On the Controllability of Nonlinear Control Systems

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In this paper, some discussions for the controllability in nonlinear systems are presented. At first various concepts about controllability are defined. In section 3, one dimensional systems are treated and sufficient conditions for controllability are obtained. In section 4, we discuss the controllability of nonlinear systems with controls appearing linearly and show that it is possible to reduce the controllability of the given system to that of some lower dimensional system. One result in section 3 is extended to n-dimensional system in the following section. In section 6, the concept of the controllability in the local sense is introduced according to L. Markus. At last, relations between various concepts of controllability are shown.

1. Introduction

The concept of controllability of linear systems was introduced by R.E. Kalman. It is now realized that the concept plays a fundamental role in the modern control theory, especially in the optimum control theory. Kalman's discussion is based on the linear algebra, and essentially restricted to linear systems[1-4].

A few authors studied the controllability of nonlinear systems. E. Roxin studied the controllability of the special types of nonlinear systems. He introduced the concept of the reachable zone and discussed the relation between optimal controls and reachable zones[5-6].

L. Markus studied the local controllability of nonlinear systems, controllability in the neighborhood of the origin. He also showed that it is possible to apply global stability theory to the controllability theory[7-8].

The generalization of the concept of controllability of linear systems to nonlinear systems was tried by H. Hermes[9]. He reduced the problem of controllability to the problem of non-integrability of some Pfaffian form, and discussed the relation between controllability and singular problems which appear in the theory of optimal control.

In this paper, we discuss the controllability of nonlinear systems with controls

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appearing linearly, by reducing the controllability of the given system to that of the auxiliary lower dimensional control system. We introduce the concept of quasi-controllability, and at first show sufficient conditions of the quasi-controllability, for some special cases. Next, a sufficient condition for controllability is shown by connecting the concepts of quasi-controllability and local controllability. In the last section various concepts of controllability are compared.

2. Definitions

We assume that the motion of the controlled system is described by a system of ordinary differential equations

$$\frac{dx_i}{dt}=f_i(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_r), \qquad (i=1, 2, \cdots, n) \tag{1}$$

or in a vector form

$$\frac{dx}{dt}=f(x, u), \tag{2}$$

where x is a state vector and u is a control vector.
The function

$$f_i(x, u),\ \frac{\partial f_i(x, u)}{\partial x_j}, \qquad (i=1, 2, \cdots, n)$$

is defined and continuous on the product space $R^n \times R^r$. In general, the function $f(x, u)$ is nonlinear with respect to both x and u.

In the case when the function $f(x, u)$ is linear with respect to u, the system (2) is called a system with controls appearing linearly and expressed as follows,

$$\dot{x}=f(x)+G(x)u \tag{3}$$

where $G(x)$ is an $n\times r$ matrix with elements $g_{ij}(x)$.
The functions

$$\frac{\partial f_i(x)}{\partial x_j},\ \frac{\partial g_{ik}(x)}{\partial x_j}, \qquad (i, j=1, 2, \cdots, n, \quad k=1, 2, \cdots, r)$$

are continuous functions of x.
Moreover if $G(x)$ is a constant matrix, the equation (3) becomes

$$\dot{x}=f(x)+Gu \tag{4}$$

where G an $n\times r$ constant matrix. Since the rank of the matrix G is an efficient number of controls, we may assume that the rank of matrix G is equal to r, in other words, the column vectors $g_1, \cdots, g_r$, of G are linearly independent. We define an $n\times r$ matrix as

$$H \equiv \begin{pmatrix} E_r \\ 0 \end{pmatrix} \tag{5}$$

where E_r is $r\times r$ unit matrix. Then, without loss of generality, we may assume that the equation (4) is of the form

$$\dot{x} = f(x) + Hu\,. \tag{6}$$

In (6) if the function $f(x)$ is also linear, we have a linear system:

$$\dot{x} = Fx + Gu\,. \tag{7}$$

where F and G are $n\times n$ and $n\times r$ constant matrix, respectively. If we take a suitable coordinate the linear system is represented by the equation

$$\dot{x} = Fx + Hu\,. \tag{8}$$

In this paper we say that a control $u(t)$ is admissible if it is continous for all t under consideration, with exception of a finite number of t at which $u(t)$ may have discontinuity of the first kind. If a certain admissible control $u(t)$ is given, the equation (2) takes the form

$$\frac{dx}{dt} = f(x, u(t))\,. \tag{9}$$

For any initial condition $x(t_0)=x^0$, the solution of the equation (9) is uniquely determined. This solution $x(t)$ will be called the solution of the system (2) corresponding to the control $u=u(t)$ for the initial condition $x(t_0)=x^0$. If the solution of the system (2) corresponding to the control $u=u(t)$ for the initial condition $x(t_0)=x^0$ satisfies the condition $x(t_1)=x^1$ at the time t_1, then we shall say that the admissible control $u(t)$ transfers the initial state x^0 to the state x^1. Since the system under consideration is time-invariant we can set always $t_0=0$.

We define several concepts with respect to the given system.

Def. 1.

For the two states $x^0=(x_1{}^0, x_2{}^0\cdots x_n{}^0)$, $x^1=(x_1{}^1\cdots x_n{}^1)$ given, if there exists some finite time $t_1>0$ and some admissible control which transfers the initial state x^0 given at the time $t=0$, to the state x^1 at t_1, we say that the state x^0 is "controllable" to x^1. In particular, in the case when x^1 is the origin we say simply that the point x^0 is controllable.

Def. 1′.

If the state x^0 is controllable to the state x^1, then the state x^1 is called "reachable" from the state x^0.

Def. 2.

The state x^0 will be called "quasi-controllable" to x^1, if in every neighborhood of x^1 there is a state which is reachble from x^0.

Def. 3.

If every state x^0 in R^n is controllable to x^1, the control system (1) is called "completely controllable" to x^1.

Def. 4.

If every state x^0 in R^n is quasi-controllable to x^1, the control system (1) is called "completely quasi-controllable" to x^1.

Def. 5.

