$	$^{4,6,7}9_{23}$	0
6	0	2_{81}	0	$^{2}4_{63}$	5_{33}	0	7_{98}	$^{2,5}8_{29}$	$^{4,5,7}9_{68}$	0
7	0	2_{68}	0	$^{2}4_{82}$	5_{34}	6_{8}	0	$^{2,5,6}8_{83}$	$^{4,5,6}9_{62}$	0
8	0	0	0	4_{42}	0	0	7_{7}	0	$^{4,7}9_{12}$	$^{9}10_{83}$
9	0	0	0	0	0	0	0	8_{61}	0	$^{8}10_{2}$
10	0	0	0	0	0	0	0	0	0	0

Fig. 10. Fundamental matrix with transition values.

We adopt the following notation as the expression of a sequence.

$$\Pi = (x_1, x_2)(x_2, x_3)\cdots(x_{n-1}, x_n)$$

That is, when $|X| = n$, Π is expressed by $(n-1)$ ordered work pairs. In case the work pair (x_i, x_j) belongs to Π, we express it by

$$(x_i, x_j) \in \Pi$$

The transition value of a complete linear sequence is expressed by the total sum of each element which belongs to Π.

The problem is to find out Π which minimize the criterion:

$$Z(\Pi) = \sum_{(i,j)\in\Pi} S(i,j)$$

Hereupon, Π must satisfy the precedence relations which are imposed on the problem, and pick out one and only one value in each row and in each column.

7.2 Algorithm

The basic method to construct the Algorithm is to break up the set of all complete linear sequences into smaller and smaller subsets without disturbing the precedence relations, to calculate for each subset a lower bound of the transition value of the best complete linear sequences therein, then to select the subset with minimum lower bound and to divide it into smaller ones.

Hereupon we must pay attention not to disturb the precedence relations. The means which settle them rationally are sequential product and linear product. The Algorithm is based on the concept of lower bound developed by little et al in picking out the proper sequences from many feasible sequences. It can not, however, be used as it is because the problem is placed under precedence relations, and therefore several new problems come to exist. The Algorithm is developed to solve the basic problem by settling these problems and introducing sequential product and linear product for the first time, and therefore it has the following characteristics in comparison with theirs.

1. Linear product applies correspondingly to the process of branching because it must be carried out without disturbing the precedence relations. That is, a pair of work elements which compose Π must be picked out from the first one.
2. The number of branches which come from a knot is equal to that of work elements capable of direct transition from the pair of work elements of its knot.
3. It is not necessary to consider a circuit. Instead of this, we must rewrite the transition value into a finite value when we get elements which can't transit directly by the transitional conditions in case of breaking the set of all complete linear sequences into smaller subsets.
4. We do not use the following value:

$\theta(i,j)$ = the value of the smallest element in row i excluding $S(i,j)$ plus the value of the smallest element in column j excluding $S(i,j)$

These are the main characteristics. Detailed results are given below.

7.2.1. Tree Diagram

The process of Branching which splits the set of all feasible complete linear sequences into disjoint subsets will be represented by a tree diagram as illustrated

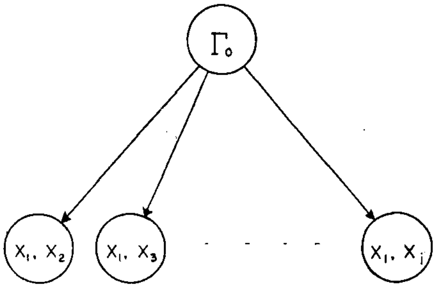


Fig. 11. Tree diagram.

in Fig. 11. The knot containing 'Γ_0' represents the set of all feasible linear sequences. So the cardinal number is equal to the total number of all feasible linear sequences. The number may be arrived at easily by the methods developed in section 6. Branching is made from knot 'Γ_0'. Pair (x_1, x_j) is the one from work element x_1 that corresponds to the starting node to work element x_i by the direct transition. Knot '(x_1, x_i)' represents the set of all complete linear sequences composed from partial linear sequence ${}^2\pi(x_1, x_i)$. Therefore, the number of the knots branched from 'Γ_0' is equal to that of the work elements capable of direct transition from x_1. The complete linear sequences included in each knot are independent of one another. The cardinal number of the set of knot '(x_1, x_i)' can be equal to the total number of all complete linear sequences which are made out from partial linear sequence ${}^2\pi(x_1, x_i)$. Therefore, the sum of the cardinal number of the sets represented by the knots of work elements capable of direct transition from x_1 is equal to the cardinal number of the knot before branching. This fact means there is no sequence lost by branching. At any stage of the process of branching, the union of the sets represented by the terminal knots is the set of all feasible complete linear sequences. This is the background of guarantee to give the optimality of the complete linear sequence.

