# Simultaneous Spectral Representations of Isotropically Correlated Random Fields

By

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(Received June 27, 1969)

The spectral representations for isotropically correlated two $l_1$- and $l_2$-vecotr random fields are given generally in terms of $l$-vector harmonics and random measures, where $l$-vector denotes a $(2l+l)$- dimensional vector in the irreducible representation space of weight $l$ of the rotation group, $l=0$ being a scalar and $l=1$ an ordinary vector. The representation is derived by making use of the multi-dimensional moving average and the previous work on $l$-vector functions. The three special cases of interest are discussed in detail; namely, two scalar random fields, two vector random fields and a scalar and a vector random field.

## 1. Introduction

In the foregoing papers [1], [2] (the latter will be referred to as I) the author has discussed the spectral representation of a homogeneous and isotropic scalar random field and that of vector random field. The representation is given in terms of solid scalar or solid vector harmonics and orthogonal random measures. In the present paper we concern ourselves with simultaneous spectral representations of random fields having isotropic correlations among them; for example, temperature and density fields of atmosphere, pressure and velocity fields of turbulent flow, electric and magnetic fields in the black-body radiation etc. We deal with two such fields simultaneously, but the case of three or more can be discussed analogously if necessary. As above examples, the following three cases are of interest to us from the point of view of physics and engineering; namely, the case of two scalar fields, that of a scalar and a vector field and that of two vector fields. To give a unified description of these cases, we will generally deal with $l_1$-vector and $l_2$-vector random fields simultaneously; $l$-vector means a $(2l+1)$-dimensional vector defined in the space $D_l$ of the irreducible representation of weight $l$ of the 3-dimensional rotation group, a scalar corresponding to $l=0$ and a vector to $l=1$ [3]. Concerning

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various definitions, notations and formulae related to the theory of rotation group and $l$-vector functions, we draw freely from a previous work of the author on the addition theorems of spherical Bessel functions and vector harmonics [4], to which we will refer as II. We will see that the $\bar{l}_1 \times l_2$-tensor addition formula of a generalized spherical Bessel function plays an important role in the isotropic correlation between the two random fields. As in the previous works, we derive the spectral representation in a somewhat formal manner from a multi-dimensional moving average; therefore we leave the general proof of the spectral representation to a future work. The main theorems of the present paper have been reported in ref [5]

## 2. Homogeneous $l$-dimensional random field

Just as a $p$-dimensional normal (Gaussian) random measure is defined in I, we define a $p$-dimensional random measure $\boldsymbol{B}(A)$ on a 3–dimensional space $R_3$, $A$ denoting a set on $R_3$ of finite measure $m(A)$. Let $\omega$ denote sample point in the sample space $\Omega$, and $E\langle\ \rangle$ the average over $\Omega$. We often delete $\omega$ from notations for simplicity. A $p$-dimensional random measure

$$\boldsymbol{B}(A) = \boldsymbol{B}(A, \omega) = \{B^{\alpha}(A, \omega), \alpha = 1, 2, \cdots, p\} \tag{2.1}$$

is a system of $p$ random variables depending on a set $A$ in such a way that

$$E\langle B^{\alpha}(A)\rangle = 0,\ E\langle B^{\alpha}(A_1)B^{\beta}(A_2)\rangle = \delta_{\alpha\beta}m(A_1 \cap A_2)\ , \tag{2.2}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker symbol. For mutually disjoint sets $A_n$, $n=1, 2, \cdots$, we can write

$$\boldsymbol{B}(A_1 + \cdots + A_m) = \boldsymbol{B}(A_1) + \cdots + \boldsymbol{B}(A_m)\ , \tag{2.3}$$

which means

$$\sum_{\alpha=1}^{p} B^{\alpha}(A_1 + \cdots + A_m) = \sum_{\alpha=1}^{p} B^{\alpha}(A_1) + \cdots + \sum_{\alpha=1}^{p} B^{\alpha}(A_m)\ . \tag{2.4}$$

Then the 'complete additivity' holds in the sense of mean convergence;

$$\boldsymbol{B}\left(\sum_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \boldsymbol{B}(A_n)\ .$$

We define a $p$-dimensional vector space $V_{\Omega}$ in which, for any pair of vectors $\boldsymbol{h}=\{h^{\alpha}\}$ and $\boldsymbol{g}=\{g^{\alpha}\}(\alpha=1, 2, \cdots, p)$, the scalar product

$$(\mathbf{h}, \mathbf{g}) = \sum_{\alpha=1}^{p} \overline{h^{\alpha}} g^{\alpha} \tag{2.5}$$

and the square length $(\boldsymbol{h}, \boldsymbol{h})$ are defined. We denote by $\boldsymbol{L}^2(R_3)$ a Hilbert space of $V_{\Omega}$-valued vector functions $\boldsymbol{f}(\boldsymbol{x})=\{f^{\alpha}(\boldsymbol{x}) \in L^2(R_3),\ \alpha=1, \cdots, p\}$, with the inner product