If, for arbitrary given $\tau > o$, there exists a certain admissible control which transfers the initial state x^0 given at the time $t=0$ to the state x^1 at the time $t=t_1<\tau$, the state x^0 is called "well controllable" to x^1.

Def. 6.

The state x^0 will be called "well quasi-controllable" to x^1, if in every neighborhood of x^1 there is a state to which x^0 is well controllable.

Def. 7.

If every state x^0 in R^n is well controllable to x^1, the control system (1) is called "completely well controllable.

Def. 8.

In the similar sense, we define the system which is "completely well quasi-controllable."

Def. 9.

If the above properties hold for every $x^1 \in R^n$, we add the term "to the whole", for example, we say that x^0 is controllable to the whole.

Remarks

1. The concept of quasi-controllability is introduced first by *E.* Roxin[5].
2. In the reference (10) the concept of total controllability is defined. The given system is totally controllable if it is completely controllable on every positive time interval $[t_0, t_1]$. In time invariant systems the definition of total controllability is identical to the definition 7 of this paper.

3. Controllability of One Dimensional System

We consider a one-dimensional system

$$\dot{x} = f(x, u_1, \cdots, u_r) \tag{10}$$

where x is a scalar-valued state, $u_1, \cdots, u_r$ are controls.

The functions $f(x, u)$ and $\dfrac{\partial f(x, u)}{\partial x}$ are defined and continuous on the product space $R^1 \times R^r$. For this control system we have a simple criterion for complete controllability.

Theorem 1.

The system (10) is completely controllable if it satisfies the following conditions:

(i) The function $f(0, u)$ can take both positive and negative values.

(ii) For every fixed $x (\neq 0)$, there exists a vector u which satisfies an inequality;

$$xf(x, u) < 0 .$$

Proof

We assume that the given initial state x_0 is positive. Let the closed interval $[0, x_0]$ be I_0, then by assumption, there exists a constant vector $u = c_\alpha \in R^r$ such that $f(\alpha, c_\alpha) < 0$ corresponding to each $\alpha \in I_0$. Then, by the continuity of the function $f(x, c_\alpha)$ with respect to x, there exists an open-neighborhood O_α of α such that $f(x, c_\alpha) < 0$ for $x \in O_\alpha$. Since this infinite number of open set O_α, $(0 \leqq \alpha \leqq x_0)$ cover the closed interval I_0, by Heine Borel's Covering Theorem we can select from O_α $(0 \leqq \alpha \leqq x_0)$ a finite number of sets $O_{\alpha_1}, O_{\alpha_2}, \cdots, O_{\alpha_N}$ $(\alpha_i > \alpha_{i+1})$ such that $I_0 \subset \sum_1^N O_{\alpha_i}$. We may assume that $O_{\alpha_i} \cap O_{\alpha_{i-1}} \neq \phi$, $x_0 \in O_{\alpha_1}$, $0 \in O_{\alpha_N}$, where ϕ is the empty set.

At first, consider the differential equation

$$\dot{x} = f(x, c_{\alpha_1})$$

with an initial condition $x(0) = x_0$, then, since $f(x, c_{\alpha_1}) < 0$ for $x \in O_{\alpha_1}$ the con rol $u = c_{\alpha_1}$ transfers the point x_0 to some point $x_1 \in O_{\alpha_1} \cap O_{\alpha_2}$ at some finite time $t = t_1$. Next, we consider the differential equation

$$\dot{x} = f(x, c_{\alpha_2})$$

with initial condition $x(t_1) = x_1$. Since $f(x, c_{\alpha_2}) < 0$ for $x \in O_{\alpha_2}$ the control $u = c_{\alpha_2}$ transfers the state x_1 to x_2 at some time $t = t_2$. Proceeding with the same processes, at last, the control $u = c_{\alpha_N}$ transfers the initial state $x_{N-1} \in O_{\alpha_{N-1}} \cap O_{\alpha_N}$ to the origin at some finite time $t = t_N$. If we use $u = u(t)$ as a control, it is clear that $u(t)$ transfers the initial state x^0 given at $t = 0$, to the origin at time $t = t_N$, where

$$u(t) = c\alpha_i \quad \text{for} \quad t_{i-1} < t \leqq t_i , \qquad i = 1, 2, \cdots, N, \quad t_0 = 0 .$$

Since this control is clearly admissible and x_0 is arbitrarily chosen the system (10) is completely controllable. Q.E.D.

With simple modifications to Theorem 1 we have a sufficient condition for complete well-controllability for the system (10).

Theorem 2.

The system (10) is completely well-controllable if it satisfies the following conditions:

(i) The same condition as in Theorem 1 holds.

(ii) Define the sets S and S_a as follows,

$$S \equiv \{x: \underset{u}{Inf} f(x, u)\ sgn\ x > -\infty\}$$

$$S_a \equiv \{x: x \in S,\ |x| < a\}$$

Then, for each $a > 0$, S_a is an empty set or consists of a finite number of points.

Proof.

Assume that the given initial point x_0 is positive. Let the closed interval $[0, x_0]$ be I_{x_0}. Without loss of generality we can assume that S_{x_0} contains only one point x_1. Then, for some $u=u_1, f(x_1, u^1) = -c_1 < 0$. Also for some $u = u^0, f(0, u^0) = -c_0 < 0$. By the continuity of the function $f(x, u)$, for arbitrary small numbers $\varepsilon_0 > 0$, $\varepsilon_1 > 0$, there is positive numbers $\delta_0 < 0$, $\delta_1 > 0$, such that

$$f(x, u^0) < \varepsilon_0 - c_0 \qquad \text{for} \quad |x| < \delta_0$$
$$f(x, u^1) < \varepsilon_1 - c_1 \qquad \text{for} \quad |x - x_1| < \delta_1 .$$

We now devide the interval I_{x_0} to subintervals $I_0, \cdots, I_3$,

where $I_0 = [0, \delta_0)$, $I_1 = [\delta_0, x_1 - \delta_1]$, $I_2 = (x_1 - \delta_1, x_1 + \delta_1)$ $I_3 = [x_1 + \delta_1, x_0]$.

