Some Contributions to Design Theory of Magnetohydrodynamic Generator Duct

By

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With the intention of giving some new contributions to the design theory of the magnetohydrodynamic generator duct, first we derive the solutions from the well-known basic quasi one-dimensional magnetohydrodynamic equations for the ideal segmented electrode Faraday generator with constant velocity in the case, where it is assumed that the working plasma fluid has the variable conductivity, which is expressed by a power-exponential formula of temperature and pressure in the state of thermal equilibrium ionization, or moreover by a power one of temperature, pressure, current and magnetic flux densities in nonequilibrium ionization. Next we introduce the numerically solvable differential equations and some solutions from the magnetohydrodynamic equations for the linear Hall generator with constant velocity in the case, where the conductivity is expressed by a power formula or a power-exponential one of temperature and pressure, and the Hall parameter by the power of one of them.

1. Introduction

As is well-known the conductivity of the working plasma fluid in the magnetohydrodynamic generator is generally a function of the temperature and pressure even in thermal equilibrium ionization. Moreover, in nonequilibrium ionization due to electron heating, which is being investigated especially in recent years, it has been confirmed theoretically and experimentally that the conductivity is varied not only due to temperature and pressure but also by the current density in plasma flow and the applied magnetic flux density. However in the flow physics or design theory of magnetohydrodynamic generator duct, the plasma conductivity mostly has been assumed to be constant, in order to simplify the theoretical analysis and numerical calculations except in the case of a few exception[1)~5)], in which the solutions of the basic one-dimentional magnetohydrodynamic equations are obtained on the assumption that the conductivity is governed by a power law of temperature and pressure.

So in the following section, first, the author tries to seek for the solutions of

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the basic flow equations in Faraday generator ducts with segmented electrodes in the two cases, where the conductivity is expressed by a power-exponential formula[5)~7)], which is deduced from, for example, Saha formula with respect to equilibrium ionization, and it is the governed by the power law[8)~13)] of temperature, pressure, current and magnetic flux densities due to nonequilibrium ionization. Next he will derive the numerically solvable differential equations and some solutions from the basic flow equations for the linear Hall generator in the case, where the fluid conductivity is expressed by a power formula or a power-exponential one of temperature and pressure, and Hall parameter by a power one[7)] of both.

In this connection, recently the flow physics in Hall generator in the case, where it is assumed that the conductivity follows the power-exponential law and Hall parameter is a function of pressure only, is discussed[6)] quantitatively by means of numerically solvable simultaneous differential equations.

2. Faraday Generator with Segmented Electrodes

2.1 Basic Equations

As is well-known, the quasi one-dimensional magnetohydrodynamic equations pertaining to the segmented electrode Faraday generator duct (Fig. 1) with slowly varying cross-sectional area and constant flow velocity are given by

$$\rho u A = \rho_0 u A_0 = \text{constant: continuity equation,} \qquad (1)$$

$$\frac{dp}{dx} = J_y B\text{: momentum equation,} \qquad (2)$$

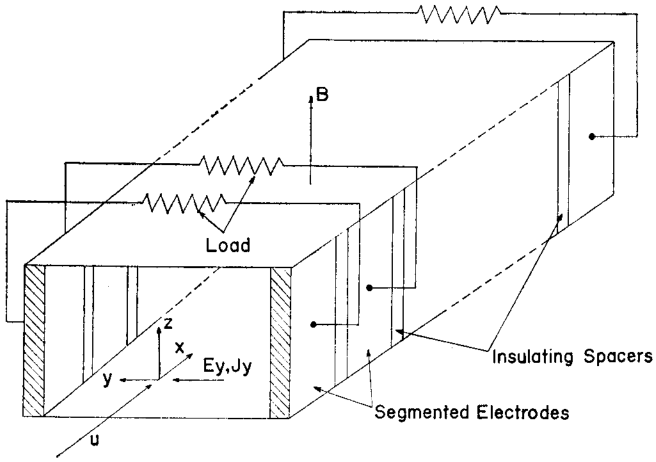

Fig. 1. Sketch of segmented electrode Faraday generator duct.