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_{R_3} (\mathbf{f}, \mathbf{g}) d\mathbf{x} = \sum_{\alpha=1}^{p} \langle f^{\alpha}, g^{\alpha}\rangle , \tag{2.6}$$

and the square norm $||\boldsymbol{f}||^2 \equiv \langle \boldsymbol{f}, \boldsymbol{f}\rangle$, where

$$\langle f^{\alpha}, g^{\alpha}\rangle \equiv \int_{R_3} \overline{f^{\alpha}(\boldsymbol{x})} g^{\alpha}(\boldsymbol{x}) d\mathbf{x} \tag{2.7}$$

defines the inner product in $L^2(R_3)$.

As $p$-dimensional Wiener integral defined in I, we can define a $p$-dimensional stochastic integral of $\boldsymbol{f} \in \boldsymbol{L}^2(R_3)$ with respect to $\boldsymbol{B}(A)$:

$$I(\mathbf{f}) = \int_{R_3} \mathbf{f}(x) d\mathbf{B}(\mathbf{x}, \omega) = \sum_{\alpha=1}^{p} \int_{R_3} f^{\alpha}(\mathbf{x}) dB^{\alpha}(\mathbf{x}, \omega) , \tag{2.8}$$

where we have put $\boldsymbol{B}(d\boldsymbol{x}) = d\boldsymbol{B}(\boldsymbol{x})$ and $B^{\alpha}(d\boldsymbol{x}) = dB^{\alpha}(\boldsymbol{x})$. $I(\boldsymbol{f})$ has following properties:

$$I(a\mathbf{f}+b\mathbf{g}) = aI(\mathbf{f})+bI(\mathbf{g}) , \tag{2.9}$$

$$E\langle I(\mathbf{f})\rangle = 0 , \tag{2.10}$$

$$E\langle \overline{I(\mathbf{f})} I(\mathbf{g})\rangle = \langle \mathbf{f}, \mathbf{g}\rangle , \tag{2.11}$$

$$E\langle |I(\mathbf{f})|^2\rangle = ||\mathbf{f}||^2 . \tag{2.12}$$

A homogeneous $l$-dimensional random field $I_i(\mathbf{x})$, $i=1, 2, \cdots, l$, on $R_3$ can be represented as a $l$-dimensional moving average in terms of $l$ $p$-dimensional stochastic integrals of $\boldsymbol{f}_i(\mathbf{x}) \in \boldsymbol{L}^2(R_3)$, $i=1, 2, \cdots, l$:

$$I_i(\mathbf{x}) = \int_{R_3} \mathbf{f}_i(\mathbf{x}-\mathbf{x}') d\mathbf{B}(\mathbf{x}') = \sum_{\alpha=1}^{p} \int_{R_3} f_i^{\alpha}(\mathbf{x}-\mathbf{x}') dB^{\alpha}(\mathbf{x}') , \tag{2.13}$$

which, by (2.11), has the correlation matrix

$$R_{ij}(\mathbf{x}) = E\langle I_i(\mathbf{x}) I_j(0)\rangle = \langle \mathbf{f}_i(\mathbf{x}-\mathbf{x}'), \mathbf{f}_j(-\mathbf{x}')\rangle_{x'} . \tag{2.14}$$

As shown in $I$ §3, the moving average (2.13) has the spectral representation

$$I_i(\mathbf{x}) = \int_{R_3} e^{2\pi i (x, y)} dM_i(\mathbf{y}, \omega), \quad i = 1, 2, \cdots, l , \tag{2.15}$$

where $dM_i(\mathbf{y}) \equiv M_i(d\mathbf{y})$ and $M_i(S)$, $S$ denoting a set on $R_3$, denotes the $l$-dimensional random spectral measure having properties

$$E\langle M_i(S)\rangle = 0 , \tag{2.16}$$

$$E\langle \overline{M_i(S)} M_j(S')\rangle = \int_{S \cap S'} F_{ij}(\mathbf{y}) d\mathbf{y} , \tag{2.17}$$

$$F_{ij}(\mathbf{y}) = (\mathbf{F}_i, \mathbf{F}_j) = \sum_{\alpha=1}^{p} \overline{F_i^{\alpha}(\mathbf{y})} F_j^{\alpha}(\mathbf{y}) . \tag{2.18}$$

The matrix-valued spectral density $F_{ij}(\mathbf{y})$ is Hermitian non-negative definite and