On the interval I_3, proceeding as in the proof of Theorem 1, we can choose a sectionally constant function $v^3(t)$ such that $f(x, v^3(t)) < -\lambda$, where $\lambda > o$ is arbitrarily given constant number. Thus the control $v^3(t)$ transfers x_0 to $x_1 + \delta_1$. Let us assume that $x(t_3) = x_1 + \delta_1$, where $x(t)$ is a solution of (10) for the control $v^3(t)$ under the initial condition $x(0) = x_0$. Similarly, on the interval I_1, we can choose a sectionally constant control $v^1(t)$ such that $f(x, v^1(t)) < -\lambda$ and the control $v^1(t)$ transfers $x(t_2) = x_1 - \delta_1$ to $x(t_1) = \delta_0$. Obviously $o < t_3 < t_2 < t_1$ and t_2 will be explicitely defined later.

We now consider the differential equation

$$\dot{x} = f(x, u^1), \quad x(t_3) = x^1 + \delta_1 .$$

Since for $x \in I_2, f(x, u^1) < \varepsilon - C_1$, the solution of the equation can be continued to the time $t=t_2$ such that $x(t_2)=x^1-\delta_1$. Similarly the control $u=u^0$ transfers the initial state

$$x(t_1) = \delta_0 \quad \text{to} \quad x(t_0) = o\,.$$

Now we define a control $u(t)$, $o\leq t\leq t_0$, as follows,

$$\begin{aligned} u(t) &= v^3(t) && 0\leq t\leq t_3 \\ &= u^1 && t_3<t\leq t_2 \\ &= v^1(t) && t_2<t\leq t_1 \\ &= u^1 && t_1<t\leq t^0 \end{aligned}$$

Then clearly this control transfers the initial point x^0 to the origin at time $t=t_0$. Moreover the required time length t_0 is such that

$$\begin{aligned} t^0 &= \int_0^{t_0} dt = \int_{x^0}^{0} \frac{dx}{f(x,\, u)} \\ &= \int_{I_3} \frac{dx}{f(x,\, v^3)} + \int_{I_2} \frac{dx}{f(x,\, u^1)} + \int_{I_1} \frac{dx}{f(x,\, v^1)} + \int_{I_0} \frac{dx}{f(x,\, u^0)} \\ &< \frac{1}{\lambda}(x^0-2\delta_1-\delta_0) + \frac{2\delta_1}{c_1-\varepsilon_1} + \frac{\delta_0}{c_0-\varepsilon_0}\,, \end{aligned}$$

Thus t_0 can be taken arbitrarily small if we select λ large enough, because δ_0 and δ_1 can be taken sufficiently small. Since x^0 is arbitrarily given the system (10) is completely well controllable. Q.E.D.

In the case when the condition (i) of the theorem is not satisfied, it is clear from the proofs of the theorems that we can show the quasi-controllability of the system. Thus, we have the following corollaries,

Corollary 1.1

The system (10) is completely quasi-controllable if it satisfies the following condition:

(i) For every fixed $x\neq o$, there exists a vector $u\in R^r$ which satisfies the inequality

$$xf(x,\, u)<o\,.$$

Corollary 2.1

The system (10) is completely well quasi-controllable if it satisfies the condition:

(i) The condition of Corollary 1.1 holds.

(ii) Define the sets S and S_a as follows,

$$S \equiv \{x: Inf f(x, u)\ sgnx > -\infty\}$$
$$S_a \equiv \{x: x \in S,\ |x| < a\}.$$

Then, for each $a > o$, S_a is an empty set or consists of a finite number of points

Remark

The condition (*ii*) of Theorem 2 or Corollary 2.1 can be generalized. In the case when S_a consists of infinite number of points, if S_a has a finite number of condensation points, the conclusion of Theorem 2 or Corollary 2.1 is valid. For example, consider a control system

$$\dot{x} = f(x, u)$$

$$\text{where} \quad f(x, u) = -x + u(x-1)^3 \sin \frac{1}{x-1}, \qquad \text{for} \quad x \neq 1$$
$$= -x \qquad\qquad , \qquad \text{for} \quad x = 1.$$

This system is completely well controllable.

4. Quasi-controllability of n-Dimensional System with Controls Appearing Linearly.

In this section we discuss the quasi-controllability of control systems with controls appearing linearly. Such a system is described by the equation (3)

$$\dot{x} = f(x) + G(x)u. \tag{3}$$

For this system we have the following lemma essentialy due to Roxin[5].

Lemma 1.

If the state x^1 is reachable from the state x^0 with respect to the system;

$$\dot{x} = G(x)u, \tag{11}$$

then the point x^0 is well quasi-controllable to x^1 with respect to the system (3).

Proof.

By the assumption there is a control $u(t)$ $o < t < t_1$ and the solution $\xi(t)$ of (11) corespording to $u = u(t)$ for the initial condition $\xi(o) = x^0$. And this solution satisfies the condition $\xi(t_1) = x^1$. We consider the equation

$$\dot{x} = \mu f(x) + G(x)u(t) \tag{12}$$

with initial condition $x(o) = x^0$ where μ is a small parameter. Since for $\mu = 0$ the solution of (12) becomes $\xi(t)$, and the right-hand side of (12) depends continuously on μ, the solution of (12) passes through the arbitrary neighborhood of x_1 at $t=t_1$, for sufficiently small μ. Let the solution of (12) be $\xi(t, \mu)$, then $\xi(t)=\xi(t, o)$. Now consider the system

$$\dot{x} = f(x) + G(x)\frac{u\left(\frac{t}{\mu}\right)}{\mu} \tag{13}$$

with initial condition $x(0)=x^0$. The solution of (13) is clearly $x=\xi\left(\frac{t}{\mu}\ \mu\right)$ and this solution satisfies $x(o)=\xi(0, \mu)=x^0$, $x(\mu t_1)=\xi(t_1, \mu)$. Since from the above discussion $\xi(t_1, \mu)$ is in the neighborhood of x^1, moreover $\mu t_1 \to o$ as $\mu \to o$, this proves that x^1 is well quasi-controllable from x^0 with respect to (13). Q.E.D.