$$\rho u c_p \frac{dT}{dx} = J_y E_y: \text{ energy equation,} \tag{3}$$

$$p = \rho RT: \text{ state equation,} \tag{4}$$

and Ohm law is expressed by

$$J_y = \sigma(E_y - uB) = -\sigma uB(1-\kappa) \tag{5}$$

In these equations and Fig. 1

A: corss-sectional area of duct,
p, T and ρ: gas pressure, temperature and density,
R: gas constant*,
$c_p = R\gamma/(\gamma-1)$: specific heat* at constant pressure,
$\gamma = c_p/c_v$: specific heat ratio*,
c_v: specific heat* at constant volume,
u: gas velocity*,
σ: electrical conductivity,
$\kappa = E_y/uB$: loading parameter*,
B: magnetic flux density*,
E_y: electric field intensity* in y-directions,
J_y: current density in y-direction, which is equal to total one,
Suffix 0 and 1: shows the quantities in duct inlet and outlet respectively,
*: shows the quantities which are assumed to be constant in analysis.

(6)

Now, the electrical conductivity is usually a function of pressure and temperature, and it can also be thought to be a function of the total current and magnetic flux densities in the nonequilibrium ionization due to strong electron heating within monatonic gases seeded with alkali metal vapors. However, up to the present, the basic flow equations have been analysed on the assumption that mostly

$$\sigma = \bar{\sigma}: \text{ mean value about presure temperature et al.,} \tag{7}$$

and at times and at most, σ is expressed with the power law as follows,

$$\sigma = cp^m T^n , \tag{8}$$

where

$$c, m \text{ and } n: \text{ constants.} \tag{8$'$}$$

In this paper, the author will give the solutions of the flow equations on the more realistic formula that

$$\sigma = cp^m T^n \exp\left(-T_i/T\right), \tag{9$_1$}$$

where

$$\left.\begin{array}{l} c,\ m \text{ and } n\text{: constants } (m=-1/2 \text{ and } n=3/4 \text{ if Saha formula can be used}),\\ T_i = \varepsilon_i/2k,\\ \varepsilon_i\text{: (equivalent) ionization energy,}\\ k\text{ : Boltzmann constant,} \end{array}\right\} \tag{9$_1'$}$$

for thermal equilibrium ionization without elevation of electron temperature, and

$$\sigma = cp^m T^n J^\mu (aB+b)^\nu \tag{10}$$

where

$$\left.\begin{array}{l} a,\ b,\ c,\ m,\ n,\ \mu,\ \nu\text{: constants,}\\ J\text{ : resultant current density} \end{array}\right\} \tag{10$'$}$$

for the nonequilibrium ionization due to electron heating. Here, from the experimental data given in the reference 12), the author pressumed the dependence of σ on B as expressed in Eq. (10). But, of course, $(aB+b)^\nu$ become a constant, when B is assumed constant as in this paper.

2.2 Solutions

(i) When $\sigma = cp^m T^n \exp(-T_i/T)$

As above mentioned, here we shall solve the one-dimensional magnetohydrodynamic equations with the scalar conductivity $\sigma = cp^m T^n \exp(-T_i/T)$.
Now using Eqs. (2) and (3), we obtain

$$\frac{dT}{dp} = \frac{\gamma_\kappa T}{p} \tag{11$_1$}$$

or non-dimentionally

$$\frac{dT^*}{dp^*} = \frac{\gamma_\kappa T^*}{p^*}, \tag{11$_2$}$$

where

$$\left.\begin{array}{l} \gamma_\kappa = (\gamma-1)\kappa/\gamma,\\ T^* = T/T_0,\\ p^* = p/p_0 . \end{array}\right\} \tag{11$'$}$$

We can easily obtain the solution of Eq. $(11)_1$ or $(11)_2$ as follows.

$$T^* = p^{*\gamma_\kappa} \tag{12}$$

This result does not depend on the conductivity σ and it is quite identical with the one in the case where σ is assumed constant. Further, using Eq.s (1), (4) and (12) we get

$$A^* = A/A_0 = \rho_0/\rho = T^*/p^* = p^{*\gamma_\kappa - 1}\,. \tag{13}$$

These relations, too, come into existance whether σ is constant or variable.