7.2.2. Lower Bound

Following the concept of lower bound, we consider how much transition value is necessary at least to complete all of the work elements which the production system must accomplish. This value may be obtained by reducing rows and columns in L_0. We express the lower bound by $l(\Gamma_0)$. $l(\Gamma_0)$ for Fig. 10 is given as follows:

$$l(\Gamma_0) = 11+5+6+20+15+29+8+7+2+4+5 = 112$$

L_0 may be transformed to a matrix with non negative elements and at least one zero in each row and column this way, it is called the reduced matrix of L_0 and expressed by L_1. The reduced matrix L_1 for Fig. 10 is given in Fig. 12. In the figure we

$L_1 =$

	1	2	3	4	5	6	7	8	9	10
1		0	77							
2			48	1	0	6	0	87		
3		0		91	23	88	18			
4			0		4	4	69	54	50	
5		46		0		4	59	52	3	
6		52		54	0		69	0	34	
7		60		74	22	0		75	49	
8				35			0		0	56
9								59		0
10										

Fig. 12. Reduced matrix.

indicate only transition values for convenience'sake

$Z(\Pi)$: the sum of the transition value of sequence Π under L_0.

$Z_{L_1}(\Pi)$: the sum of the transition value of sequence Π under L_1.

Then,
$$Z(\Pi) = l(\Gamma_0) + Z_{L_1}(\Pi)$$

Therefore, we can say that the complete linear sequence with the smallest sum of the transition value under L_0 has also the smallest one under L_1. Expansion of this consideration gives the algorithm to solve the basic problem.

7.2.3. Flow Chart

A flow chart of the algorithm for the problem is shown in Fig. 13. Block 1 arranges the problem in the form to which algorithm can apply. Block 2 calculates $l(\Gamma_0)$ and makes the first knots 'Γ_0' of the tree diagram. Block 3 makes branching. If the direct transition from x_1 to x_i can occur under L_1, the lower bound $l\{^2\pi(x_1, x_i)\}$, which a set of the complete linear sequences composed of $^2\pi(x_1, x_i)$ have, increases by the transition value from x_1 to x_i, *i.e.*, $S_{L_1}(x_1, x_i)$ because $l\{^2\pi(x_1, x_i)\}$ includes $^2\pi_1(x_i, x_1)$. Suffix L_1 in S shows the transition value under the matrix L_1. Then, row x_1 and column x_i are crossed off because they are no longer needed. Moreover, by selecting $^2\pi(x_1, x_i)$, the elements which can not transit directly from x_i come to exist. Therefore their values are transformed into infinite values. That is, if $c(i, p)$ in each box of row x_i does not satisfy the following condition:

$$c(i, p) \in S\{^2\pi(x_1, x_i)\}$$

we set $S(i, p) = \infty$.

The $(n-1, n-1)$ matrix at the result is transformed to the reduced matrix $L_2(x_i)$ which has at least one zero in each row and column. The sum of reducing