For the special case, assume that the matrix $G(x)$ is constant. Let $G(x)=(g_1, \cdots, g_r)$ where g_i $(i=1, \cdots, r)$ is a constant n-dimensional column vector. From Lemma 1 we have the following lemma.

Lemma 2.

Let S be the subspace of R^n spanned by the constant vecters $g_1, \cdots, g_r$. Then for any given pair of point x^0, x^1 such that the vector $z \equiv x^1 - x^0$ is on S, x^0 is well quasi-controllable to x^1 with respect to (4).

Proof.

Let $\xi(t)$ be the differentiable curve on the subspace S such that $\xi(0)=0$, $\xi(t_1)=z$, then the tangent vector of $\xi(t)$ is expressed with a linear combination of g_i, $(i=1, \cdots, r)$;

$$\frac{d\xi}{dt} = \sum_1^r u_i g_i$$

If we change the paramenter t from 0 to t_1, then the function $u_i(t)$ is determined. Thus we have an equation

$$\frac{dx}{dt} = Gu(t)$$

The solution of this equation for the initial condition $x(o)=x^0$ satisfies the condition $x(t_1)=x^1$. This shows that x^1 is reachable from x^0 with respect to (11). Then from Lemma 1, x^0 is well quasi-controllable to x^1 with respect to (4). Q.E.D.

In the case when the matrix G is constant, some intersting results are obtained by reducing the quasi-controllability of the system to that of some lower dimensional system. We treat the control system (4)

$$\dot{x} = f(x) + Gu\,. \tag{4}$$

Instead of (4), we can treat the equation (6)

$$\dot{x} = f(x) + Hu\,, \tag{6}$$

without loss of generality, where H is of the form given at (3). We can rewrite (6) as follows,

$$\dot{y} = \phi(y, z) + E_r u \tag{12}$$

$$\dot{z} = \psi(y, z) \tag{13}$$

where $y=(x_1, x_2, \cdots, x_r)$, $z=(x_{r+1}, \cdots, x_n)$, $\phi=(f_1, \cdots, f_r)$, $\psi=(f_{r+1}, \cdots, f_n)$. Applying Lemma 2 to the system (12), (13), the next lemma holds clearly.

Lemma 3.

Let two points x^0, x^1 be $x^0=(y^0, z^0)=(x_1^0, \cdots, x_n^0)$, $x^1=(y^1, z^0)=(x_1^1, x_2^1, \cdots, x_r^1, x^0_{r+1} \cdots x_n^0)$, then x^0 is well quasi-controllable to x^1.

Corresponding to the system (12), (13) we define the $(n\text{-}r)$-dimensional control system

$$\dot{z} = \psi(v, z)\,, \tag{14}$$

where $z=(x_{r+1}, \cdots, x_n)$ is an $(n\text{-}r)$-dimensional state vector, and $v=(v_1 \cdots v_r)$ is an r-dimensional control vector. Between the controllability of the system (6) and that of the system (14) there are some relations.

Theorem 3

If the given system (6) is completely controllable, then system (14) is completely controllable. If (14) is completely quasi-controllable then (6) is completely quasi-controllable. Moreover the same statement is valid if the word "controllable" is replaced by the word "well-controllable".

Before proving the theorem, we notice the next lemmas.

Lemma 4

If a state x^1 is reachable from x^0 with respect to the system (1), then x^0 is quasi-controllable to x^1 with continuously differentiable control.

Proof.

By the hypothesis there exists a sectionally continuous control $u^0(t)$ and the corresponding solution $x^0(t)$ of (1) such that $x^0(0)=x^0$ and $x^0(T)=x^1$. Let t_i $(t_i<t_{i+1},\ i=1, 2, \cdots, k)$ be the points of discontinuity of $u^0(t)$. Now, define the interval I_i as follows, I_i: $t_i-\varepsilon\Delta_i \leq t \leq t_i+\varepsilon\Delta_i$, where Δ_i $(i=1, 2, \cdots, k)$, an fixed constants and ε is a small parameter. Then, for an arbitrary $\varepsilon>0$, there is continuous control $u^*(t)$ such that

$$u^*(t) = u^0(t)\,, \qquad \text{if } t \text{ does not belong to any } I_i,$$
$$= v(t)\,, \qquad \text{if } t \text{ belongs to some } I_i,$$

and $|u^*(t)| < M$, where M is a constant which is independent of ε. Let $x^*(t)$ be the solution of (1) corresponding to $u^*(t)$. Using continuous dependence of the solution of (1) on the initial condition, it can be shown that $x^*(t)$ is defined on the interval $0 \leqq t \leqq T$ and that $x^*(t)$ converges uniformly to $x^0(t)$ on the interval $0 \leq t \leq T$, as ε tends to zero. Since $u^*(t)$ is continuous, for arbitrary small $\eta > 0$, there exists a continuously differentiable control $u^{**}(t)$ such that

$$|u^*(t) - u^{**}(t)| < \eta \qquad \text{for} \quad 0 \leqq t \leqq T.$$

Let $x^{**}(t)$ be the solution of (1) corresponding to the control $u^{**}(t)$.
By the continuous dependence of the solution of (1) on the parameter, $x^{**}(t)$ uniformely converges to $x^*(t)$ on the interval $0 \leqq t \leqq T$ as η tends to zero. Since

$$|x^0(t) - x^{**}(t)| \leq |x^0(t) - x^*(t)| + |x^*(t) - x^{**}(t)|\,,$$

if we take $\varepsilon > 0$, $\eta > 0$ sufficiently small then the left side of the above inequality can be made arbitrarily small. Q.E.D.

Proof of the Theorem 3.

