Buoyancy Effects on Forced Convection Flow and Heat Transfer

By

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(Received January 28, 1969)

Buoyancy effects on a forced convection laminar flow with uniform internal heat generation along a vertical flat plate are theoretically studied. Solutions are obtained by means of the integral equation method (the approximate Kármán-Pohlhausen method) and are compared with the solutions obtained by the series expansion method in which the stream function and the temperature function are expanded into power series in terms of the parameter Gr_x^*/Re_x^3 for $Pr \geqq 1$, whereas in terms of the parameter $Gr_x^*/(Re_x^3 Pr)$ for $Pr<1$. Velocity and temperature distributions of fluid are presented for various values of the governing parameters.

1. Introduction

Fluids with internal heat generation are commonly encountered in nuclear reactors. If the heat-generating fluids have low flow velocity, density differences arise as a result of differences in temperature, and the effects of buoyancy forces may not be ignored.

Many investigators have studied combined laminar flow of forced and free-convection and heat transfer of heat-generating fluids. Heat transfer by this combined laiminar flow with uniform internal heat generation in vertical tube was studied by Hallman(1). Iqbal(2) has dealt with the influence of free convection on forced flow in a horizontal circular tube. But they solved their problems by assuming that the velocity and temperature distributions are fully developed.

The purpose of this study is to analyze the velocity and temperature distributions of fluid flowing along a vertical plate by means of the integral equation method which Michiyoshi et al.(3) have undertaken and to compare the appoximate solutions by the integral equation method with the solutions by the series expansion method.

When a fluid flows without internal heat generation along a flat plate having a constant temperature, Sparrow & Gregg(4) and Szewczyk(5) showed that the buoyancy effects on a forced laminar convection flow along a vertical plate are

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governed by the parameter Gr_x/Re_x^2, and Mori[6] and Sparrow & Minkowycz[7] showed that the buoyancy effects on a forced convection flow over a horizontal plate are governed by $Gr_x/Re_x^{5/2}$. If other cases of the fluid flowing with internal heat generation along the flat plate or with a constant heat flux at the wall are further examined, the buoyancy effects are found to be governed by the parameters given in Table 1.

Table 1. Parameters representing buoyancy effects on forced convection flow.

internal heat generation	boundary condition	vertical flat plate	horizontal flat plate
$Q=0$	$T_w=$constant	$\frac{Gr_x}{Re_x^2}$	$\frac{Gr_x}{Re_x^{5/2}}$
	$q=$constant	$\frac{Gr_x\dagger}{Re_x^{5/2}}$ 3)	$\frac{Gr_x\dagger}{Re_x^3}$
$Q=$constant	$q=0$	$\frac{Gr_x^*}{Re_x^3}$ $(P\geqq 1)$ 1) $\frac{Gr_x^*}{Re_x^3 Pr}$ $(Pr<1)$	$\frac{Gr_x^*}{Re_x^{7/2}}$
	$T_w=T_0$	$\frac{Gr_x^*}{Re_x^3}$ 2)	$\frac{Gr_x^*}{Re_x^{7/2}}$

1): Chapter 2, 2): Appendix 1, 3): Appendix 2.

We analyze mainly the heat transfer in the case of a fluid flowing with uniform internal heat generation along a vertical flat plate which is thermally insulated by means of both integral equation method and series expansion method in the following chapter. The solutions obtained by both methods are compared with each other. Other cases are dealt with in Appendices.

2. Analysis

Let us consider a two-dimensional geometry such as illustrated in Fig. 1. The vertical plate is thermally insulated, *i.e.*, $q=0$. The fluid contacts with the leading

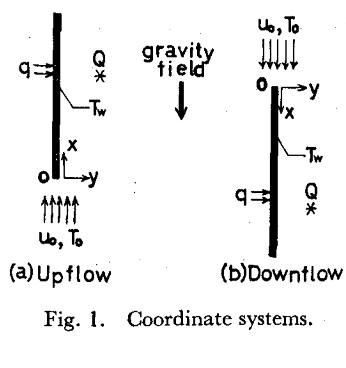


Fig. 1. Coordinate systems.

edge ($x=0$) of plate with a specific uniform velocity u_0 and uniform temperature T_0, and flows with uniform internal heat generation along the plate.

The equations of continuity, momentum and energy which govern combined laminar flow of forced and free-convection and heat transfer are respectively

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}\pm g\beta(T-T_\infty)\,, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}+\frac{Q}{c\rho}\,, \tag{3}$$

where the plus and minus signs in Eq. (2) refer respectively to the upflow and downflow illustrated in Fig. 1 (a) and (b). The order of double sign is the same below. Except the density differences needed to form the buoyancy term, the variation of fluid properties has been neglected. Viscous dissipation has also been deleted. The boundary conditions are

$$\begin{aligned} &y=0\ :\ u=v=0\,, \quad \partial T/\partial y=0\,,\\ &y=\infty :\ u=u_0\,, \qquad T=T_0+Qx/(u_0c\rho)\,. \end{aligned} \tag{4}$$

2.1 The Integral Equation Method

We first introduce the boundary layer thickness and solve this problem. The velocity and temperature distributions may be written as polynomials satisfying the boundary conditions

$$\frac{u}{u_0}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\pm\frac{\gamma\delta^2\delta_T^2}{12}\left[\frac{1}{2}\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2+\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right], \tag{5}$$

$$\theta=\frac{T-T_0}{Qx/(u_0c\rho)}=1+\frac{u_0\delta_T^2}{ax}\left[\frac{1}{6}-\frac{1}{2}\left(\frac{y}{\delta_T}\right)^2+\frac{1}{3}\left(\frac{y}{\delta_T}\right)^3\right], \tag{6}$$

where δ and δ_T are respectively the hydrodynamic and thermal boundary layer thicknesses and γ is the factor accounting for the effects of buoyancy forces:

$$\gamma=\frac{Pr\,g\beta Q}{u_0c\rho\nu^2}\,. \tag{7}$$

$\gamma=0$ implies absence of the buoyancy effects.