$F_i^\alpha(\mathbf{y})$ is the Fourier transform of $f_i^\alpha(\mathbf{x})$:

$$f_i^\alpha(\mathbf{x}) = \int_{R_3} e^{2\pi i(x\cdot y)} F_i^\alpha(\mathbf{y})d\mathbf{y}, \quad \overline{F_i^\alpha(\mathbf{y})} = F_i^\alpha(-\mathbf{y}) . \tag{2.19}$$

The correlation matrix $R_{ij}(\boldsymbol{x})$ has the spectral representation

$$R_{ij}(\mathbf{x}) = \int_{R_3} F_{ij}(\mathbf{y})e^{-2\pi i(x\cdot y)}d\mathbf{y} , \tag{2.20}$$

with the inversion formula

$$F_{ij}(\mathbf{y}) = \int_{R_3} R_{ij}(\mathbf{x})e^{2\pi i(x\cdot y)}d\mathbf{x} . \tag{2.21}$$

Since $F_{ij}(\boldsymbol{y})$ is non-negative definite Hermitian matrix, we have

$$\det\,(F_{ij}) = \begin{vmatrix} F_{11}(\mathbf{y}) \cdots F_{1l}(\mathbf{y}) \\ \cdots\cdots\cdots\cdots \\ F_{l1}(\mathbf{y}) \cdots F_{ll}(\mathbf{y}) \end{vmatrix} \geqslant 0 . \tag{2.22}$$

A $l$-dimensional random field $I_i(\boldsymbol{x})$ is called degenerate if

$$\det\,(F_{ij}) = 0 \tag{2.23}$$

holds for almost all $\boldsymbol{y}$, and the largest rank of $(F_{ij})$ for which the set of $\boldsymbol{y}$ has non-zero measure is called the rank of the spectral density matrix, which turns out to be the degree of freedom of $I_i(\boldsymbol{x})$. For $F_{ij}(\boldsymbol{y})$ given by (2.18), the rank $r$ is equal to that of $F_i^\alpha(\boldsymbol{y})$. Hence, if $p<l$, (2.15) is always degenerate. If $r<l$, $l-r$ functions from among $l$ $p$-dimensional vector functions $\boldsymbol{F}_i=\{F_i^\alpha,\ \alpha=1,\ 2,\cdots,\ p\}$ can be expressed as, say,

$$\mathbf{F}_i(\mathbf{y}) = \sum_{j=1}^{r} A_{ij}(\mathbf{y})\mathbf{F}_j(\mathbf{y}) , \quad i = r+1, \cdots, l , \tag{2.24}$$

and hence,

$$\mathbf{f}_i(\mathbf{x}) = \sum_{j=1}^{r} \int_{R_3} a_{ij}(\mathbf{x}-\mathbf{x}')\mathbf{f}_j(\mathbf{x}')d\mathbf{x}' , \tag{2.25}$$

where $a_{ij}(\boldsymbol{x})$ denotes the Fourier transform of $A_{ij}(\boldsymbol{y})$. Thus

$$I_i(\mathbf{x}) = \sum_{j=1}^{r} \int_{R_3} a_{ij}(\mathbf{x}-\mathbf{x}')I_j(\mathbf{x}')d\mathbf{x}' , \quad i = r+1, \cdots, l ; \tag{2.26}$$

that is, when a homogeneous $l$-dimensional random field is degenerate with rank $r$, its $l-r$ components can be derived from other $r$ by means of a linear operation commuting with spatical translations. In what follows, we put $l=p$ without loss of generality.

## 3. Isotropically correlated two random fields

In the following sections we consider $l$-vector random field defined in the representation space $D_l$. A $l_1$-vector and a $l_2$-vector random fields have $(2l_1+1)$ and $(2l_2+1)$ components respectively. When viewed differently, these two $l$-vectors together can be regarded as a $(2l_1+1)+(2l_2+1)$-dimensional vector in the sum-space of representation $D_{l_1}+D_{l_2}$, which we call $l_1+l_2$-vector for convenience. The correlation function of this random field, therefore, is a tensor in the product space

$$(\overline{D_{l_1}+D_{l_2}})\times(D_{l_1}+D_{l_2}) = \overline{D_{l_1}}\times D_{l_1}+\overline{D_{l_2}}\times D_{l_1}+\overline{D_{l_1}}\times D_{l_2}+\overline{D_{l_2}}\times D_{l_2}\,. \quad (3.1)$$

The canonical component[1] of the correlation tensor in each subspace $D_\lambda\times D_\mu(\lambda,\ \mu = l_1,\ l_2)$ is defined by

$$R^{\lambda\mu}_{\dot{m}n}(\mathbf{x}) = E\langle\overline{I_{(\lambda)m}(\mathbf{x})}I_{(\mu)n}(0)\rangle$$
$$m = -\lambda,\cdots,\lambda,\quad n = -\mu,\cdots,\mu\,, \quad (3.2)$$

The correlation tensors of $l_1$- and $l_2$-vector random fields themselves are defined in in the first and the last space on the right-hand side of (3.1), while the mutual correlation tensors are defined in the second and the third space.

