An Algorithm for Solving the Weighted Distribution Linear Programs with Zero-One Variables

By

Hisashi MINE and Hiroyuki NARIHISA*

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Recently, very considerable efforts have been devoted to integer programming. In practical point of view, zero-one integer programming is important for solving the actual integer programming problems.

For these problems, various approaches have been proposed by many researchers in this field. However, the fundamental idea for solving these problems is based on the additive algorithm for solving linear programs with zero-one variables proposed by Egon Balas in 1965.

In this paper, we propose an algorithm for solving the weighted distribution linear programming problem with zero-one variables. This algorithm is also an extension of the additive algorithm, but is more powerful than that of Egon Balas for the structured problem as the weighted distribution linear programming problem with zero-one variables.

1. Introduction

Recently, there are many papers concerning integer programming problems, especially, zero-one integer programming problems. For these problems, various approaches have been proposed. The fundamental idea for solving these problems is based on the additive algorithm for solving linear programs with zero-one variables proposed by Egon Balas.

In this paper, we propose an algorithm for solving the weighted distribution linear programming problem with zero-one variables. The algorithm is also an extension of the additive algorithm, but is more powerful than that of Egon Balas for the weighted distribution linear programming problem with zero-one variables.

2. Formulation of problem

Without loss of generality, the linear programming problem with zero-one variables can be expressed as;

* Department of Applied Mathematics and Physics

Find $\boldsymbol{X}$ such as

$$\text{Minimize } z = \boldsymbol{CX} \qquad (\boldsymbol{C} \geqq 0), \tag{1}$$

$$\text{Subject to } \boldsymbol{AX}+\boldsymbol{Y} = \boldsymbol{B}, \tag{2}$$

$$x_j = 0 \quad \text{or} \quad 1 \qquad (j \in N), \tag{3}$$

$$\boldsymbol{Y} \geqq 0, \tag{4}$$

where $\boldsymbol{X}=(x_j)$ is an n components column vector, $\boldsymbol{C}=(c_j)$ is an n components row vector, $\boldsymbol{A}=(a_{ij})$ is an $m \times n$ matrix, $\boldsymbol{B}=(b_i)$ is an m components column vector, $\boldsymbol{Y}=(y_i)$ is an m components nonnegative slack column vector, and $N=\{1, 2,\ldots, n\}$.

Moreover, $\boldsymbol{A}$ is assumed to be expressed as the following structured form.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{A}_r \\ & \bar{\boldsymbol{A}} & \end{pmatrix}, \tag{5}$$

where $\boldsymbol{A}_p$ ($p \in \boldsymbol{P}=\{1, 2,\cdots, r\}$) is an $m_p \times n_p$ matrix, $\bar{\boldsymbol{A}}$ is an $m_\beta \times n$ matrix.

In this case, denoting $\sum_{i=1}^{r} m_i = m_\alpha$, we have the following relations.

$$m = m_\alpha + m_\beta, \quad \sum_{i=1}^{r} n_i = n. \tag{6}$$

Additionally, defining the following sets,

$${}_pN = \{ \sum_{i=1}^{p-1} n_i+1, \quad \sum_{i=1}^{p-1} n_i+2,\cdots, \quad \sum_{i=1}^{p-1} n_i+n_p\}, \tag{7}$$

$$M_p = \{ \sum_{i=1}^{p-1} m_i+1, \quad \sum_{i=1}^{p-1} m_i+2,\cdots, \quad \sum_{i=1}^{p-1} m_i+m_p\}, \tag{8}$$

we have

$$\bigcup_{p \in \boldsymbol{P}} {}_pN = N, \tag{9}$$

$$M_\alpha \cup M_\beta = M, \tag{10}$$

where $\cup$ denotes union of sets, $M_\beta=\{m_\alpha+1, m_\alpha+2,\cdots, m_\alpha+m_\alpha=m\}$ and M_α is defined as

$$M_\alpha = \bigcup_{p \in \boldsymbol{P}} M_p. \tag{11}$$

In the course of search for the optimal solution, we start from the $n+m$ dimension solution vector $\boldsymbol{U}^0=(\boldsymbol{X}^0, \boldsymbol{Y}^0)=(0, \boldsymbol{B})$ and obtain a new solution vector by assigning each of x_j ($j \in N$) zero or one according to some given criterion. After successive iterations, finally we shall obtain the optimal solution vector.