We assume (6) is compoetely controllable and show that (14) is completely controllable. From the hypothesis there is an admissible control $u^0(t) = (u_1^{\,0}(t)\,, \cdots, u_r^{\,0}(t))$ which transfers the initial state $x^0 = (x_1^{\,0}\,, \cdots, x_n^{\,0})$ to the origin in a finite time. Let $x(t; u^0)$ be the solution of (6) corresponding to the control $u = u^0(t)$ for the initial condition $x(0; u) = x^0$, then $x(t_1; 0) = o$ at some finite time $t = t_1$. In the system (14) we take the function $v(t) = (x_1(t; u^0)\,, \cdots, x_r(t; u^0))$ as a control. Then obviously the solution of (14) with an initial condition $z(0) = z^0 = (x^0{}_{r+1}\,, \cdots, x_n^{\,0})$ satisfies the relation $z(t_1) = 0$. Since x^0 is arbitrary, the system (14) is completely controllable. Now, we assume (14) is completely quasi-controllable and show that (6) is completely quasi-controllable. From the assumption there exists an admissible control function $v^0(t) = (v_1^{\,0}(t)\,, \cdots, v_r^{\,0}(t))$ which transfers the initial state $z(0) = z^0 = (x^0{}_{r+1}, \cdots, x_n^{\,0})$ to a given neighorhood of the origin at some finite time $t = t_1$. Then by Lemma 4, there exists a continuously differential control $v^*(t)$ and the corresponding solution of (14) $z(t; v^*)$ such that $z(t_1; v^*)$ is in the small neighborhood of the origin of the space R^{n-r}. In the control system (6), assume that the control law is determined by $u^0(t) = \dfrac{dv^*}{dt} - \phi(v^*(t), z(t; v^*))$, then from the uniqueness of the solution (14) this control $u^0(t)$ transfers the initial state $\tilde{x}^0 = (v^*(0), z^0)$ to the $\tilde{x}^1 = (v^*(t_1), z(t_1; v^*))$. Let $x^0 = (y^0, z^0)$ be a given initial point, then from Lemma 3 x^0 is well quasi-controllable to $\tilde{x}^0$ and $\tilde{x}^1$ is also well quasi-controllable to the point $x^1=(o,\ z(t_1;\ v^*))$. Then, owing to the continuity of the solutions of the differential equation to the initial condition it is easy to prove that x^0 is quasi-controllable to the origin.

The proof of last statement of the theorem is also the same as the above proof. Q.E.D.

From the above pıoof it is obvious that if (6) is completely quasi-controllable, then (14) is also completely quasi-controllable. Thus we have the next corollary.

Corollary 3

A necessary and sufficient condition for the system (6) to be completely quasi-controllable is that the system (14) is completely quasi-controllable.

In particular, if $f(x)$ in (6) is linear with respect to x there is well known criterion of complete controllability due to Kalman[1-4]. Consider a linear system (7)

$$\dot{x} = Fx+Gu \tag{7}$$

then the next lemma holds.

Lemma 5. (Kalman)

A constant system (7) is completely controllable if and only if the $n\times nr$ matrix

$$(G,\ FG\ ,\cdots,F^{n-1}G) \tag{15}$$

has rank n. Moreover, the system (7) is completely well-controllable if it is completely controllable.

Now, assume that the control equation is transformed to the form of (8)

$$\dot{x} = Fx+Hu\ , \tag{8}$$

and the matrix F is expressed as follows,

$$F=\begin{bmatrix} F_{11}\ , & F_{12} \\ F_{21}\ , & F_{22} \end{bmatrix}$$

where F_{12}, F_{12}, F_{22} and F_{22} are $n\times r$, $r\times(n-r)$ $(n-r)\times r$ and $(n-r)\times(n-r)$ constant matrices, respectively. Then the system (8) is expressed as follows,

$$\begin{aligned} \dot{y} &= F_{11}y+F_{12}z+E_r u\ , \\ \dot{z} &= F_{21}y+F_{22}z\ , \end{aligned} \tag{16}$$

where $y=(x_1\cdots x_r)$ and $z=(x_{r+1}\cdots x_n)$ are r-dimensional and $(n-r)$-dimensional vectors, respectively.

Corresponding to the system (16), we consider the following control system;

$$\dot{z} = F_{22}z+F_{21}v \tag{17}$$

where z is an $(n-r)$-dimensional state vector, v is an r-dimensional control vector. Then we get the following theorem correwponding to Theorem 3.

Theorem 4.

The constant system (16) is completely controllable if and only if the system (17) is completely controllable.

Proof.

If the system (17) is completely controllable, from Theorem 10 it is also completely well-controllable. Therefore, by Theorem 3 the system (16) is completely well quasi-controllable. If a constant system is completely well quasicontrollable, it is completely controllable (see Theorem 10)

We shall study in detail the case when the system has $(n-1)$ controls. In this case from Corollary 1.1. Corollary 2.1 and Theorem 3, we obtain a sufficient condition for the system to be completely quasi-controllable.

Theorem 5.

Consider a control system described by the control equation (6). Then, if the function $f_n(x)$ in (6) satisfies the condition:

(i) For every fixed x_n, there exists a $(n-1)$-dimensional vector v which satisfies an inequality,

$$x_n f_n(v, x_n) < 0 ,$$

then the system (6) is completely quasi-controllable.

Proof.

From Theorem 3 a sufficient condition for the given system (6) to be completely quasi-controllable is that a one-dimensional control system

$$\dot{x}_n = f_n(v_1, \cdots, v_{n-1}, x_n) \tag{18}$$

is completely quasi-controllable. Then, from Corollary 1.1 it is obvious that the condition of the above theorem is sufficient for the system (18) to be completely quasi-controllable.

Theorem 6.

If the function $f_n(x)$ satisfies the following conditions:

(i) The condition (i) of Theorem 5 is satisfied.

(ii) Define the sets S and S_a as follows,

$$S \equiv \{x_n : \underset{u}{Inf} f_n(x, u)\, sgn x > -\infty\}$$
$$S_a \equiv \{x_n : x_n \in S,\ |x_n| < a\}\ . \tag{19}$$

Then, for each $a>0$, S_a is an empty set or consists of a finite number of points.

Then the system (6) is completely well quasi-controllable.

Proof.