Next substituting Eq. (5) into (2), we have

$$\frac{dp}{dx} = -\sigma u B^2(1-\kappa)\,. \qquad (14)_1$$

As we can modify the expression $(9)_1$ as follows,

$$\sigma = \sigma_0 p^{*\,m+n\gamma_\kappa} \exp\{-T_i^*(p^{*-\gamma_\kappa}-1)\} \qquad (9)_2$$

where

$$T_i^* = T_i/T_0 \qquad (9)_2'$$

Eq. $(14)_1$ is transformed into the following equation.

$$\frac{dp^*}{dx^*} = \frac{1}{F(p^*)}\,, \qquad (14)_2$$

where

$$\left.\begin{aligned} F(p^*) &= -\frac{\bar\sigma^*}{1-p_1^*}p^{*-(m+n\gamma_\kappa)}\exp\{T_i^*(p^{*-\gamma_\kappa}-1)\}\,,\\ \bar\sigma^* &= \bar\sigma/\sigma_0\,,\\ x^* &= x/l\,,\\ l &= \frac{p_0(1-p_1^*)}{\bar\sigma u B^2(1-\kappa)}:\ \text{duct length obtained when } \sigma=\bar\sigma\,,\\ p_1^* &= p_1/p_0\,. \end{aligned}\right\} \qquad (14)_2'$$

The solution of Eq. $(14)_2$ becomes

$$x^* = \int_1^{p^*} F(p^*)\,dp^*\,. \tag{15}$$

Henceforth by integrating numerically this equation about p^*, we can obtain the numerical values of x^* to evaluate the x^*-p^* characteristics in the generator duct.

(ii) When $\sigma = cp^m T^n J^\mu (aB+b)^\nu$

In the case of the nonequilibrium ionization, too, we can obtain Eq. $(11)_1$ to

(13), because they are not dependent on the conductivity. Now substituting Eq. (10) into (5) gives

$$J'_y = cp^m T^n J'^{\mu}_y (aB+b)^{\nu} uB(1-\kappa)\,, \tag{16}$$

where

$$J'_y = -J_y = J\,. \tag{16$'$}$$

So, if we express the inlet current density with J_0, neglecting the so-called inlet relaxation, we have

$$J_0 = J'_{y0} = cp_0^m T_0^n J'^{\mu}_{y0}(aB+b)^{\nu} uB(1-\kappa)\,. \tag{17}$$

From Eq.s (16) and (17), we can derive

$$J'^*_y = -J^*_y = p^{*m_\mu} T^{*n_\mu} = p^{*m_n+n_\mu\gamma_\kappa}\,, \tag{18}$$

where

$$\left.\begin{aligned} J'^*_y &= -J^*_y = -J_y/J'_{y0}\,, \\ m_\mu &= m/(1-\mu)\,, \\ n_\mu &= n/(1-\mu)\,. \end{aligned}\right\} \tag{18$'$}$$

By means of Eq.s (2) and (18), we can introduce

$$\frac{dp^*}{dx^*} = -\frac{p^{*m_\mu+n_\mu\gamma_\kappa}}{\bar{J}_y}\,, \tag{19}$$

where

$$\left.\begin{aligned} \bar{J}'^*_y &= \bar{J}'_y/J'_{y0}\,, \\ \bar{J}'_y &= -\bar{J}_y: \text{ mean current density.} \end{aligned}\right\} \tag{19$'$}$$