If δ_T may be assumed to be equal to δ, Eqs. (5) and (6) become respectively

$$\frac{u}{u_0}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\pm\frac{\gamma\delta^4}{12}\left[\frac{1}{2}\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2+\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right], \tag{8}$$

$$\theta = 1+\frac{u_0\delta^2}{ax}\left[\frac{1}{6}-\frac{1}{2}\left(\frac{y}{\delta}\right)^2+\frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]. \quad (9)$$

Integrating Eqs. (2) and (3) with respect to y with Eq. (1) held in view, we obtain

$$\frac{d}{dx}\int_0^\delta u^2dy-u_{y=\delta}\frac{d}{dx}\int_0^\delta udy = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}\pm\int_0^\delta g\beta(T-T_{y=\delta_T})dy\,, \quad (10)$$

$$\frac{d}{dx}\int_0^{\delta_T}uTdy-T_{y=\delta_T}\frac{d}{dx}\int_0^{\delta_T}udy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0}+\int_0^{\delta_T}\frac{Q}{c\rho}dy\,. \quad (11)$$

2.1.1 Case of $\delta \geqq \delta_T$

Inserting Eqs. (5) and (6) into Eqs. (10) and (11) and taking account of $\delta \geqq \delta_T$, we have

$$\left[\frac{39}{280}\mp r\delta^2\delta_T^2\left(\frac{1}{1120}\pm\frac{r\delta^2\delta_T^2}{12096}\right)\right]\frac{d\delta^2}{dx}\mp r\delta^4\left[\frac{1}{1680}\pm\frac{r\delta^2\delta_T^2}{15120}\right]\frac{d\delta_T^2}{dx}$$
$$= \frac{\nu}{u_0}\left[3\pm r\delta^2\delta_T^2\left(\frac{1}{12}-\frac{1}{6}\frac{\delta_T}{\delta}\right)\right], \quad (12)$$

$$\left[\left\{\frac{1}{112}\left(\frac{\delta_T}{\delta}\right)^5-\frac{3}{80}\left(\frac{\delta_T}{\delta}\right)^3\right\}\pm r\delta^2\delta_T^2\left\{\frac{1}{960}\left(\frac{\delta_T}{\delta}\right)^3-\frac{1}{4032}\left(\frac{\delta_T}{\delta}\right)^5\right\}\right]\frac{d\delta^2}{dx}$$
$$+\left[\left\{\frac{3}{20}\frac{\delta_T}{\delta}-\frac{1}{56}\left(\frac{\delta_T}{\delta}\right)^3\right\}\pm r\delta^2\delta_T^2\left\{\frac{1}{160}\frac{\delta_T}{\delta}-\frac{7}{1080}\left(\frac{\delta_T}{\delta}\right)^2+\frac{1}{504}\left(\frac{\delta_T}{\delta}\right)^3\right\}\right]\frac{d\delta_T^2}{dx}$$
$$= \frac{a}{u_0}\left[\left\{2-\frac{3}{2}\frac{\delta_T}{\delta}+\frac{1}{4}\left(\frac{\delta_T}{\delta}\right)^3\right\}\mp r\delta^2\delta_T^2\left\{\frac{1}{24}\frac{\delta_T}{\delta}-\frac{1}{18}\left(\frac{\delta_T}{\delta}\right)^2+\frac{1}{48}\left(\frac{\delta_T}{\delta}\right)^3\right\}\right], \quad (13)$$

with the boundary conditions

$$x = 0: \quad \delta = \delta_T = 0\,. \quad (14)$$

If δ can be substituted for δ_T in Eqs. (12) and (13), we obtain

$$\left[\frac{39}{280}\mp r\delta^4\left(\frac{1}{672}\pm\frac{r\delta^4}{6720}\right)\right]\frac{d\delta^2}{dx} = \frac{\nu}{u_0}\left[3\mp\frac{r\delta^4}{12}\right], \quad (15)$$

$$\left[\frac{29}{280}\pm\frac{11r\delta^4}{4320}\right]\frac{d\delta^2}{dx} = \frac{a}{u_0}\left[\frac{3}{4}\mp\frac{r\delta^4}{144}\right]. \quad (16)$$

Eq. (16) is identical with the results obtained by Michiyoshi et al.[3]

2.1.2 Case of $\delta \leqq \delta_T$

In the case of $\delta \leqq \delta_T$ we have the following equations in place of Eqs. (12) and (13):

$$\left[\frac{39}{280}\mp r\delta^2\delta_T^2\left(\frac{1}{1120}\pm\frac{r\delta^2\delta_T^2}{12096}\right)\right]\frac{d\delta^2}{dx}\mp r\delta^4\left[\frac{1}{1680}\pm\frac{r\delta^2\delta_T^2}{15120}\right]\frac{d\delta_T^2}{dx}$$
$$=\frac{\nu}{u_0}\left[3\mp r\delta^2\delta_T^2\left\{\frac{1}{4}-\frac{1}{3}\left(\frac{\delta}{\delta_T}\right)^2+\frac{1}{6}\left(\frac{\delta}{\delta_T}\right)^3\right\}\right], \tag{17}$$

$$\left[\left\{\frac{1}{16}-\frac{1}{16}\left(\frac{\delta_T}{\delta}\right)^2-\frac{1}{35}\frac{\delta}{\delta_T}\right\}\pm\frac{r\delta^2\delta_T^2}{24}\left\{\frac{1}{24}\left(\frac{\delta_T}{\delta}\right)^2-\frac{1}{24}+\frac{2}{105}\frac{\delta}{\delta_T}\right\}\right]\frac{d\delta^2}{dx}$$
$$+\left[\left\{\frac{1}{4}\frac{\delta_T}{\delta}-\frac{1}{8}+\frac{1}{140}\left(\frac{\delta}{\delta_T}\right)^3\right\}\pm\frac{r\delta^2\delta_T^2}{24}\left\{\frac{1}{18}-\frac{1}{60}\left(\frac{\delta}{\delta_T}\right)^2+\frac{1}{315}\left(\frac{\delta}{\delta_T}\right)^3\right\}\right]\frac{d\delta_T^2}{dx}$$
$$=\frac{a}{u_0}\left[\frac{3}{4}\mp\frac{r\delta^2\delta_T^2}{144}\right]. \tag{18}$$

If δ_T is assumed to be edual to δ, Eqs. (17) and (18), of course, become respectively Eqs. (15) and (16).