When the mutual correlations are isotropic tensor fields[2], then we say that the two random fields are isotropically correlated. In the following we deal with homogeneous and isotropic random fields which are isotropically correlated, and hence the correlation tensor of the $l_1+l_2$-vector random field is an isotropic tensor field in the product space (3.1).

Applying Lemmata 2 and 3 of II, Appendix II, to (2.20) and (2.21), we obtain the following result:

THEOREM 1. (*Spectral representation of correlation tensor*) For the homogeneous and isotropic $l_1$- and $l_2$-vector random fields which are isotropically correlated with each other, both the correlation tensor $R^{\lambda\mu}_{\dot{m}n}$ and the corresponding spectral density tensor $F^{\lambda\mu}_{\dot{m}n}$ are isotropic $\bar{\lambda}\times\mu$-tensor fields in $\bar{D}_\lambda\times D_\mu$ $(\lambda,\ \mu=l_1,\ l_2)$. They are represented as 'diagonal' matrices in canonical components:

$$R^{\lambda\mu}_{\dot{m}n}(\mathbf{x}) = \delta_{mn}R^{\lambda\mu}_{n}(r)\,,\quad r = |\mathbf{x}|\,, \quad (3.3)$$
$$F^{\lambda\mu}_{\dot{m}n}(\mathbf{y}) = \delta_{mn}F^{\lambda\mu}_{n}(t)\,,\quad t = |\mathbf{y}|\,, \quad (3.4)$$
$$\lambda,\ \mu = l_1,\ l_2,\quad m = -\lambda,\cdots,\lambda,\quad n = -\mu,\cdots,\mu\,,$$

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1 See II, Appendix I.

2 For definition of isotropic tensor field, see I, §4. General form of isotropic tensor field as well as its Fourier transform are given in II, Appendix II.

where $F_n^{\lambda\mu}$ is a non-negative definite Hermitian matrix with respect to the indices $\lambda$, $\mu$. $R^{\lambda\mu}$ and $F^{\lambda\mu}$ satisfy the following symmetries

$$\overline{R_m^{\lambda\mu}(r)} = (-1)^{\lambda-\mu} R_{-m}^{\mu\lambda}(r)\,, \tag{3.5}$$

$$F_n^{\lambda\mu}(t) = \overline{F_n^{\mu\lambda}(t)}\,, \quad F_n^{\lambda\lambda}(t) \geqslant 0\,. \tag{3.6}$$

The correlation tensor has the following spectral representation in terms of the generalized spherical Bessel function[3]

$$R_m^{\lambda\mu}(r) = 4\pi \sum_{n=-L}^{L} \int_0^\infty \overline{j_{mn}^{\mu\lambda}(2\pi tr)} F_n^{\lambda\mu}(t) t^2 dt\,, \quad m = -L, \cdots, L \tag{3.7}$$[4]

with the inversion formula

$$F_n^{\lambda\mu}(t) = 4\pi \sum_{m=-L}^{L} \int_0^\infty j_{mn}^{\mu\lambda}(2\pi tr) R_m^{\lambda\mu}(r) r^2 dr\,, \quad n = -L, \cdots, L \tag{3.8}$$

where $L$ denotes the smaller of $\lambda$ and $\mu$.

The random fields have the following representations in terms of solid $l$-vector harmonics[5] and random spectral measures. The derivation of the theorem will be given in section 5.

THEOREM 2. (*Spectral representation of random field*) The above random fields have the following simultaneous spectral representations:

$$\mathbf{I}_{(\lambda)}(r, \theta, \varphi) = \sqrt{4\pi} \sum_{n=-\lambda}^{\lambda} \sum_{l=0}^{\infty} \sum_{s=-l}^{l} \int_0^\infty \mathbf{J}_{(\lambda)n}^{ls}(2\pi tr, \theta, \varphi) dM_{(\lambda)n}^{ls}(t, \omega)\,, \quad \lambda = l_1, l_2\,, \tag{3.9}$$

where $dM_{(\lambda)n}^{ls}(t) = M_{(\lambda)n}^{ls}(dt)$ and $M_{(\lambda)n}^{ls}(\Delta)$ denotes the random spectral measure defined on the half-line $T$, $0 \leqslant t < \infty$, having properties

$$E\langle M_{(\lambda)n}^{ls}(\Delta)\rangle = 0\,, \tag{3.10}$$

$$E\langle \overline{M_{(\lambda)n}^{ls}(\Delta)} M_{(\mu)n'}^{l's'}(\Delta')\rangle = \delta_{nn'}\delta_{ll'}\delta_{ss'} 4\pi \int_{\Delta\cap\Delta'} F_n^{\lambda\mu}(t) t^2 dt\,, \quad \lambda, \mu = l_1, l_2,\ n = -L, \cdots, L\,, \tag{3.11}$$

$\Delta$ and $\Delta'$ being any intervals on $T$.