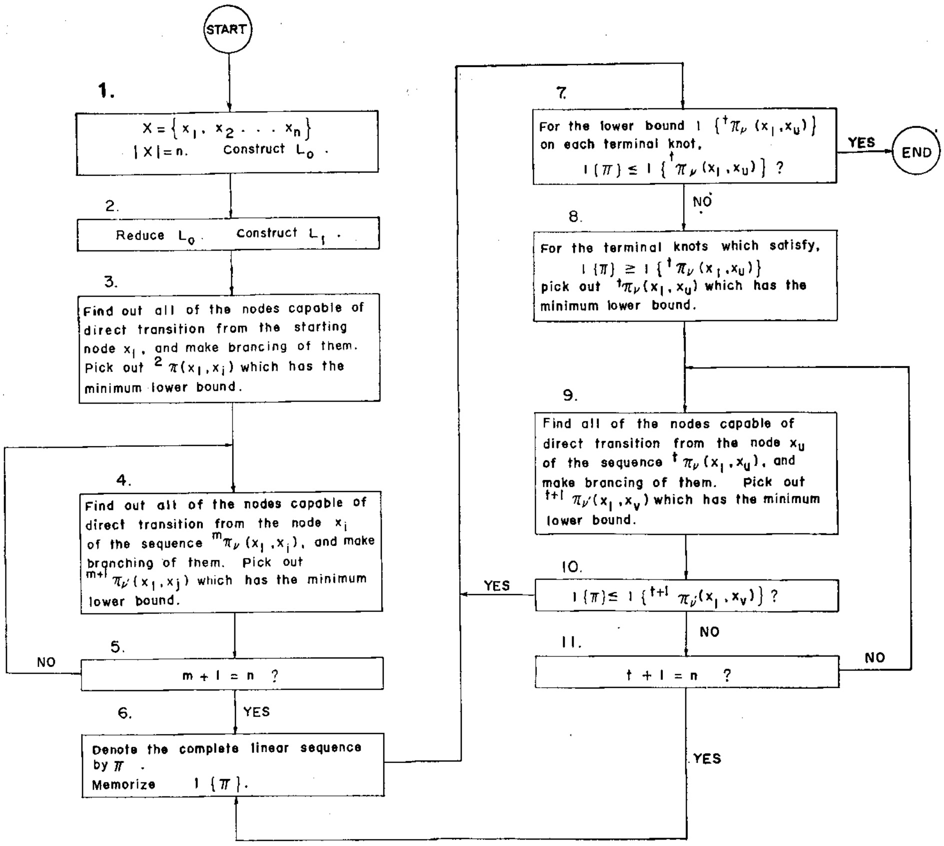


Fig. 13. Flow chart.

constants $H_{L_1}(x_1, x_i)$ increases $l\{^2\pi(x_1, x_j)\}$ as a lower bound corresponding to L_1. Terefore,

$$l\{^2\pi(x_1, x_i)\} = l(\Gamma_0) + S_{L_1}(x_1, x_i) + H_{L_1}(x_1, x_i)$$

As there are generally several work elements capable of direct transition from x_1, the lower bound for each of them must be calculated the same way. We pick out $^2\pi(x_1, x_i)$ which has the minimum lower bound. There is no guarantee of the optimality of the complete linear sequence composed of partial linear sequence $^2\pi(x_1, x_i)$, but at this step $^2\pi(x_1, x_i)$ has the highest possibility of optimality. Block 4 does the following operations:

(1) Find out work element x_j capable of direct transition from x_i of the selected partial linear sequence $^m\pi(x_1, x_i)$ $(m \geqq 2)$

(2) Delete row x_i and column x_j from $L_m(x_i)$. As there are generally several reduced matrices L_m, we denote L_m capable of direct transition to x_i by $L_m(x_i)$ to distinguish them clearly.

(3) Set $S(j, q)=\infty$ for $c(j, q)$ in each box of row x_j for the following condition:

$$c(j, q) \in S\{^m\pi(x_1, x_i)\}$$

(4) Reduce the matrix at the result and make $L_{m+1}(x_j)$. Seek $S_{L_m(x_i)}(x_i, x_j)$, $H_{L_m(x_i)}(x_i, x_j)$, and calculate

$$l\{^{m+1}\pi(x_1, x_i)\} = l\{^m\pi(x_1, x_i)\}+S_{L_m(x_i)}(x_i, x_j)+H_{L_m(x_i)}(x_i, x_j)$$

make a branch of knot '(x_i, x_j)'.

(5) Perform the above mentioned operations (1)–(4) about all of the work elements capable of direct transition from x_i.

(6) Find out knot '(x_i, x_j)' which gives the minimum lower bound and select linear sequence $^{m+1}\pi(x_1, x_j)$.