2.2 The Series Expansion Method

2.2.1 Power series in terms of Gr_x^*/Re_x^3

We note that Eq. (1) may be satisfied by a stream function ψ, defined in the usual way as

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}. \tag{19}$$

In the absence of the buoyancy effects, the solution of these equations would correspond to the purely forced convection situation. Inasmuch as our aim here is to find the deviations from forced convection due to buoyancy effects, it is natural to seek solutions in the form of a series of which the first term is the purely forced convection solution. So we write

$$f=\frac{\psi}{\sqrt{\nu x u_0}}=f_0(\eta)\pm\frac{Gr_x^*}{Re_x^3}f_1(\eta)+\cdots, \tag{20}$$

$$\theta=\frac{T-T_0}{Qx/(u_0c\rho)}=\theta_0(\eta)\pm\frac{Gr_x^*}{Re_x^3}\theta_1(\eta)+\cdots, \tag{21}$$

where η, Gr_x^*, and Re_x are, respectively, the Blasius similarity variable, modified Grashof number, and Reynolds number. These quantities are defined as

$$\eta=y\sqrt{\frac{u_0}{\nu x}}, \quad Gr_x^*=\frac{g\beta Qx^5}{\lambda\nu^2}, \quad Re_x=\frac{u_0x}{\nu}. \tag{22}$$

The functions f_0 and θ_0 are associated with the purely forced convection problem, while f_1, θ_1, $\cdots$ give the deviation due to buoyancy effects.

When Eqs. (20) and (21) are substituted in Eqs. (19), (2) and (3), and terms are grouped according to powers of Gr_x^*/Re_x^3, there result the following ordinary differential equations for f_0, θ_0, f_1 and θ_1:

$$f_0''' + \frac{1}{2} f_0 f_0'' = 0 , \tag{23a}$$

$$\theta_0'' + Pr\left(\frac{1}{2} f_0 \theta_0' - f_0' \theta_0 + 1\right) = 0 , \tag{23b}$$

$$f_1''' + \frac{1}{2} f_0 f_1'' + \frac{5}{2} f_1 f_0'' - 2 f_0' f_1' + \frac{1}{Pr} (\theta_0 - 1) = 0 , \tag{24a}$$

$$\theta_1'' + Pr\left(\frac{1}{2} f_0 \theta_1' + \frac{5}{2} f_1 \theta_0' - 3 f_0' \theta_1 - f_1' \theta_0\right) = 0 . \tag{24b}$$

with boundary conditions

$$\begin{aligned} &f_0(0) = f_1(0) = 0 , \quad f_0'(0) = f_1'(0) = 0 , \quad \theta_0'(0) = \theta_1'(0) = 0 , \\ &f_0'(\infty) = 1 , \quad f_1'(\infty) = 0 , \quad \theta_0(\infty) = 1 , \quad \theta_1(\infty) = 0 . \end{aligned} \tag{25}$$

The solutions of Eqs. (23a) and (23b) were respectively given by Blasius[8] and Michiyoshi et al.[3] Numerical solutions for Eqs. (24a) and (24b) have been obtained by means of the Kyoto University Digital Computer (KDC-II), for Prandtl numbers of 0.01, 0.1, 0.72, 1.0, 5.0, and 10.0 using the Runge-Kutta numerical integration procedure. The numerical values of $f_n''(0)$ and $\theta_n(0)$ are tabulated in Table 2.

Table 2. $f_n''(0)$ and $\theta_n(0)$.

Pr	$f_0''(0)$	$f_1''(0)$	$\theta_0(0)$	$\theta_1(0)$
0.01	0.3320575	/	1.158668	/
0.1	0.3320575	7.179953	1.520468	−3.898502
0.72	0.3320575	2.267490	2.383997	−3.102781
1.0	0.3320575	1.837776	2.614146	−2.913710
5.0	0.3320575	0.6016313	4.278591	−1.891867
10.0	0.3320575	0.3565922	5.354188	−1.480628

2.2.2 Power series in terms of $Gr_x^*/(Re_x^3 Pr)$

Following a procedure similar to that presented in Section 2.2.1., the stream function and the temperature function are now expanded into a power series in terms of the parameter $Gr_x^*/(Re_x^3 Pr)$, i.e.,

$$f = \frac{\psi}{\sqrt{\nu x u_0}} = f_0(\eta) \pm \frac{Gr_x^*}{Re_x^3 Pr} f_1(\eta) + \cdots , \tag{26}$$

$$\theta = \frac{T - T_0}{Qx/(u_0 c\rho)} = \theta_0(\eta) \pm \frac{Gr_x^*}{Re_x^3 Pr} \theta_1(\eta) + \cdots . \tag{27}$$

Substituting these power-series expansions into Eqs. (19), (2) and (3) and grouping terms according to powers of $Gr_x^*/(Re_x^3 Pr)$, we obtain the following set of ordinary differential equations:

$$f_0''' + \frac{1}{2} f_0 f_0'' = 0\,, \tag{28a}$$

$$\theta_0'' + Pr\left(\frac{1}{2} f_0 \theta_0' - f_0' \theta_0 + 1\right) = 0\,, \tag{28b}$$

$$f_1''' + \frac{1}{2} f_0 f_1'' + \frac{5}{2} f_1 f_0'' - 2 f_0' f_1' + \theta_0 - 1 = 0\,, \tag{29a}$$

$$\theta_1'' + Pr\left(\frac{1}{2} f_0 \theta_1' + \frac{5}{2} f_1 \theta_0' - 3 f_0' \theta_1 - f_1' \theta_0\right) = 0\,. \tag{29b}$$

The boundary conditions are

$$\begin{aligned} &f_0(0) = f_1(0) = 0\,,\quad f_0'(0) = f_1'(0) = 0\,,\quad \theta_0'(0) = \theta_1'(0) = 0\,, \\ &f_0'(\infty) = 1\,,\quad f_1'(\infty) = 0\,,\quad \theta_0(\infty) = 1\,,\quad \theta_1(\infty) = 0\,. \end{aligned} \tag{30}$$

The zeroth approximation, i.e., the purely forced convection equations (28a) and (28b) are identical with Eqs. (23a) and (23b). Numerical solutions of Eqs. (29a) and (29b) have been obtained on KDC-II for various Prandtl numbers. The numerical values of $f_n''(0)$ and $\theta_n(0)$ are tabulated in Table 3.