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3 For definition, see II, §2.2.

4 According to II, (A. 46), the Fourier transform of a $\lambda \times \mu$-tensor $R_{\dot{m}n}^{\lambda\mu}$ is given by $i^{\mu-\lambda} F_{\dot{m}n}^{\lambda\mu}$. However, we take $F_{mn}^{\lambda\mu}$ as the spectral density tensor to supress the useless factor $i^{\lambda-\mu}$ without affecting the Hermitian property. For $\lambda = \mu = l_1 = l$, (3.7) reduces to I, (4.12) for a single vector random field.

5 For definition, see II, §3.

The spectral representation of correlation tensor (3.7) can be immediately obtained from (3.9): We multiply two expressions for arbitrary observation points and take the average using (3.11). We then recover (3.3), (3.7) on making use of the tensor addition theorem given by II, (4.1). In particular when $\lambda=\mu=l_1=1$, (3.9) agrees with the formerly obtained result I, (5.17) for a single vector random field.

## 4. Examples of isotropically correlated random fields

We write down the above obtained result for the three cases individually, which are of interest to us as mentioned in the introduction.

4.1. *Scalar-Scalar* ($l_1=l_2=0$). We descriminate the two scalar fields by the indices $\lambda=1, 2$, instead of $l_1$, $l_2$ both of which equal 0 in the present cases. There would be no confusion in doing this. The representation (3.9) can be written in terms of solid scalar harmonics[6];

$$I_\lambda(r, \theta, \varphi) = \sqrt{4\pi} \sum_{l=0}^{\infty} \sum_{s=-l}^{l} \int_0^\infty J^{ls}(2\pi tr, \theta, \varphi) dM_{(\lambda)}^{ls}(t, \omega) , \qquad \lambda = 1, 2 , \tag{4.1}$$

$$E\langle \overline{M_{(\lambda)}^{ls}(\Delta)} M_{(\mu)}^{l's'}(\Delta') \rangle = \delta_{ll'} \delta_{ss'} 4\pi \int_{\Delta \cap \Delta'} F^{\lambda\mu}(t) t^2 dt , \tag{4.2}$$

$$R^{\lambda\mu}(r) = 4\pi \int_0^\infty j_0(2\pi tr) F^{\lambda\mu}(t) t^2 dt . \tag{4.3}$$

An example of two such scalar fields is furnished by the temperature and density fields of atmosphere which are statistically correlated with each other. The random fields in this case may be considered as nondegenerate.

4.2. *Vector-Vector* ($l_1=l_2=1$). As above, the two vector random fields are labelled by $\lambda=1, 2$. The corresponding spectral representation is given by (3.9) with $l$-vector harmonic $\mathbf{J}_{(\lambda)n}^{ls}$ replaced by vector harmonic $\boldsymbol{J}_n^{ls}$, $n=-1, 0, 1$[7]. The isotropic correlation tensor

$$R_{mn}^{\lambda\mu}(\mathbf{x}) = \delta_{mm} R_n^{\lambda\mu}(r), \quad m, n = -1, 0, 1, \quad \lambda, \mu = 1, 2, \tag{4.4}$$

when expressed in the matrix with row and column numbered by $n=-1, 0, 1$, corresponding to the canonical bases in $D_1+D_1$, has the form

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6 For definition, see II, (5.20).

7 See II, §5.1.

$$\left(\begin{array}{ccc|ccc} R^{11}_{-1} & & & R^{12}_{-1} & & \\ & R^{11}_{0} & & & R^{12}_{0} & \\ & & R^{11}_{1} & & & R^{12}_{1} \\ \hline \overline{R^{12}_{1}} & & & R^{22}_{-1} & & \\ & \overline{R^{12}_{0}} & & & R^{22}_{0} & \\ & & \overline{R^{12}_{-1}} & & & R^{22}_{1} \end{array}\right) \tag{4.5}$$