Solving Eq. (19), we have

$$p^* = \left[1 - \frac{\{1-(m_\mu+n_\mu\gamma_\kappa)\}(1-p_1^*)x^*}{\bar{J}'^*_y}\right]^{\frac{1}{1-(m_\mu+n_\mu\gamma_\kappa)}}. \tag{20}$$

As is able to be presumed from Eq.s (5), (8) and (18), this solution has the same form as the one in the case where $\bar{\sigma}$ is given by Eq. (8). p^* in this case is obtained as follows[1].

$$p^* = \left[1 - \frac{\{1-(m+n\gamma_\kappa)\}(1-p_1^*)x^*}{\bar{\sigma}^*}\right]^{\frac{1}{1-(m+n\gamma_\kappa)}}. \tag{21}$$

3. Hall Generator

3.1 Basic Equations

As is well-known, the plasma flow in the ideally segmented electrode Hall

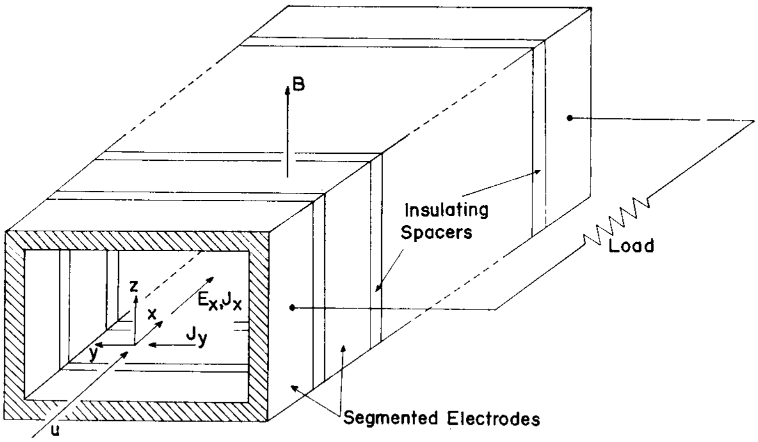

Fig. 2. Sketch of segmented electrode Hall generator duct.

generator duct (Fig. 2) with constant velocity is decribed by the following set of equations.

$$\rho u A = \rho_0 u A_0 : \text{ continuity equation,} \tag{22}$$

$$\frac{dp}{dx} = J_y B : \text{ momentum equation,} \tag{23}$$

$$\rho u c_p \frac{dT}{dx} = J_x E_x : \text{ energy equation,} \tag{24}$$

$$J_x A = J_{x0} A_0 : \text{ current continuity condition,} \tag{25}$$

$$p = \rho R T : \text{ state equation,} \tag{26}$$

$$J_x = \frac{\sigma}{1+\beta^2}(E_x + \beta u B) = \frac{\sigma \beta u B (1-\kappa_h)}{1+\beta^2}, \tag{27}$$

$$J_y = \frac{\sigma}{1+\beta^2}(\beta E_x - u B) = -\frac{\sigma u B (1+\kappa_h \beta^2)}{1+\beta^2}. \tag{28}$$

In these equations and Fig. 2

$$\left.\begin{array}{l} \beta : \text{ Hall parameter with respect to electron,} \\ \kappa_h = -E_x/\beta u B : \text{ loading parameter,} \\ E_x : \text{ electric field intensity in } x\text{-direction,} \\ J_x : \text{ current density in } x\text{-direction.} \end{array}\right\} \tag{29}$$

The other symbols show the same as the ones defined in Eq.s (6). Here, we can assume that β is expressed as follows.

$$\beta = \lambda B p^{m'} T^{n'}, \tag{30}$$

where

m' and n': constants ($m'=-1$ and $n'=1/2$ for inert gas seeded with alkali metal). (30)′

In the following analyses, we shall employ this formula for the Hall parameter. Some analyses and numerical discussion are already done in regard to the case, where σ and β are assumed constant. In the references (2) and (3), the basic equations in the case, where σ and β are expressed by Eq.s (8) and (30) respectively, are a little discussed numerically. Moreover in refrence (6), the case, where σ is expressed by Eq. $(9)_1$ and β is assumed as follows,

$$\beta = \lambda' p^{m'}, \tag{31}$$

where

λ' and m' : constants, (31)′

is numerically discussed.