Table 3. $f_n''(0)$ and $\theta_n(0)$.

Pr	$f_0''(0)$	$f_1''(0)$	$\theta_0(0)$	$\theta_1(0)$
0.01	0.33206	0.23237	1.1543	−0.044232
0.1	0.33206	0.71799	1.5205	−0.38984
0.72	0.33206	1.6326	2.3840	−2.2340
1.0	0.33206	1.8377	2.6141	−2.9135
2.5	0.33206	2.4762	3.4383	−5.8401
5.0	0.33206	3.0083	4.2786	−9.4599
10.0	0.33206	3.5663	5.3542	−14.808

The numerical values of $f_1''(0)$ and $\theta_1(0)$ divided by Prandtl numbers are equal to the values of $f_1''(0)$ and $\theta_1(0)$ given in Table 2 respectively. It is seen that the value of $-\theta_1(0)$ becomes greater with decreasing Pr and $-\theta_1(0) > \theta_0(0)$ for $Pr \leqq 1$ in Table 2, while in Table 3 $-\theta_1(0)$ becomes greater with increasing Pr and $-\theta_1(0) > \theta_0(0)$ for $Pr \geqq 1$. So the solutions may be obtained by expanding the stream function and the temperature function into power series in terms of the parameter Gr_x^*/Re_x^3 for $Pr \geqq 1$, whereas for $Pr < 1$ in terms of the parameter $Gr_x^*/(Re_x^3 Pr)$.

3. Results and Discussions

We first examine the temperature distributions of heat-generating fluid flowing along the thermally insulated vertical flat plate by forced convection alone obtained by the integral equation method. As a result the dimensionless wall temperature $\theta(0)$ is related approximately to the Prandtl number Pr by the expression

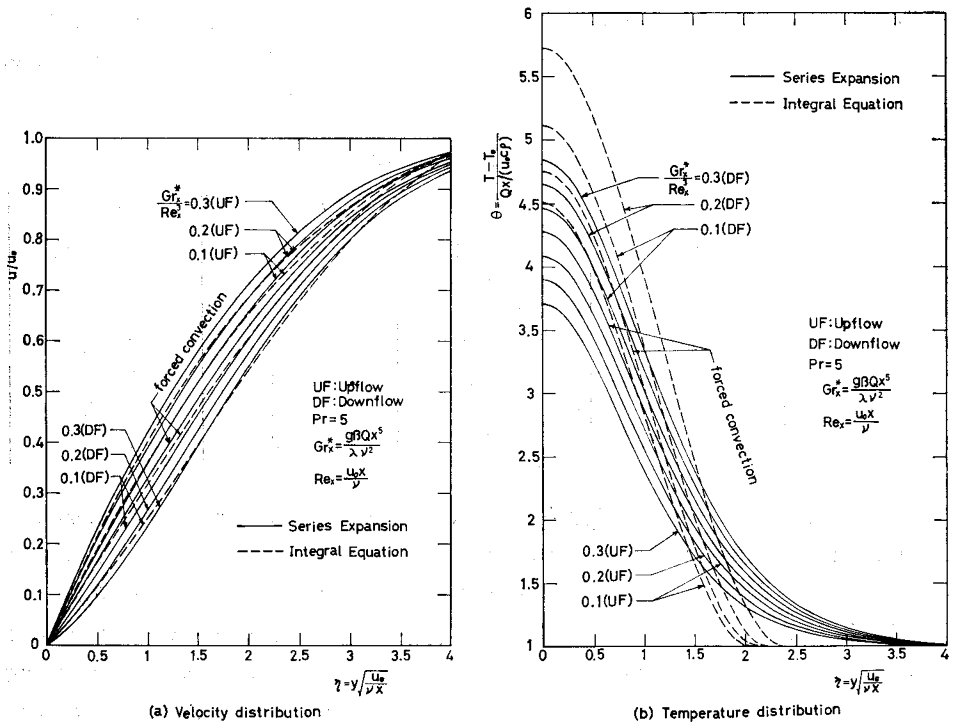
$$\theta(0) = 3.03\,Pr^{0.328} \qquad (0.72 \leqq Pr \leqq 10)\,, \tag{31a}$$

$$\theta(0) = 1.72\,Pr^{0.0664} \qquad (0.001 \leqq Pr \leqq 0.025)\,. \tag{31b}$$

These values of $\theta(0)$ are about 10 per cent higher than the rigorous solutions given by Michiyoshi et al.(3)

Figure 2 shows the velocity and temperature distributions of fluid of $Pr=5$ obtained for combined laminar flow of free convection and forced convection. The broken lines represent the solutions obtained by the integral equation method, and the solid lines are the solutions obtained by the series expansion method with the parameter Gr_x^*/Re_x^3.