In this problem, we call the constraints which correspond to the matrix

$A_p(p \in P)$ of (5) as constraint p and call the set of these constraints as the constraints α and finally we call the constraints which correspond to the matrix $\bar{A}$ of A as constraints β.

3. Outline of the additive algorithm

As our algorithm is an modification of Balas', we shall describe the basic idea of the additive algorithm in this section.

An $(n+m)$-dimensional vector $U=(X, Y)$ is called a solution, if it satisfies (2) and (3); a feasible solution, if it satisfies (2), (3), and (4); and an optimal solution, if it satisfies (1), (2), (3), and (4).

Let P^s denote the linear programming problem defined by (1), (2), (4) and the constraints

$$x_j \geqq 0, \quad (j \in N) \tag{3 a}$$

$$x_j = 1, \quad (j \in J_s) \tag{3 b$_s$}$$

where J_s is a subset of N. P^0 is meant by the ordinary linear programming problem with $J_0=\phi$.

We start from P^0 with $U^0=(X^0, Y^0)=(0, B)$, which is obviously a dual feasible solution to P^0 (because $C \geqq 0$)

The basis of the solution U^0 consists of the unit-matrix $I_{(m)}=(e_i)(i \in M)$, e_i being the i-th unit vector. For some $y_i^0<0$, we choose, a vector a_{j1} such that $a_{ij1} <0$, to introduce into the basis. But instead of introducing a_{j1} in place of a vector e_i in the basis, as we do in the usual dual simplex method, we add to P^0 the constraint $x_{j1}=1$, which is slightly modified as the form $-x_{j1}+y_{m+1}=-1$ with an artificial variable y_{m+1} in practice. Thus we obtain the problem P^1 with $J_1=\{j_1\}$ defined by (1), (2), (3a), (4) and the additional constraint

$$x_{j1} = 1 . \tag{3 b$_1$}$$

It is easy to see that the set $x_j=0$ $(j \in N), y_i=b_i(i \in M)$, is a dual feasible solution to P^1. In the extended basis $I_{(m+1)}=(e_i)$ $(i=1, \cdots, m+1)$, the $(m+1)$st unit vector e_{m+1} corresponds to y_{m+1}. We introduce a_{j1} at the place of this unit vector e_{m+1}, and thus x_{ji} takes the value 1 in the new solution to P^1 that obviously remains daul feasible. As the artificial variable y_{m+1}, which becomes 0, does not play any role henceforth, it must be abandoned and the new solution can be written as $U^1=(X^1, Y^1)=(x_1^1, \cdots, x_n^1, y_1^1, \cdots, y_m^1)$.

Given the additional constraint, the pivot operation around the element–1 yields the algebraic addition $B\text{-}a_{j1}$. Thus, the new dual feasible solution $U^1=(X^1, Y^1)$ to P^1 is

$$x_j^1 = \begin{cases} 1 & (j = j_1) \\ 0 & (j = N-\{j_1\}), \end{cases}$$
$$y_i^1 = b_i - a_{ij_1} \qquad (i \in M).$$

As the operations to be carried out at each iteration consist of only additions and subtractions, this algorithm is called additive one.

If the solution-vector $\boldsymbol{U}^1$ still has negative components, then according to the above mentioned rules we choose another vector $\boldsymbol{a}_{j_2}$ to introduce into the basis, and we add to P^1 the new constraint $x_{j_2}=1$, in the form $-x_{j_2}+y_{m+2}=-1$, y_{m+2} being another artificial variable. This yields the problem P^2, consisting of (1), (2), (3a), and (4) and the additional constraint set ($3b_2$), made up of $x_{j_1}=1$, $x_{j_2}=1$. The set $x_{j_1}=1$, $x_j=0$ $[j \in (N-\{j_1\})]$, $y_i=b_i-a_{ij_1}$ $(i \in M)$, $y_{m+2}=-1$, is a dual feasible solution to P^2. The vector $\boldsymbol{a}_{j_2}$ is now introduced in place of $\boldsymbol{e}_{m+2}$, and x_{j_2} takes the value 1 in the new solution to P^2, which remains dual feasible. As the artificial variable y_{m+2} does not play any role henceforth, it must be dropped as in the case of y_{m+1}, and the new dual feasible solution to P^2 is $\boldsymbol{U}^2=(\boldsymbol{X}^2, \boldsymbol{Y}^2)$, where