So in this section, from the above basic equations we derive the differential equations of the first order, of which the numerical solution can be obtained by the Runge-Kutta or the other digital calculation methods, for the case, where σ is expressed by Eq. (8) or $(9)_1$ and β by Eq. (30).

Moreover we shall introduce the solutions of the basic equations in the case, where about Hall parameter we can assume as follows.

$$\beta^2 \gg 1 , \tag{32}$$

$$\kappa_h \beta^2 \gg 1 . \tag{33}$$

In this connection, when only $\beta^2 \gg 1$ can be assumed, it is very difficult that we derive the solutions of the basic equations, and we so must tre1t the differential equations to obtain the numerical solution as in the case, where Eqs. (32) and (33) are not able to be presumed.

3.2 Solutions

(i) When $\sigma = cp^m T^n$

Using Eq.s. (22) to (28), we can get the following differential equation.

$$\frac{dT}{dp} = \frac{\gamma' \kappa_h (1-\kappa_h) T}{(\beta^{-2}+\kappa_h) p}, \tag{34}$$

where

$$\left.\begin{aligned} \kappa_h &= 1 - \frac{\beta J_{x0} p T_0}{\sigma u B p_0 T}(1+\beta^{-2}) , \\ \gamma' &= (\gamma-1)/\gamma . \end{aligned}\right\} \tag{34'}$$

Substituting Eq.s (8) and (30) in (34) yields

$$\frac{dT^*}{dp^*}=r'J^*_{x0}p^{*-m+m'}T^{*-n+n'}(1+\beta^{-2})[1+\beta^{-2}/\{1-J^*_{x0}p^{*-m+m'+1}T^{*-n+n'-1}\times(1+\beta^{-2})\}]^{-1}$$
$$=r'\{1-J^*_{x0}p^{*-m+m'+1}T^{*-n+n'-1}(1+\beta^{-2})\}(J^{*-1}_{x0}p^{*m-m'}T^{*n-n'}-p^*T^{*-1}) \quad (35)$$

where

$$\left.\begin{aligned} J^*_{x0} &= J_{x0}/\sigma_0 uB/\beta_0\,,\\ \beta &= \beta_0 p^{*m'}T^{*n'}\,,\\ \beta_0 &= \lambda B p_0^{m'}T_0^{n'}\,. \end{aligned}\right\} \quad (35)'$$

Eq. (35) can be numerically solved by a digital calculation method, and so we are able to determine the quantitative relations between p^* and T^* i.e. the local gas pressure and temperature.

Next, the substitution of Eq. (28) in (23) gives

$$\frac{dp^*}{dx^*}=\frac{1}{F_{h10}(p^*,T^*)}\,, \quad (36)$$

where

$$\left.\begin{aligned} F_{h10}(p^*,T^*) &= -\{\bar{l}^*p^{*m}T^{*n}(1-J^*_{x0}p^{*-m+m'+1}T^{*-n+n'-1})\}^{-1}\,,\\ l^* &= \bar{l}/l_0\,,\\ l_0 &= p_0/\sigma_0 uB^2\,,\\ \bar{l} &: \text{ duct length}^{6)14)} \text{ obtained when } \sigma=\bar{\sigma}\,. \end{aligned}\right\} \quad (36)'$$

under the support of the other basic equations.

From Eq. (36), we have

$$x^*=\int_1^{p^*}F_{h10}(p^*,T^*)\,dp^*\,. \quad (37)$$

By using the numerical relation between p^* and T^*, which is acquired by Eq. (35), we can carry out digitally the integration of Eq. (37) and ascertain the quantitative relations between p^* and x^* moreover x^* and T^* in the generator duct.