It can be seen that the velocity distribution near the wall becomes steeper for



(a) (b)

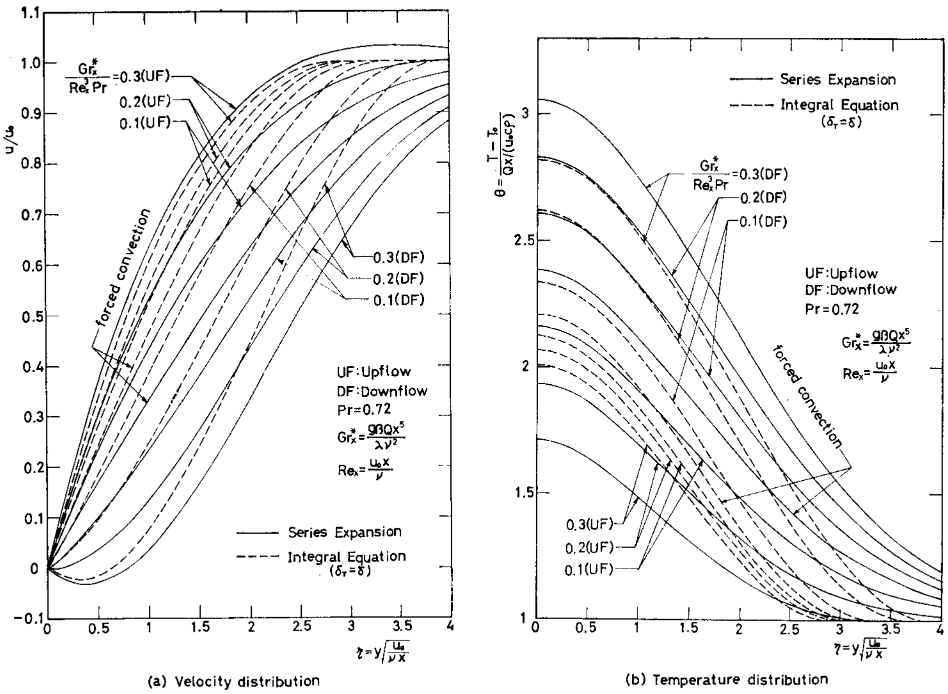
Fig. 2. Buoyancy effects on forced convection flow and heat transfer ($Pr=5$).

upflow (or flatter for downflow) with larger values of Gr_x^*/Re_x^3. This fact agrees with the results obtained by Szewczyk[5] in the case of a constant wall temperature. The solutions by the integral equation method are also in good agreement with those by the series expansion method.

Figure 2(b) shows that on account of the effects of buoyancy forces the temperature near the wall is lower for upflow (or higher for downflow) compared with the purely forced convection. The wall temperature obtained by the integral equation method is higher than that by the series expansion method, and the former temperature distribution is steeper than the latter. This is because the boundary layer thickness is estimated to be too thin in the integral equation method.

Figure 3 shows the velocity and temperature distributions for Prandtl number of 0.72. The broken lines are the distributions obtained by the integral equation method, assuming that the thermal boundary layer thickness δ_T is equal to the hydrodynamic δ. The solid lines are the solution by the series expansion method with the parameter $Gr_x^*/(Re_x^3 Pr)$.

Figure 3 (a) shows that for large values of $Gr_x^*/(Re_x^3 Pr)$ the buoyancy effects cause flow reversal in a part of the boundary layer of downflow, as shown by the



(a) (b)

Fig. 3. Buoyancy effects on forced convection flow and heat transfer (Pr=0.72).

negative values of u/u_0. The solutions by the integral equation method agree fairly with those by the series expansion method. Although it is seen in Fig. 3(b) that in the case of upflow the buoyancy effects obtained by the integral method are weaker than those by the series expansion method, agreement will be improved if more higher power terms in the power-series expansion are used.

4. Conclusion

The effects of buoyancy forces on a forced convection laminar flow with uniform internal heat generation along a thermally insulated vertical flat plate were analyzed. Comparison of the integral equation method with the series expansion method has yielded the following conclusions:

(1) The stream and temperature functions may be expanded into a power series in terms of the parameter Gr_x^*/Re_x^3 for $Pr \geqq 1$, whereas in terms of $Gr_x^*/(Re_x^3 Pr)$ for $Pr<1$.

(2) The solutions by the integral equation method agree fairly with those by the series expansion method.

(3) For large values of the parameter $Gr_x^*/(Re_x^3 Pr)$ in Fig. 3(a) the buoyancy effects cause flow reversal in a part of the boundary layer of downflow.

(4) The buoyancy effects on the temperature distribution result in the lower wall temperature for upflow and in the higher wall temperature for downflow.

Nomenclature

a	:	Thermal diffusivity of fluid
c	:	Specific heat of fluid
f	:	Dimensionless stream function
f_n	:	nth stream function
g	:	Acceleration due to gravity
Gr_x	:	Grashof number defined as $g\beta(T-T_\infty)x^3/\nu^2$
Gr_x^*	:	Modified Grashof number defined by Eq. (22)
$Gr_x^\dagger$	:	Modified Grashof number defined by Eq. (46)
Pr	:	Prandtl number
q	:	Wall heat flux
Q	:	Heat generating rate per unit volume
Re_x	:	Reynolds number defined by Eq. (22)
T	:	Temperature
u	:	Velocity in the x-direction
v	:	Velocity in the y-direction

x : Coordinate along plate measured from leading edge
y : Coordinate normal to plate measured away from plate
β : Coefficient of thermal expansion of fluid
γ : Buoyancy effects defined by Eq. (7)
Γ : Buoyancy effects defined by Eq. (53)
δ : Hydrodynamic boundary layer thickness
δ_T : Thermal boundary layer thickness
η : Dimensionless coordinate defined by Eq. (22)
θ : Dimensionless temperature function defined by Eq. (6)
θ_n : nth temperature function of θ
Θ : Dimensionless temperature function defined by Eq. (45)
Θ_n : nth temperature function of Θ
λ : Thermal conductivity of fluid
ν : Kinematic viscosity of fluid
ρ : Density of fluid
ψ : Stream function
(Subscripts)
w : Conditions at wall ($y=0$)
0 : Conditions at inlet ($x=0$)
∞ : Conditions outside boundary layer

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Appendix 1

Case of Q=constant and $T_w=T_0$:

The fluid contacts with the leading edge of plate with a specific uniform velocity u_0 and uniform temperature T_0, and flows with uniform internal heat generation along the vertical plate whose temperature is kept at T_0.