If this process is carried far enough, a complete linear sequence will be eventually produced. Block 5 checks whether a complete linear sequence is arrived at or not. In Block 6, the complete linear sequence is denoted by Π and the value $l\{\Pi\}$ is memorized. Block 7 checks if the complete linear sequence has a value less than or equal to the lower bounds on all terminal knots of the tree diagram. If there are terminal knots of values less than $l\{\Pi\}$, the terminal knot which has the minimum value is picked out in Block 8. Block 9 makes a branch and does the same kind of operations mentioned in Block 4. In Block 10, if the value after branching is larger than $l\{\Pi\}$, go to Block 7. If the value after branching is less than $l\{\Pi\}$, go to Block 11. Block 11 checks whether a complete linear sequence is produced, and go to Block 9 or Block 6 with the result. When the condition of Block 7 is satisfied finally, an optimal complete linear sequence can be gotten.

Now, we shall apply the algorithm to the above example. The work elements capable of direct transition from the first element 1 are 2 and 3, and therefore row 1 and column 2, and row 1 and column 3 are deleted from L_1. Fig. 14 and Fig. 15 show $L_2(2)$ and $L_2(3)$ respectively. In Fig. 14 the values of the boxes of column 5, 6, 7, 8 in row 2 are set to infinite values. In Fig. 15 the value of the box of column 4 in row 3 is set to an infinite value.

$$S_{L_1}(1, 2) = 0 \quad H_{L_1}(1, 2) = 19$$
$$S_{L_1}(1, 3) = 77 \quad H_{L_1}(1, 3) = 4$$

Therefore,

$$l\{^2\pi(1, 2)\} = 112+0+19 = 131$$
$$l\{^2\pi(1, 3)\} = 112+77+4 = 193$$

Partial linear sequence $^2\pi(1, 2)$ must be picked out. Fig. 16 shows the results of

$L_2(2) =$

	1	3	4	5	6	7	8	9	10
2		47	0						
3			73	5	70	0			
4		0		4	4	69	54	50	
5			0		4	59	52	3	
6			54	0		69	0	34	
7			74	22	0		75	49	
8			35			0		0	56
9							59		0
10									

Fig. 14.

$L_2(3) =$

	1	2	4	5	6	7	8	9	10
2			1	0	6	0	87		
3		0		23	88	18			
4				0	0	65	50	46	
5		46	0		4	59	52	3	
6		52	54	0		69	0	34	
7		60	74	22	0		75	49	
8			35			0		0	56
9							59		0
10									

Fig. 15.

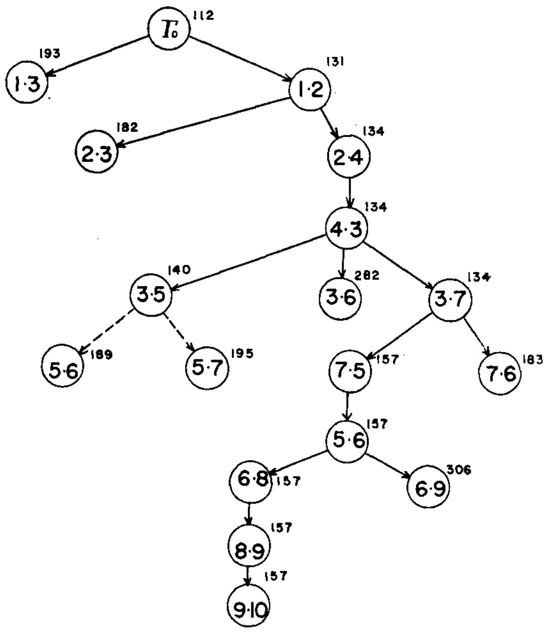


Fig. 16. Tree diagram.

calculation after Block 7. The number on the right of each knot shows the lower bound. The following complete linear sequence is eventually obtained at first:

$$\Pi = 1 \cdot 2 \cdot 4 \cdot 3 \cdot 7 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 10 \,.$$

In the figure, the results checked up by the Blocks after 7 are shown in dotted line. In the final analysis, it is shown that the complete linear sequence obtained first is the optimum solution. Fig. 1 has 246 complete linear sequences as mentioned in section 6.

The number of the steps used to determine an optimum complete linear sequence is 17.