After we obtain the numerical solutions of p^* and T^* vs. x^*, we can get the digital relation of, for example, A^* vs. x^* by the following equation

$$A^*=p^{*-1}T^*\,, \quad (38)$$

which is derived from Eq.s (22) and (26), (35) and (37).

Next, when we can assume that $\beta^2\gg 1$ and $\kappa_h\beta^2\gg 1$, Eq. (35) are simplified as follows.

$$\frac{dT^*}{dp^*} = r' J_{x0}^* p^{*-m+m'} T^{*-n+n'} . \tag{39}$$

Then the solution of the equation becomes

$$T^* = \left\{ 1 + \frac{n-n'+1}{m-m'-1} r' J_{x0}^* (1 - p^{*-m+m'+1}) \right\}^{1/(n-n'+1)} . \tag{40}$$

Moreover although the form of Eq. (36) is not transformed even in the case, where $\beta^2 \gg 1$ and $\kappa_h \beta^2 \gg 1$, $F_{h10}(p^*, T^*)$ becomes the function of only p^* due to Eq. (40), namely

$$\frac{dp^*}{dx^*} = \frac{1}{F_{h11}(p^*)} , \tag{41}$$

where

$$F_{h11}(p^*) = \{\bar{l}^* p^{*m} T^{*n} (1 - J_{x0}^* p^{*-m+m'+1} T^{*-n+n'-1})\} , \tag{41$'$}$$

T^*: given by Eq. (40).

Solving Eq. (41), we have

$$x^* = \int_1^{p^*} F_{h11}(p^*) \, dp^* . \tag{42}$$

(ii) when $\sigma = c p^m T^n \exp(-T_i/T)$

From Eq.s $(9)_1$, (30), (34), we can derive the following differential equation.

$$\begin{aligned} \frac{dT^*}{dp^*} &= r' J_{x0}^* p^{*-m+m'} T^{*-n+n'} (1+\beta^{-2}) \exp\{T_i^*(T^{*-1}-1)\} \\ &\quad \times [1+\beta^{-2}/\{1 - J_{x0}^* p^{*-m+m'+1} T^{*-n+n'-1} (1+\beta^{-2}) \exp(T_i^* T^{*-1} - 1)\}]^{-1} \\ &= r' [1 - J_{x0}^* p^{*-m+m'+1} T^{*-n+n'-1} (1+\beta^{-2}) \exp\{T_i^*(T^{*-1}-1)\}] \\ &\quad \times [J_{x0}^{*-1} T^{*m-m'} p^{*n-n'} \exp\{T_i^*(T^{*-1}-1)\} - p^* T^{*-1}]^{-1} . \end{aligned} \tag{43}$$

By substitution of Eq. (27) in (24), we have

$$\frac{dp^*}{dx^*} = \frac{1}{F_{h20}(p^*, T^*)} , \tag{44}$$

where

$$\begin{aligned} F_{h20}(p^*, T^*) &= -[\bar{l}^* p^{*m} T^{*n} \exp\{-T_i^*(T^{*-1}-1)\} \\ &\quad \times \{1 - J_{x0}^* p^{*-m+m'+1} T^{*-n+n'-1} \exp(T_i^* \overline{T_i^{*-1} - 1})\}] . \end{aligned} \tag{44$'$}$$

Hence we can obtain the following solution

$$x^* = \int_1^{p^*} F_{h20}(p^*, T^*) \, dp^* . \tag{45}$$

By means of Eq.s (43) and (45), we can get the quantitative relation between p^* and T^* and the one between p^* and x^* moreover T^* and x^* with the aid of appropriate digital computations.