With respect to the coordinate system shown in Fig. 1, the equations governing combined laminar flow of forced and free-convection and heat transfer are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 , \tag{32}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}\pm g\beta(T-T_\infty) , \tag{33}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}+\frac{Q}{c\rho} , \tag{34}$$

where the plus and minus signs in Eq. (33) refer respectively to the upflow and downflow. The properties of the fluid are assumed to be constant, except the density differences needed to form the buoyancy term. The boundary conditions are

$$\begin{aligned} y = 0 : \quad & u = v = 0 , \quad T = T_0 , \\ y = \infty : \quad & u = u_0 , \quad T = T_0+Qx/(u_0 c\rho) . \end{aligned} \tag{35}$$

Following a procedure similar to that presented in Section 2.2.1. and expanding the stream and temperature functions into a power series in terms of Gr_x^*/Re_x^3, we obtain the following set of ordinary differential equations:

$$f_0'''+\frac{1}{2}f_0 f_0'' = 0 , \tag{36a}$$

$$\theta_0''+Pr\left(\frac{1}{2}f_0\theta_0'-f_0'\theta_0+1\right) = 0 , \tag{36b}$$

$$f_1'''+\frac{1}{2}f_0 f_1''+\frac{5}{2}f_1 f_0''-2f_0' f_1'+\frac{1}{Pr}(\theta_0-1) = 0 , \tag{37a}$$

$$\theta_1''+Pr\left(\frac{1}{2}f_0\theta_1'+\frac{5}{2}f_1\theta_0'-3f_0'\theta_1-f_1'\theta_0\right) = 0 , \tag{37b}$$

with boundary conditions

$$\begin{aligned} & f_0(0) = f_1(0) = 0 , \quad f_0'(0) = f_1'(0) = 0, \quad \theta_0(0) = \theta_1(0) = 0 , \\ & f_0'(\infty) = 1 , \quad f_1'(\infty) = 0 , \quad \theta_0(\infty) = 1, \quad \theta_1(\infty) = 0 . \end{aligned} \tag{38}$$

Table 4. $f_n''(0)$ and $\theta_n'(0)$.

Pr	$f_0''(0)$	$f_1''(0)$	$\theta_0'(0)$	$\theta_1'(0)$
0.01	0.3320575	−140.0893	0.1109723	−1.607827
0.1	0.3320575	−10.81024	0.3664485	−0.2286015
0.72	0.3320575	−0.6809172	1.156209	−0.004574791
1	0.3320575	−0.3514885	1.418029	−0.01987674
5	0.3320575	0.1012887	3.989876	−0.1456143
10	0.3320575	0.09555734	6.295007	−0.1923763

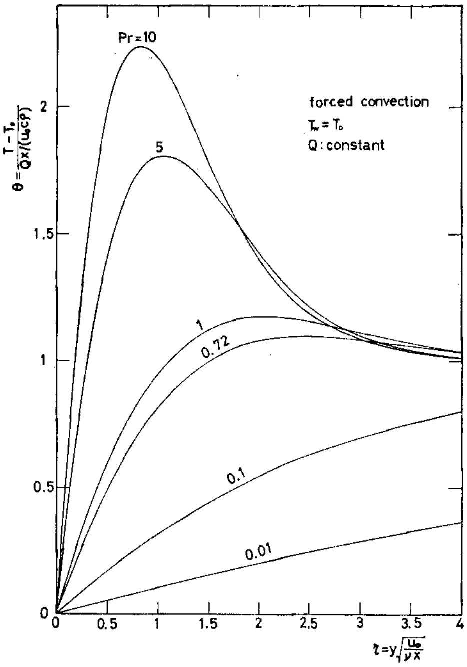
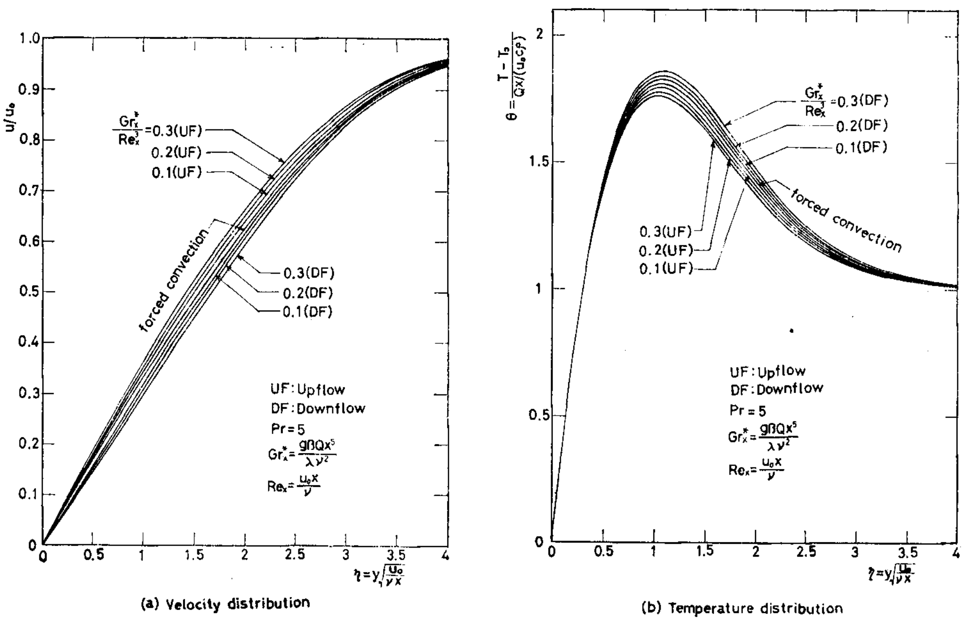


Fig. 4. Temperature distribution under forced flow.



(a) (b)

Fig. 5. Buoyancy effects on forced convection flow and heat transfer ($Pr=5$).

Numerical solutions of Eqs. (36) and (37) have been obtained on KDC-II for various Prandtl numbers. The numerical values of $f_n''(0)$ and $\theta_n'(0)$ are tabulated in Table 4.