7.3 A more generalized problem

In the basic problem, the times of work elements themselves were not considered. It is because the times may often be regarded to be independent of sequence. In this section, the problem, in case the times depend on sequence, is handled. This arises, for instance, in the following situations: 'There are several methods for a work element, and the time, called operation value, is different from one another. Hence, a transition value is given to a pair of methods for a work element and another work element. What sequence shall we make out to complete all of the work elements in the smallest time using proper methods without disturbing precedence relations?

The algorithm developed above can also cope with such a problem sufficiently. Here, the operation value of a method (a) to carry out a work element (i) is expressed by $T(i_a)$, and the transition value from the method (a) of the work element (i) to the method (b) of the work element (j) by $T(i_a, j_b)$. If we introduce the following equation,

$$S(i_a, j_b) = T(i_a) + T(i_a, j_b) \qquad (*)$$

the above-mentioned algorithm can be applied after a little modification by making out a matrix $(S(i_a, j_b))$. In this case it is necessary for a precedence diagram to have only one ending node in addition to one starting node by arranging. But if the node of last rank in the precedence diagram is only one, the above-mentioned modification is not necessary, introducing the following instead of the equation $(*)$ for the node of last rank.

$$S(m_c, n_d) = T(m_c) + T(m_c, n_d) + T(n_d)$$

Now, there are, for example, several methods (a, b, c) for each node in the precedence diagram as shown in Fig. 17 and the matrix for them is given in Fig. 18. Fig. 19 shows the result of calculation obtained by the algorithm after a little modification. In the figure, the simbols a, b, c on the arrows of the inside of the knot show the methods for the precedent knot. This means that a better sequence can be obtained by the method on the arrow in comparison with other methods. In this case, the optimum complete linear sequence is not the first, but the second:

$$\Pi = 1 \cdot 3_b \cdot 2_c \cdot 5 \cdot 7_b \cdot 4 \cdot 6_b \cdot 9_b \cdot 8_a \cdot 10$$

In Fig. 17, the number of the distinct feasible complete linear sequences is 53136.

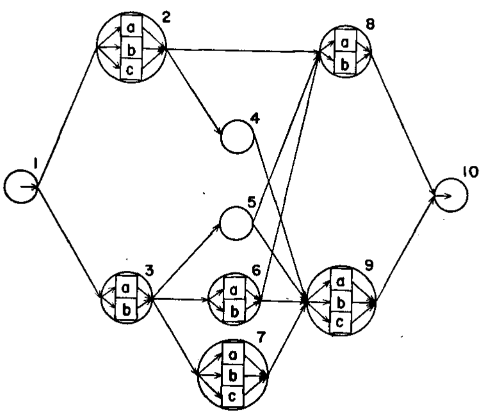


Fig. 17. Precedence diagram.

		1	2			3		4	5	6		7			8		9			10
			a	b	c	a	b			a	b	a	b	c	a	b	a	b	c	
1			11	88	53	6	9													
2	a					6	97	68	82	34	8	83	25	40	58	40				
	b					20	28	62	42	7	12	63	34	39	2	92				
	c					75	61	61	2	73	36	85	67	28	50	49				
3	a		15	19	74			61	38	93	73	68	76	23						
	b		5	64	12			80	58	35	6	88	73	48						
4						44	6		93	55	39	26	27	70	98	76	68	78	36	
5			77	82	96			60		17	18	48	16	34	92	19	52	98	84	
6	a		24	10	70			59	62			42	53	67	14	95	29	84	65	
	b		50	64	7			49	54			14	18	50	54	18	82	23	79	
7	a		75	73	51			12	53	67	51				11	67	73	22	32	
	b		76	18	36			16	28	25	82				70	54	87	49	48	
	c		79	17	37			64	21	91	15				14	52	11	39	7	
8	a							12				14	79	86			50	52	49	41
	b							44				17	38	6			93	47	10	62
9	a														91	22				63
	b														3	9				94
	c														62	42				60
10																				

Fig. 18.