It is clear from Corollary 2.1, Theorem 3, and the Proof of Theorem 5.
In the case when the control equation is of the form (4), we can find a vector g_n which is orthogonal to each column vector of G, g_i, $(i=1, \cdots, n-1)$ and has unit length. Define a matrix $\tilde{G}$ as $\tilde{G} \equiv (g_1, \cdots, g_{n-1}, g_n)$ and the inverse matrix of $\tilde{G}$ as $\tilde{G}^{-1} \equiv K = (k_1, \cdots, k_n)'$, then clearly the vector k_n and the vector g_n are identical, we transform the vector x to y by the transformation

$$x = \tilde{G}y$$

then the given system (4) is transformed to he system;

$$\dot{y} = Kf(\tilde{G}y) + KGu\ . \tag{20}$$

Since $KG = \begin{bmatrix} E_{n-1} \\ 0 \end{bmatrix}$, we can apply Theorem 5 and Theorem 6 to the system (20).

Corollary 5.1

A sufficient condition for (4) to be completely quasi-controllable is that for every fixed value of $y_n = g_n'x$ there exists a $(n-1)$-dimensional vector v such that

$$y_n g_n' \tilde{f}(v, y_n) < 0 \tag{21}$$

where $\tilde{f}(y) = f(\tilde{G}y)$.

Proof.

Since $g_n = k_n$, $KG = \begin{bmatrix} E_{n-1} \\ 0 \end{bmatrix}$ from (20)we obtain

$$\dot{y}_n = g_n' \tilde{f}(y)$$

Then from Theorem 5 the condition (21) is obtained.

Corollary 6.1

If the following conditions are satisfied the system (4) is completely well quasi-controllable.

(i) The condition of Corollary 5.1 is satisfied.

(ii) Define the sets S and S_a as follows,

$$S \equiv \{y_n : \underset{v}{Inf}\, g_n' \tilde{f}(v_1, \cdots, v_{n-1}, y_n)\, sgn\, y_n > -\infty\}$$

$$S_a \equiv \{y_n : y_n \in S,\ |y_n| < a\}\,.$$

Then, for each $a>0$, S_a is an empty set or consists of a finite number of points.

Proof.

It is obvious from the proof of Corollary 6.1 and Theorem 6.

As an application of Corollary 5.1, we consider the linear system (8). From Corollary 5.1 if for every y_n the condition

$$y_n g_n' \tilde{f}(v_1, \cdots, v_{n-1}, y_n) < 0 \tag{22}$$

is satisfied for some $v_1, \cdots, v_{n-1}$, then the system (8) is completely quasi-controllable, where,

$$\tilde{f}(y) = f(\tilde{G}y) = F\tilde{G}y\,.$$

Let $\tilde{G}=(G, g_n)$, then (22) become;

$$y_n g_n F(G, g_n)\begin{bmatrix} v \\ y_n \end{bmatrix} < 0 \tag{23}$$

where $g_n'F(G, g_n)\begin{bmatrix} v \\ y_n \end{bmatrix} = g_n'(FG, Fg_n)\begin{bmatrix} v \\ y_n \end{bmatrix} = g_n'(FGv + Fg_n y_n)$,

so (23) is satisfied if and only if all of row vectors of $g_n'FG$ are not equal to zero, in other words,

$$g_n'FG = 0 \tag{24}$$

Note that the relation (24) is equivalent to the condition

$$\text{rank}\ (G, FG) = n$$

so, from Lemma 5, the condition (24) is necessary and sufficient for the system to be completely controllable.

Example 1.

Consider the case when ψ in the equation (13) is linear. The system equation become

$$\begin{aligned} \dot{y} &= \phi(y, z) + E_r u \\ \dot{z} &= F_1 y + F_2 z\,, \end{aligned} \tag{25}$$

where $y=(x_1, \cdots, x_r)$, $z=(x_{r+1}, \cdots, x_n)$ and F_1 s $(n-r)\times r$ matrix and F_2 is

$(n-r)\times(n-r)$ matrix. Then from Theorem 3 and Lemma 5, this system is completely quasi-controllable if

$$\text{rank}\,(F_1, F_2F_1, \cdots, F_2^{n-r-1}F_1) = n-r\,.$$

Example 2.
consider a higher order system

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x = u\,, \tag{26}$$

where $x^i = \frac{dx^i}{dt^i}$, a_i is a function of x, $\dot{x}$,$\cdots$, x^{n-1}, $(i=1, \cdots, n)$. If we put $x=x_1$, $\dot{x}=x_2$,$\cdots$, $x^{n-1}=x_n$ then the system (26) is equivalent tothe system;

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \quad\vdots \\ \dot{x}_n = -(a_1x_n + a_2x_{n-1} + \cdots + a_{n-1}x_2 + a_nx_1) + u \end{cases}$$

Since the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \quad\vdots \\ \dot{x}_{n-1} = v \end{cases}$$

is completely controllable from Lemma 5, the system (26) is completely quas - controllable.

Example 3.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3) + u \\ \dot{x}_2 &= x_1 + f_2(x_2, x_3) \\ \dot{x}_3 &= x_2 + f_3(x_3) \end{aligned} \tag{27}$$

This system is completely quasi-controllable if the system

$$\begin{aligned} \dot{x}_2 &= f_2(x_2, x_3) + v \\ \dot{x}_3 &= x_2 + f_3(x_3) \end{aligned} \tag{28}$$

is comple ely quasi-controllable where v is a control. The system (28) is completely quasi-controllable since $\dot{x}_3 = f_3(x_3) + w$ is completely quasi-controllable where w is control. Thus the system (27) is shown to be completely quasi-controllable.

5. Quasi-Controllability of General Nonlinear Control Systems

In this section Corollary 1.1 obtained in the section 3 is extended to general nonlinear systems, in other words, a sufficient condition of complete quasi-controllability for n-dimensional systems is shown.

Condider an n-dimensional system

$$\dot{x} = f(x, u) \tag{2}$$

For preliminary, we state a simple lemma due to E. Roxin[5].

Lemma 6.

If x^1 is quasi-reachable from x^0 and x^2 is quasi-reachable from x^1, then x^2 is quasi-reachable from x^0.