Figure 4 shows the zeroth temperature functions, i.e., the purely forced convection temperature distributions for various Prandtl numbers. It is seen that for large Prandtl numbers the temperature distribution has a peak in a part of the boundary layer and that the peak becomes higher with increasing *Pr*.

The velocity and temperature distributions obtained for combined laminar flow of free and forced-convection are shown in Fig. 5(a) and (b) in the case of $Pr=5$. In this case, it can be seen that the buoyancy effects are negligibly small.

Appendix 2

Case of $Q=0$ and $q=$constant:

A fluid flows without internal heat generation along a vertical flat plate with a constant heat flux q to the fluid at the wall. Using the coordinate system shown in Fig. 1, the equations of mass, momentum and energy are respectively

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{39}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}\pm g\beta(T-T_\infty)\,, \tag{40}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}\,, \tag{41}$$

where the plus and minus signs in Eq. (40) refer, respectively, to the upflow and downflow. The boundary conditions are

$$\begin{aligned} y=0\ &:\ u=v=0, \quad q=-\lambda(\partial T/\partial y)\,, \\ y=\infty &:\ u=u_0\,, \qquad T=T_0 \end{aligned} \tag{42}$$

A2.1. The Series Expansion Method

By introducing the stream function defined by

$$u=\frac{\partial\psi}{\partial y}\,, \quad v=-\frac{\partial\psi}{\partial x}\,, \tag{43}$$

the continuity equation (39) is satisfied.

The stream function and the temperature function are expanded into

$$f=\frac{\psi}{\sqrt{\nu x u_0}}=f_0(\eta)\pm\frac{Gr_x^\dagger}{Re_x^{5/2}}f_1(\eta)+\cdots\,, \tag{44}$$

$$\Theta = \frac{T-T_0}{\dfrac{q}{\lambda}\sqrt{\dfrac{\nu x}{u_0}}} = \Theta_0(\eta) \pm \frac{Gr_x^\dagger}{Re_x^{5/2}}\Theta_1(\eta) + \cdots , \tag{45}$$

where η, $Gr_x^\dagger$ and Re_x are, respectively, the Blasius similarity variable, modified Grashof number and Reynolds number. These quantities are defined as

$$\eta = y\sqrt{\frac{u_0}{\nu x}}, \quad Gr_x^\dagger = \frac{g\beta q x^4}{\lambda \nu^2}, \quad Re_x = \frac{u_0 x}{\nu}. \tag{46}$$

The functions f_0 and Θ_0 are associated with the purely forced convection problem, while f_1, Θ_1, $\cdots$ give the deviations from the forced convection due to buoyancy effects.

When these expansions are substituted into Eqs. (43), (40) and (41) and terms are grouped according to powers of $Gr_x^\dagger/Re_x^{5/2}$, the following set of ordinary differential eqations is obtained for f_0, Θ_0, f_1 and Θ_1:

$$f_0''' + \frac{1}{2}f_0 f_0'' = 0 , \tag{47a}$$

$$\Theta_0'' + \frac{Pr}{2}(f_0\Theta_0' - f_0'\Theta_0) = 0 , \tag{47b}$$

$$f_1''' + \frac{1}{2}f_0 f_1'' + 2f_1 f_0'' - \frac{3}{2}f_0' f_1' + \Theta_0 = 0 , \tag{48a}$$

$$\Theta_1'' + \frac{Pr}{2}(f_0\Theta_1' + 4f_1\Theta_0' - 4f_0'\Theta_1 - f_1'\Theta_0) = 0 , \tag{48b}$$

with boundary conditions

$$\begin{aligned} &f_0(0) = f_1(0) = 0 , \quad f_0'(0) = f_1'(0) = 0 , \quad \Theta_0'(0) = -1 , \quad \Theta_1'(0) = 0 , \\ &f_0'(\infty) = 1 , \quad f_1'(\infty) = 0 , \quad \Theta_0(\infty) = \Theta_1(\infty) = 0 . \end{aligned} \tag{49}$$

Numerical solutions of Eqs. (47) and (48) have been obtained on KDC-II for various Prandtl numbers. The numerical values of $f_n''(0)$ and $\Theta_n(0)$ are tabulated in Table 5.

Table 5. $f_n''(0)$ and $\Theta_n(0)$.

Pr	$f_0''(0)$	$f_1''(0)$	$\Theta_0(0)$	$\Theta_1(0)$
0.01	0.3320575	20.37142	12.89298	−58.21213
0.1	0.3320575	6.621191	4.983698	−11.77136
0.72	0.3320575	2.450472	2.439788	−2.781628
1	0.3320575	2.059825	2.178790	−2.150799
5	0.3320575	0.8377988	1.263940	−0.5607496
10	0.3320575	0.5547342	1.002121	−0.3019560

For purely forced convection, the dimensionless wall temperature $\Theta_0(0)$ is related approximately to Pr by

$$\Theta_0(0) = 2.18 Pr^{-0.338} \qquad (0.72 \leqq Pr \leqq 10)\,, \tag{50a}$$

$$\Theta_0(0) = 1.67 Pr^{-0.445} \qquad (0.0025 \leqq Pr \leqq 0.025)\,. \tag{50b}$$

A2.2. The Integral Equation Method

When the boundary layer thickness is introduced, the velocity and temperature distributions may be written as polynomials satisfying the boundary conditions

$$\frac{u}{u_0} = 2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \pm \frac{\Gamma\delta^2\delta_T}{12}\left[\frac{y}{\delta} - 3\left(\frac{y}{\delta}\right)^2 + 3\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right], \tag{51}$$

$$\Theta = \frac{T - T_0}{\frac{q}{\lambda}\sqrt{\frac{\nu x}{u_0}}} = \delta_T\sqrt{\frac{u_0}{\nu x}}\left[\frac{1}{2} - \frac{y}{\delta_T} + \left(\frac{y}{\delta_T}\right)^3 - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^4\right], \tag{52}$$

where δ and δ_T are respectively the hydrodynamic and thermal boundary layer thicknesses and Γ is the factor accounting for the buoyancy effects:

$$\Gamma = \frac{Pr\,g\beta q}{u_0 c\rho\nu^2}\,. \tag{53}$$

$\Gamma = 0$ implies absence of the buoyancy effects.