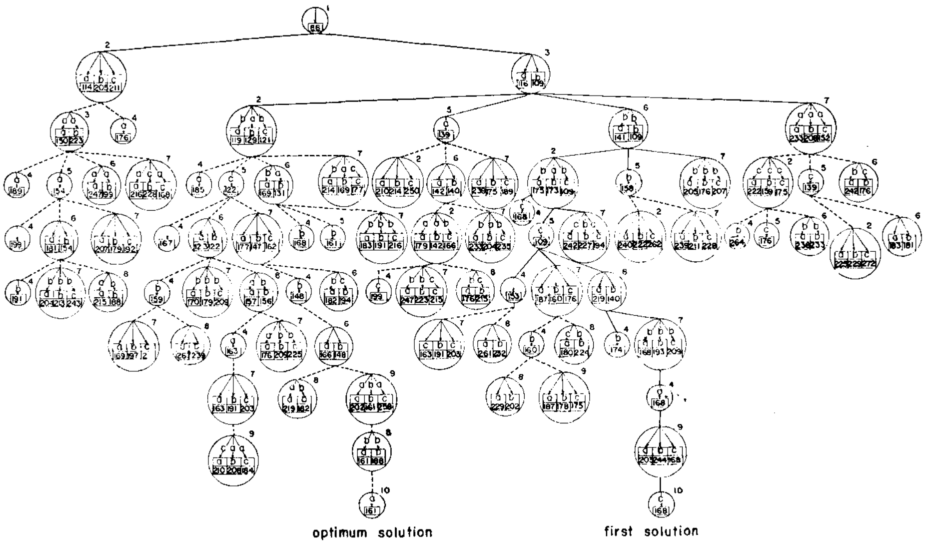


Fig. 19. Tree diagram.

The number of the steps used for an optimum complete linear sequence is 83. This fact shows that the above-mentioned algorithm is also effective for the extended problem. Note that the great portion of the number 83 of calculation is spent to assure optimality. This means that an approximate solution can be got more easily. In general, the number of calculations increases acceleratedly with increasing number of work elements.

Therefore we must compıomise on an approximate solution in case of being limited in calculation time. However, the algorithm developed here generally can also give a very suitable approximate solution in a limitted time.

7.4 Discussion

In this section the problem of minimum transition value and the extension often encountered in the practical production system were handled as an example of determining an optimal linear sequence. In other words, the algorithm effective to any kind of the problem has been developed, settling precedence relations rationally by the establishing method of linear sequences, introducing the concept of lower bound on the operation for picking out suitable sequences from tremendous feasible solutions, and then settling the several problems which come to exist by introducing the concept of lower bound. The algorithm can assure optimality. The algorithm can cope with the case of being limited in calculation time, and also give a suitable approximate solution.

8. Conclusion

By arranging the process necessary to establish the systematical method of sequences step by step, we introduced fundamental matrix which makes the precedence diagram into the form available to theoretical analysis. Sequential product was also introduced as the operational method by which precedence relations can be handled rationally, and linear product as product by which we can establish all of the feasible sequences without overlapping. In relation to this problem the seeking method of the total number of complete linear sequences was considered to give various multipliers on the occasion of calculation. Finally, the problem of minimum transition value restricted by precedence relations was handled as an example of determining an optimal linear sequences.

The concept of direct transition is essential to fundamental matrix. This matrix has come to exist by arranging the conditions of transition in the precedence diagram. It is the first attempt to give a matrix the information like $c(i, j)$. The representative method of matrices in the past can only make clear whether a transition can occur or not. Fundamental matrix introduced in this paper can also make clear in what cases a transition may occur and in what cases a transition may not occur. Sequential product has come to exist by excluding the troubles which we encounter in case of establishing a linear sequence of high cardinal number successively. We could clarify the process through which a sequence of high cardinal number could be made from a sequence of lower cardinal number. By using sequential product, we can do the operation about precedence relations in the most simple form and never make out unproper sequences which do not satisfy precedence restrictions. Feasible sequences are established easily, systematically and very mechanically by linear product. In the practical problem the number of feasible solutions is generally tremendous even if precedence restrictions exist as shown in the example. This indicates that the technique of the operation, which pick out the suitable sequences from tremendous feasible sequences, is substantial to find an optimum complete linear sequence subject to a certain criterion. The algorithm developed to solve the problem of minimum transition value restricted by precedence relations is effective to any kind of problem and can be assured.

In the final analysis, linear product gives the background to which the technique is applied in the process of making sequences of high cardinal number successively and therfore it can be widely applied to the practical problems.

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