The corresponding tensor has the spectral representation

$$R^{\lambda\mu}_{m}(r) = 4\pi \sum_{n=-1}^{1} \int_0^\infty \overline{j^{1}_{mn}(2\pi tr) F^{\lambda\mu}_{n}(t)}\, t^2 dt\,, \quad m = -1, 0, 1\,, \tag{4.6}$$

where $j^{l}_{mn} \equiv j^{l1}_{mn}$[8]. An example of isotropically correlated two vector random fields is given by the electric and magnetic fields in the black-body radiation. A vector random field $\mathbf{I}_2 = rot\ \mathbf{I}_1$ was already discussed in I, §6.3. When $\mathbf{I}_1$ and $\mathbf{I}_2$ are considered simultaneously, the corresponding spectral density tensor has the matrix form

$$\left(\begin{array}{ccc|ccc} |F_{-1}|^2 & & & -2\pi t\,|F_{-1}|^2 & & \\ & |F_0|^2 & & & 0 & \\ & & |F_1|^2 & & & 2\pi t\,|F_1|^2 \\ \hline -2\pi t\,|F_{-1}|^2 & & & |2\pi t F_{-1}|^2 & & \\ & 0 & & & 0 & \\ & & 2\pi t\,|F_1|^2 & & & |2\pi t F_1|^2 \end{array}\right) \tag{4.7}$$

for which $det\,|F_{mn}| = 0$, showing the degeneracy of 6–dimensional random field $\boldsymbol{I}_1 + \boldsymbol{I}_2$.

4.3. *Vector-Scalar* ($l_1 = 1$, $l_2 = 0$). Representations (3.7) and (3.9) may be written down separately for two fields:

$$\mathbf{I}(r, \theta, \varphi) = \sqrt{4\pi} \sum_{n=-1}^{1} \sum_{l=0}^{\infty} \sum_{s=-l}^{l} \int_0^\infty \mathbf{J}^{ls}_{n}(2\pi tr, \theta, \varphi)\, dM^{ls}_{n}(t)\,, \tag{4.8}$$

$$I(r, \theta, \varphi) = \sqrt{4\pi} \sum_{l=0}^{\infty} \sum_{s=-l}^{l} \int_0^\infty J^{ls}(2\pi tr, \theta, \varphi)\, dM^{ls}(t)\,, \tag{4.9}$$

$$E\langle \overline{M^{ls}_{n}(\Delta)} M^{l's'}_{n'}(\Delta')\rangle = \delta_{nn'}\delta_{ll'}\delta_{ss'} 4\pi \int_{\Delta \cap \Delta'} |F_n(t)|^2 t^2 dt\,, \tag{4.10}$$

$$E\langle \overline{M^{ls}(\Delta)} M^{l's'}(\Delta')\rangle = \delta_{ll'ss'} 4\pi \int_{\Delta \cap \Delta'} |F(t)|^2 t^2 dt\,, \tag{4.11}$$

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8 For explicit representation, see II, (5.2). It is called the spherical Bessel vector function. Se also I, §4.2.

$$E\langle \overline{M_n^{ls}(\Delta)M^{l's'}(\Delta')}\rangle = \delta_{n0}\delta_{ll'ss'}4\pi \int_{\Delta\cap\Delta'} F^{12}(t)t^2dt\,, \tag{4.12}$$

where we have put $F_n^{11}(t)=|F_n(t)|^2$ and $F^{22}(t)=|F(t)|^2$. Then nonvanishing components of the correlation tensor have the representations

$$R_m(r) = 4\pi\sum_{n=-1}^{1}\int_0^\infty \overline{j_{mn}^1(2\pi tr)}\,|F_n(t)|^2t^2dt\,, \quad m = -1\ 0,\ 1\,, \tag{4.13}$$

$$R(r) = 4\pi\int_0^\infty j_0(2\pi tr)\,|F(t)|^2t^2dt\,, \tag{4.14}$$

$$M(r) = 4\pi\int_0^\infty j_1(2\pi tr)F^{12}(t)t^2dt\,, \tag{4.15}$$

the correlation tensor having the matrix form

$$\left(\begin{array}{ccc|c} R_{-1} & & & \\ & R_0 & & -M \\ & & R_1 & \\ \hline & \bar{M} & & R \end{array}\right) \tag{4.16}$$

An example of correlated vector and scalar random fields is given by the velocity and the pressure fields of atmosphere. Degenerated examples are provided by $\boldsymbol{I}_1=grad\ I$, $I_2=I$ and $\boldsymbol{I}_1=\boldsymbol{I}$, $I_2=div\ \boldsymbol{I}$; the former case was discussed in I, §6.1. as potential random field. The corresponding spectral density tensors have the forms respectively

$$\left(\begin{array}{ccc|c} 0 & & & \\ & |2\pi tF|^2 & & 2\pi t\,|F|^2 \\ & & 0 & \\ \hline & 2\pi t\,|F|^2 & & |F|^2 \end{array}\right), \quad \left(\begin{array}{ccc|c} |F_{-1}|^2 & & & \\ & |F_0|^2 & & -2\pi t\,|F_0|^2 \\ & & |F_1|^2 & \\ \hline & -2\pi t\,|F_0|^2 & & |2\pi tF_0|^2 \end{array}\right) \tag{4.17}$$

## 5. Derivation of the spectral representation

We denote $k=(2l_1+1)+(2l_2+1)$ in the following. In this paper we derive the spectral representation (3.9) from a $k$-dimensional moving average (2.13), where $p=k$ as mentioned at the end of section 2. The procedure essentially follows that of I; the argument used for a vector is now applied to a $l_1+l_2$-vector. Hence we describe the procedure briefly.