Integrating Eqs. (40) and (41) with respect to y with Eq. (39) held in view, we obtain

$$\frac{d}{dx}\int_0^\delta u^2 dy - u_{y=\delta}\frac{d}{dx}\int_0^\delta u dy = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \pm \int_0^\delta g\beta(T - T_{y=\delta_T})dy\,, \tag{54}$$

$$\frac{d}{dx}\int_0^{\delta_T} uT dy - T_{y=\delta_T}\frac{d}{dx}\int_0^{\delta_T} u dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0}\,. \tag{55}$$

If the Prandtl number is not far from unity, the thermal boundary layer thickness δ_T may be assumed to be equal to the hydrodynamic δ. Substituting δ for δ_T in Eqs. (51), (52) and (55), we obtain

$$\frac{u}{u_0} = 2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \pm \frac{\Gamma\delta^3}{12}\left[\frac{y}{\delta} - 3\left(\frac{y}{\delta}\right)^2 + 3\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right], \tag{56}$$

$$\Theta = \delta\sqrt{\frac{u_0}{\nu x}}\left[\frac{1}{2} - \frac{y}{\delta} + \left(\frac{y}{\delta}\right)^3 - \frac{1}{2}\left(\frac{y}{\delta}\right)^4\right], \tag{57}$$

$$\frac{d}{dx}\int_0^\delta uT dy - T_{y=\delta}\frac{d}{dx}\int_0^\delta u dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0}\,. \tag{58}$$

Inserting Eqs. (56) and (57) into Eq. (58), we have

$$\left(\frac{37}{630}\pm\frac{55}{24192}\Gamma\delta^3\right)\frac{d\delta^2}{dx}=\frac{a}{u_0}. \tag{59}$$

Integrating under $\delta=0$ at $x=0$, we obtain

$$x=\frac{u_0\delta^2}{a}\left(\frac{37}{630}\pm\frac{11}{12096}\Gamma\delta^3\right). \tag{60}$$

If the buoyancy effects are neglected, $\Gamma=0$, then we have

$$x=\frac{37}{630}\frac{u_0\delta^2}{a}. \tag{61}$$

The dimensionless wall temperature thus can be expressed by

$$\Theta(0)=\frac{1}{2}\sqrt{\frac{630}{37}\frac{1}{Pr}}\doteqdot 2.06. \tag{62}$$

This value is in good agreement with $\Theta_0(0)=2.18$ for $Pr=1$ obtained from Eq. (50a).

Figure 6 shows the buoyancy effects on the velocity and temperature distributions in the case of $Pr=0.72$. The solid lines are the solutions by the series

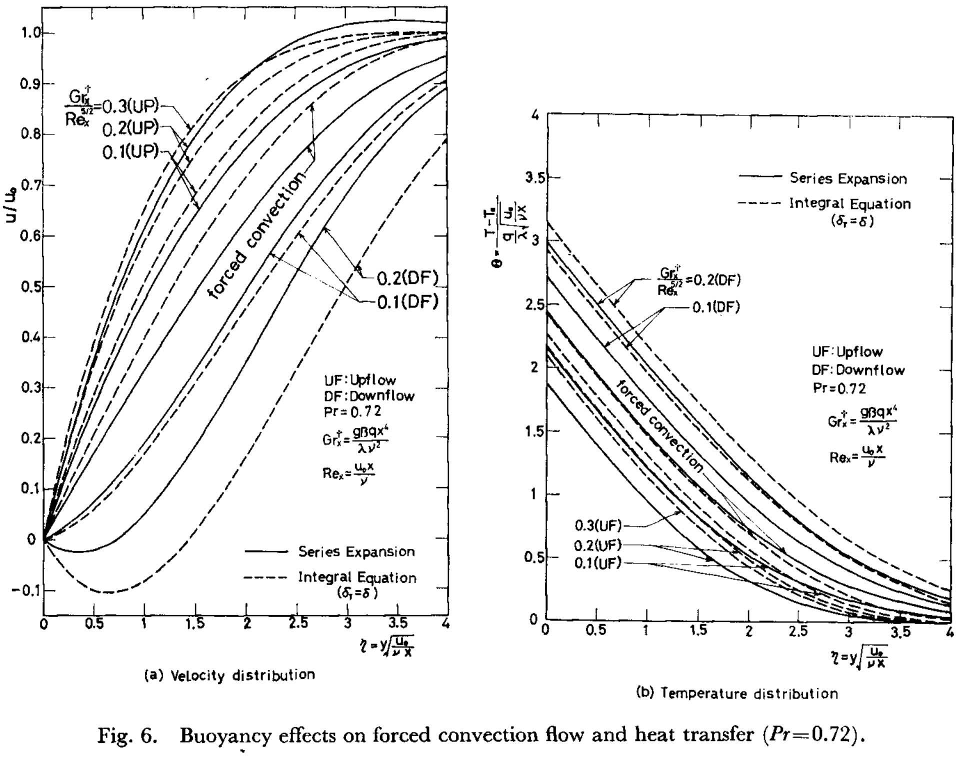


Fig. 6. Buoyancy effects on forced convection flow and heat transfer ($Pr=0.72$).

expansion method and the broken lines are those by the integral equation method with assumption of $\delta_T=\delta$. The parameter $Gr_x^\dagger/Re_x^{5/2}$ represents the buoyancy effects.

The velocity distribution shown in Fig. 6(a) becomes steeper for upflow (or flatter for downflow) with larger values of $Gr_x^\dagger/Re_x^{5/2}$. The buoyancy effects also cause flow reversal for large $Gr_x^\dagger/Re_x^{5/2}$ in downflow. It is seen that on account of the buoyancy effects the temperature near the wall is lower for upflow (or higher for downflow) in Fig. 6(b). The buoyancy effects obtained by the integral equation method are smaller for upflow (or larger for downflow) compared with those by the series expansion method.