Using this lemma we shall prove the following theorem.

Theorem 7.

Assume hat there is a positive definite scalar function $v(x)$ which satisfies the following conditions:

(i) The function $v(x)$ has continuous partial derivatives.

(ii) $\lim\limits_{|x|\to\infty} v(x) = \infty$

(iii) For each fixed $x(\neq 0)$, there exists a constan vector $u \in R^r$ such that $\frac{\partial v}{\partial x} f(x, u) < 0$.

Then the control system (2) is completely quas -con rollab'e.

Proof.

Let x^0 be a given initial state, and define the set R_{x^0} as the set of $x \in R^n$ which is reachable from x^0 in a finite time. If the origin is not quasi-reachable from x^0, then $\inf\limits_{x\in R_{x^0}} v(x) \neq 0$. So we shall show that $\inf\limits_{x\in R_{x^0}} v(x) = 0$. Assume $\inf\limits_{x\in R_{x^0}} v(x) = \alpha > 0$, then there exists a sequence $S_1 = (x^1, x^2, \cdots, x^i, \cdots)$, $S_1 \subset R_{x^0}$ such that $\lim\limits_{i\to\infty} v(x^i) = \alpha$. From the sequence S_1 we can select a sequence $S_2 = (x^{i_1}, x^{i_2}, \cdots,)$, which converges to some point x^α. This point x^α is quasi-reachable from x^0 and clearly $v(x^\alpha) = \alpha$. From the hypothesis of the theorem there is a neighborhood o_α of x^α and a vector $u^\alpha \in R^r$ such that;

$$\frac{\partial v(x)}{\partial x} f(x, u^\alpha) < 0 , \qquad \text{for} \quad x \in o_\alpha .$$

We consider the differential equation;

$$\dot{x} = f(x, u^\alpha) .$$

with initial condi ion x^α at $t=0$. This solution $x(t; u^\alpha)$ exists at some time interval, and since the time derivative of the function $v(x)$ along the solution of the equation (2) is negative for $x \in o_\alpha$;

$$\frac{dv(x(t;\, u^\alpha))}{dt} = \frac{\partial v(x(t;\, u^\alpha))}{\partial x} f(x(t;\, u^\alpha),\, u^\alpha) < 0$$

for t such that $x(t;\, u^\alpha) \in o_\alpha$.

There is some time $t = t_1$ such that $v(x(t_1;\, u^\alpha)) < \alpha$. If we put $x^\beta = x(t_1;\, u^\alpha)$, then x^β is reachable from x^α and $v(x^\alpha) > v(x^\beta)$. Since x^α is quasi-reachable from x^0. So, in the neighborhood of x^β there is a point x^γ which is reachable from x^0 and $v(x^\gamma) < \alpha$. This contradicts the assumption, $\inf_{x \in R_{x^0}} v(x) = \alpha$. Hence, $\inf_{x \in R_{x^0}} v(x) = 0$, and the theorem is proved. Q.E.D.

Corollary 7.1

Assume that there exists a positive definite scalar function $v(x)$ and r-dimensional vector function $u(x) = (u_1(x), \cdots, u_r(x))$ which satisfy the following conditions:

(i) The condition (i), (ii) of Theorem 7 hold.

(ii) $u_i(x)$ is continuous $(i = 1, \cdots, r)$.

(iii) $\dfrac{\partial v(x)}{\partial x} \cdot f(x,\, u(x)) < 0$ for $x = 0$.

Then, the system (2) is completely quasi-controllable.

This is the result obtained by L. Markus[8]. The proof is clear from Theorem 7.

Example

$$\dot{x}_1 = f_1(x) + x_1 u$$
$$\dot{x}_2 = f_2(x) + x_2 u$$

Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, then,

$$\frac{\partial v}{\partial x} f(x) = x_1 f_1(x) + x_2 f_2(x) + (x_1^2 + x_2^2)\, u\,.$$

Since this function $V(x)$ satisfies the conditions of Theorem 7, this system is completely quasi-controllable.

6. Local Controllability and Complete Controllability of Nonlinear Systems

Consider the nonlinear systems

$$\dot{x} = f(x,\, u) \tag{2}$$

In this section the origin is assumed to be the stationary point of the controlled system. In other words, the right hand side of (2) satisfies the condition;

$$f(o, o) = o$$

For this system we define the concept of local controllability.

Definition. If there exists a neighborhood $U \subset R^n$ of the origin such that each point $x^0 \in U$ can be transfered to the origin in R^n in a finite time interval, using an admissible control function, then the system (2) is said to be locally completely controllable.

A sufficient condition for local complete controllability is known[7-8].

Lemma 7.

Condider the control system in R^n,

$$\dot{x} = f(x, u)$$

where $f(x, u)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial u}$ are continuous in $R^n \times \Omega$.

The control restraint $\Omega \subset R^r$ contains the origin in its interior.

Assume:

(1) $f(o, o) = o$

(2) rank $(B, AB, \cdots, A^{n-1}B) = n$,

where $A = \frac{\partial f(o, o)}{\partial x}$, $B = \frac{\partial f(o, o)}{\partial u}$,

Then the system is locally completely controllable.

From the definitions of local controllability and quasi-controllability we have at once the following theorem.

Theorem 8.

If control system (2) is completely quasi-controllable and locally completely controllable, then the system is completely controllable.

We consider the same examples as in the section 4.

Example 1.

The system (25) is completely quasi-controllable if

$$\text{rank } (F_1, F_2F_1 \cdots F_2^{n-r-1}F_1) = n- \tag{29}$$

Consider the system

$$\begin{aligned} \dot{y} &= Ay + Bz + E_r u \\ \dot{z} &= F_1 y + F_2 z \end{aligned} \tag{30}$$

where $A = \dfrac{\partial f^1(o, o)}{\partial y}, \quad B = \dfrac{\partial f^1(o, o)}{\partial z},$

If the system (30) is completely controllable then the system (25) is locally completely controllable. On the other hand, from Theorem 4 the system (30) is completely controllable if the relation (29) holds. Therefore the relation (29) is a sufficient condition for the system (25) to be completely controllable.