The spectral density tensor (2.18) is an isotropic tensor field having the form (3.4). Therefore, we must find $k$ $V_\Omega$-valued ($k$-dimensional) vector functions $\{F_m^\alpha(\boldsymbol{y}),\ \alpha=1, \cdots, k\}$ $m=-\lambda, \cdots, \lambda,\ \lambda=l_1,\ l_2$, which satisfy

$$F^{\lambda\mu}_{\dot{m}n}(\mathbf{y}) = \sum_{\alpha=1}^{k} \overline{F^{\lambda\alpha}_{m}(\mathbf{y})F^{\mu\alpha}_{n}(\mathbf{y})} = \delta_{mn}F^{\lambda\mu}_{n}(t)\,, \tag{5.1}$$
$$m = -\lambda, \cdots, \lambda, \quad n = -\mu, \cdots, \mu\,.$$

Let us regard $V_\Omega$ as another representation space $D_{l_1}+D_{l_2}$, and divide the vector components $\{F^{\lambda\alpha}_{m},\ \alpha=1, \cdots, k\}$ into the canonical components belonging to either $l_1$- or $l_2$-vector, which we write as

$$F^{\lambda\alpha}_{m}(\mathbf{y}) = \varphi^{\mu\lambda}_{m\dot{\alpha}}(\mathbf{y})\,, \quad \mu = l_1, l_2, \quad \alpha = -\mu, \cdots, \mu\,, \tag{5.2}$$

where the superscript $\alpha$ denoting a covariant component is lowered and dotted. Let $V_R$ stand for the $k$-dimensional vector space with respect to the subscript $m$. Furthermore, we regard $\varphi^{\lambda\mu}_{\dot{\alpha}m}$, with respect to the subscripts $m$, $\alpha$ and the superscripts $\lambda$, $\mu$, as an isotropic tensor field in the product space $V_R\times V_\Omega=(D_{l_1}+D_{l_2})\times\overline{(D_{l_1}+D_{l_2})}$:

$$\varphi^{\lambda\mu}_{m\dot{\alpha}}(\mathbf{y}) = \delta_{m\alpha}\varphi^{\lambda\mu}_{m}(t)\,. \tag{5.3}$$

Then we find that (5.2) satisfies (5.1) with

$$F^{\lambda\mu}_{n}(t) = \sum_{\nu=l_1,l_2} \overline{\varphi^{\lambda\nu}_{n}(t)}\varphi^{\mu\nu}_{n}(t)\,, \quad \lambda, \mu = l_1, l_2\,, \tag{5.4}$$
$$n = -L, \cdots, L\,,$$

where $L$ denotes the smaller of $\lambda$ and $\mu$, and that $F^{\lambda\mu}_{n}(t)$ is a nonnegative definite Hermitian matrix with respect to the indices $\lambda$ and $\mu$. Now that $\varphi^{\lambda\mu}_{m\dot{\alpha}}(\boldsymbol{y})$, namely $F^{\lambda\alpha}_{m}(\boldsymbol{y})$, is an isotropic tensor field, its Fourier transform $f^{\alpha}_{m}(\boldsymbol{x})$ is another isotropic tensor field having the representation[9]

$$f^{\lambda\mu}_{m\dot{\alpha}}(\mathbf{x}) = \delta_{m\alpha}f^{\lambda\mu}_{m}(r)\,, \tag{5.5}$$

$$f^{\lambda\mu}_{m}(r) = 4\pi i^{\lambda-\mu}\sum_{n=-L}^{L}\int_0^\infty j^{\mu\lambda}_{mn}(2\pi tr)\varphi^{\lambda\mu}_{n}(t)t^2dt\,,$$
$$\lambda, \mu = l_1, l_2, m = -L, \cdots, L\,. \tag{5.6}$$

To supress the trivial factor $i^{\lambda-\mu}$, we replace $i^{\lambda-\mu}\varphi^{\lambda\mu}$ by $\varphi^{\lambda\mu}$ and $i^{\lambda-\mu}F^{\lambda\mu}$ by $F^{\lambda\mu}$ in what follows. The Hermitian property of $F^{\lambda\mu}$ is never destroyed by doing so.