Next, when we can assume the relations in Eq.s (32) and (33), Eq. (43) is reduced as follows.

$$\frac{dT^*}{dp^*} = r' J_{x0}^* p^{*-m+m'} T^{*-n+n'} \exp\{T_i^*(T^{*-1}-1)\}. \tag{46}$$

The solution of the above equation is given by

$$p^* = \left[1 - \frac{n-n'-1}{r' J_{x0}^*} \int_1^{T^*} T^{*n-n'} \exp\{T_i^*(T^{*-1}-1)\} dT^*\right]^{1/(-m+m'+1)}. \tag{47}$$

Like this, as the solution has been introduced, in which p^* is expressed as the function of T^*, although we can determine the numerical relation between p^* and x^* by utilizing Eqs. (45) and (47), let us derive the relation between T^* and x^* in stead of the one between p^* and x^*, which are obtained already. So transforming Eq. (24), we obtain the following equation.

$$\frac{dT^*}{dx^*} = \frac{1}{F_{h21}(T^*)}, \tag{48}$$

where,

$$F_{h21}(T^*) = -[r'\bar{l}^* J_{x0}^* p^{*m'} T^{*n'}\{1 - J_{x0}^* p^{*-m+m'+1} T^{*-n+n'-1} \times \exp(T_i^* \overline{T^{*-1}-1})\}]^{-1}, \tag{48$'$}$$

p^* : given by Eq. (47).

Eq. (48) has the following solution.

$$x^* = \int_1^{T^*} F_{h21}(T^*) dT^*. \tag{49}$$

4. Conclusions

The author could derive the solutions from the basic quasi one-dimentional magnetohydrodynamic equations in Faraday generator duct in the cases, where the plasma fluid with constant velocity are in the thermal equilibrium and non-equilibrium ionization states. Next he could derive the numerically solvable differential equations and some solutions from the basic equations in Hall generator in the case, where the gas is in the state of equilibrium. Although the forms of the three kinds of solutions about Faraday generators relatively resemble one another, in particular the forms of the solutions with respect to the nonequilibrium ionization are similar to the ones in the case, where the conductivity is governed by the power law in the equilibrium ionization. Especially in Faraday generator, the relation between temperature and pressure does not depend on the conductivity of the fluid. On the other hand, in regard to Hall generator, the relation between temperature and pressure is very complex and troublesome, as it depends on not only the conductivity but also Hall parameter. However the author could find the numerically solvable differential equations and some solutions from the basic equations for Hall generator with constant flow velocity.

References

1) G.W. Sutton, A. Sherman: "Engineering MHD", McGraw-Hill Book Co., 405 (1965).
2) V.I. Kowbasiuk, S.A. Medin, V.A. Prokudin, S.A. Stepanov: MHD Power Generation ENEA, **II**, 718~719 (1964).
3) V.I. Kowbasiuk, S.A. Medin: Electricity from MHD, IAEA, Vienna, **I**, 431~437 (1966).
4) T. Honam: Docter Thesis in Kyoto Univ., 115~117, Aug. (1966).
5) T. Honma: Convention Records at the Annual Meeting of I.E.E.J., 239~240, April (1967).
6) T. Honma: J.I.E.E. Japan. **88**, 1065~1072 (1968).
7) M. Okada and Y. Arata: "Plasma Engineering", Nikkan Kogyo Sinbun Ltd., Tokyo, 649~651 (1965).
8) G. Brederlow, W. Feneberg and R. Hodgson: Electricity from MHD, IAEA, Vienna, **II**, 29~37 (1966).
9) J.L. Kerrebrock: Engine ring Aspects of MHD, Columbia Univ. Press, New York, 327~346 (1962).
10) C. Carter: Brit. J. Appl. Phys., **17**, 863~871 (1966).
11) T. Noguchi, Y. Eshima and T. Sakaguchi: Convention Records at the Annual Meeting in Kansai District of I.E.E.J., 98, Oct. (1966).
12) T. Noguchi and others: Bulletin of the Engineering Research Institute Kyoto Univ., **31**, 13~14, March (1967).
13) F.J. Hale and J.L. Kerrebrock: AIAA J., **2**, 461~469, March (1964).
14) J. Umoto: Unpublished paper.