Example 2.
The system (26) is completely quasi-controllable and locally completely quasi-controllable, hence completely controllable.

Example 3.
The system (27) is known to be completely quasi-controllable. For this system

$$A = \frac{\partial f(o, o)}{\partial x} = \begin{bmatrix} a_{11}, & a_{12}, & a_{13} \\ 1 & a_{22}, & a_{23} \\ o & 1 & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ o \\ o \end{bmatrix}$$

where $a_{ij} = \dfrac{\partial f^i(o, o)}{\partial x_j}$, $(i, j = 1, 2, 3)$. Since rank $(B, AB, A^2B) = 3$, the system (27) is locally completely controllable. Therefore it is completely controllable.

7. Relations between the various kinds of the controllability

In this section we study the relations between the various concepts of the controllability, which is defined in the section 1. At first consider the linear system

$$\dot{x} = Fx + Gu \tag{7}$$

By Kalman, it is proved that the system (7) is expressed, with an appropriate linear transformation, as follows,

$$\dot{y} = F_{11} y + F_{12} z + Hu \tag{31}$$

$$\dot{z} = F_{22} z, \tag{32}$$

where the vector y is controllable and the sum of dimensions of the vectors y and z is equal to n. If the system is transformed to this form, we have at once a necessary and sufficient condition for complete quasi-controllability.

Theorem 9.

A necessary and sufficient condition for complete quasi-controllability of (7) is that the system is completely controllable or the system (32) is asymptotically stable.

Theorem 10.

For linear systems, the concepts of complete controllability, complete well controllability, complete well quasi-controllability, complete controllability to the whole, and complete quasi-controllability to the whole are equivalent.

Proof.

From Lemma 5, if the system is completely controllable, then completely well controllable. If the system is not completely controllable, then the dimension of the vector z in (32) is not equal to zero. Since the system (32) is autonomous it cannot be completely well quasi-controllable. In other words, if the system is completely well quasi-controllable, it is also completely controllable. If the system is completely controllable, it is also completely controllable to the whole. Assume that the system is completely quasi-controllable to the whole, but not completely controllable, then, of course, it is completely quasi-controllable to the origin. Therefore the system (32) must be asymptotically stable. This contradcts the assumption. Hence, the dimension of the vector z must be zero. Q.E.D.

From the above theorem it is known that in linear systems we distinguish only two kinds of controllability, that is, complete controllability and complete quasi-controllability.

The relations between various concepts are shown in Fig. 1.

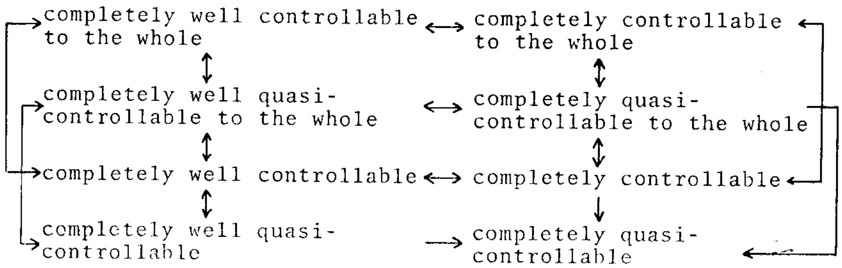

Fig. 1.

Consider a l near system

$$\dot{x}_1 = -x$$

$$\dot{x}_2 = u$$

This system is not completely controllable but completely quasi-controllable.

For nonlinear systems several relations which hold for linear systems, do not hold. For example, consider one dimensional systems

Example 1.

$$\dot{x} = g(u) ,$$

where $|g(u)| < K$ and $ug(u) > 0$ for $u \neq 0$. This system is clearly completely controllable, but is not completely well-controllable.

Example 2.

$$\dot{x} = -(x+1)u_1^2 - (x-1)u_2^2$$

This system is completely controllable. But, since

$$x\dot{x} < 0 \qquad \text{for all} \quad |x| > 1 ,$$

it is not completely controllable to the whole.

Example 3.

$$\dot{x} = ux$$

This system is completely well quasi-controllable but is not completely controllable.

These relations are shown in Fig. 2.

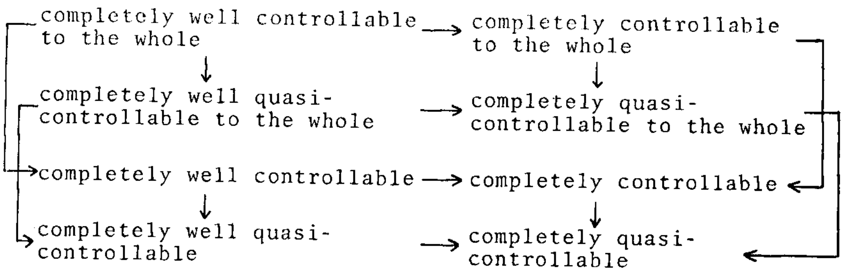

Fig. 2.

8. Conclusion

The concepts of controllability, quasi-controllability, well-controllability, etc, are introduced, and sufficient conditions for this controllability are obtained.

At first one-dimensional systems are considered and some sufficient conditions for controllability are obtained. One of these results are extended to n-dimensional nonlinear systems in the section 5 and a simple criterion for complete quasi-controllability was obtained. But systems to which this criterion is applicable are restricted. In many cases we cannot discuss directly the controllability of general n-dimensional nonlinear systems. So, we treated some special type of nonlinear systems, the system which is nonlinear with respect to x but linear with respect to u. In section 4 the controllability of such systems was discussed by reducing the discussion of the given system to that of some lower dimensional system. In the case when the origin is the critical point of the control equation with $u=0$, the concept of local controllability is important.

It is also shown that many of the concepts of the controllability defined in this paper are equivalent to each other if the system is linear, and there is essentially two kinds of controllability, complete controllability and complete quasi-controllability.

In this paper the control u is assumed to have no restraint, except in Lemma 9. But in reality the control is necessarily restrained in some means. Hence, it seems important to discuss the controllability with restrained controls.

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