In order to integrate the moving average (2.13), we put

$$\boldsymbol{\rho} = \mathbf{x}-\mathbf{x}'\,,$$
$$\rho = |\mathbf{x}-\mathbf{x}'| = \sqrt{r^2+r'^2-2rr^2\cos\Theta}\,, \tag{5.7}$$

where $\Theta$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}'$, and apply the tensor addition formula

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9 See II, Appendix II.

II, (4.1) to the tensor function (5.5):

$$\sum_{n=-L}^{L} \mathbf{e}_{(\lambda)m}(\boldsymbol{\rho})\overline{e_{(\mu)m}(\boldsymbol{\rho})}f_m^{\lambda\mu}(\rho) =$$

$$= (4\pi)^2 \sum_n \sum_l \sum_s \int_0^\infty \mathbf{J}_{(\lambda)n}^{ls}(2\pi tr, \theta, \varphi)\overline{\mathbf{J}_{(\mu)n}^{ls}(2\pi tr', \theta, \varphi')}\varphi_n^{\lambda\mu}(t)t^2dt$$

$$= \sqrt{4\pi}\sum_{n=-L}^{L}\sum_{l=0}^{\infty}\sum_{s=-l}^{l}\int_0^\infty \mathbf{J}_{(\lambda)n}^{ls}(2\pi tr, \theta, \varphi)\overline{d\mathbf{E}_{(\mu)n}^{ls}(t)f^{\lambda\mu}}, \tag{5.8}$$

where we have put $d\boldsymbol{E}_{(\mu)n}^{ls}(t) \equiv \boldsymbol{E}_{(\mu)n}^{ls}(dt)$ and

$$\overline{\mathbf{E}_{(\mu)n}^{ls}(\Delta)f^{\lambda\mu}} \equiv (4\pi)^{3/2}\int_\Delta \overline{\mathbf{J}_{(\mu)n}^{ls}(2\pi tr, \theta, \varphi)}\varphi_n^{\lambda\mu}(t)t^2dt, \tag{5.9}$$

for any interval $\Delta$ on $T$. Eq. (5.9) is a $\bar{\mu}$-vector function and we consider a sum-vector with $\mu=l_1, l_2$, namely, a $\bar{l}_1+\bar{l}_2$-vector function in the $k$-dimensional vector space $V_\Omega$. Then the inner product of two such $V_\Omega$-valued vector functions becomes

$$\langle \overline{\mathbf{E}_{(l_1)n}^{ls}(\Delta)f^{\lambda l_1}}+\overline{\mathbf{E}_{(l_2)n}^{ls}(\Delta)f^{\lambda l_2}},\ \overline{\mathbf{E}_{(l_1)n'}^{l's'}(\Delta')f^{\mu l_1}}+\overline{\mathbf{E}_{(l_2)n'}^{l's'}(\Delta')f^{\mu l_2}}\rangle$$
$$= \delta_{nn'}\delta_{ll'}\delta_{ss'}4\pi\sum_{\nu=l_1,l_2}\int_{\Delta\cap\Delta'}\overline{\varphi_n^{\lambda\nu}(t)}\varphi_n^{\mu\nu}(t)t^2dt, \tag{5.10}$$

by virtue of II, (3.8). The $k$-dimensional stochastic integral of the $V_\Omega$-valued function[10]

$$M_{(\lambda)n}^{ls}(\Delta) = I(\overline{\mathbf{E}_{(l_1)n}^{ls}(\Delta)f^{\lambda l_1}}+\overline{\mathbf{E}_{(l_2)n}^{ls}(\Delta)f^{\lambda l_2}})$$
$$\lambda = l_1, l_2, \quad n = -\lambda, \cdots,$$

then gives a random measure on $T$ having properties

$$E\langle \overline{M_{(\lambda)n}^{ls}(\Delta)}M_{(\mu)n'}^{l's'}(\Delta')\rangle = \delta_{nn'}\delta_{ll'}\delta_{ss'}4\pi\sum_{\nu=l_1,l_2}\int_{\Delta\cap\Delta'}\overline{\varphi_n^{\lambda\nu}(t)}\varphi_n^{\mu\nu}(t)t^2dt,$$
$$\lambda, \mu = l_1, l_1, \tag{5.12}$$

by virtue of (2.11) and (5.10). Consequently, if we make a direct-sum of $l_1$- and $l_2$-vectors corresponding to $\mu=l_1, l_2$ on both the sides of (5.8) and take the $k$-dimentional stochastic integral of $l_1+l_2$-vector function termwise, then we obtain the spectral representations (3.9).

**Acknowledgement**

The author is gratefully indebted to Professor J. Ikenoue for his advice and encouragement during this research.

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[10] The $k$-dimensional stochastic integral as defined by (2.8) is easily modified so as to apply to a $V_\Omega$-valued function expressed in terms of canonical component. See I, §5.1, 5.2.

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