$$x_j = \begin{cases} 1 & (j = j_1, j_2) \\ 0 & [j \in (N-\{j_1, j_2\})], \end{cases}$$
$$y_i^2 = y_i^1 - a_{ij_2} \qquad (i \in M).$$

This procedure is repeated until either a solution $\boldsymbol{U}^s$ with all nonnegative components is obtained, or any solution to P^s does not exist. If a nonnegative vector $\boldsymbol{U}^s=(\boldsymbol{X}^s, \boldsymbol{Y}^s)$ is obtained, it is a feasible solution to P^s.

The procedure is started again from a solution $\boldsymbol{U}^p$ for some $p<s$ according to the backtracking idea, introducing a suitable vector into the basis, until either another feasible solution $\boldsymbol{U}^t$ such that $z_t<z_s$ is obtained (z_p being the value of z for $\boldsymbol{U}^p$) $(p=0, 1, \cdots)$, or evidence is obtained of the absence of such solutions.

The sequence $\boldsymbol{U}^q (q=0, 1, \cdots)$ converges towards an optimal solution. This procedure might be called a pseudo-dual algorithm, because, as in the dual simplex method, it starts with a dual feasible solution and then successively approaches the primal feasible solution holding at all times the property of dual feasibility. However, a real dual simplex method never takes place; the dual simplex criterion for choosing the vector to enter the basis is not used, nor any of the vectors $\boldsymbol{e}_i$ $(i \in M)$ are ever eliminated from the basis in the sense of being replaced by another vector.

4. Some definitions and fundamental idea of the algorithm

Assume that we obtain a solution vector $\boldsymbol{U}^p=(\boldsymbol{X}^p, \boldsymbol{Y}^p)$ after p-th iteration, As each constraint of the set (2) contains exactly one component of Y, a solution $\boldsymbol{U}^p=(\boldsymbol{X}^p, \boldsymbol{Y}^p)$ is uniquely determined by the set $J_p=\{j \mid j\in N,\ x^p{}_j=1\}$. That is to say, if

$$x_j{}^p = \begin{cases} 1 & (j\in J_p) \\ 0 & [j\in(N-J_p)]\ , \end{cases} \tag{12}$$

then

$$y_i{}^p = b_i - \sum_{i\in J_p} a_{ij} \qquad (i\in M) \tag{13}$$

As already shown, the additive algorithm generates a sequence of solutions. For the s-th term $\boldsymbol{U}^s$ of this sequence,

$$\boldsymbol{U}^s = U(j_1, \cdots, j_l) = (\boldsymbol{X}^s, \boldsymbol{Y}^s)\ , \tag{14}$$

where

$$\{j_1, \cdots, j_l\} = \{j \mid j\in N,\ x_j{}^s = 1\} = J_s\,.$$

Therefore, if we obtain a solution $\boldsymbol{U}^s=(\boldsymbol{X}^s, \boldsymbol{Y}^s)$ after s-th iteration, then the values of $\boldsymbol{X}^s$ and $\boldsymbol{Y}^s$ are given as follows;

$$x_j{}^s = \begin{cases} 1 & j\in J_s \\ 0 & j\in(N-J_s) \end{cases} \tag{15}$$

$$y_i{}^s = b_i - \sum_{j\in J_s} a_{ij} \qquad (i\in M)\ , \tag{16}$$

where $x_j{}^s$ and $y_i{}^s$ are elements of the vector $\boldsymbol{X}^s$ and $\boldsymbol{Y}^s$ respectively.

Let z_q denote the value of the objective function at q-th iteration for the feasible solution. Then we define the following Z_s.

$$z_s = \{z_q \mid q\leqq s,\ U^q\geqq 0\} \tag{17}$$

The smallest element of this set is called the ceilling for the solution vector $\boldsymbol{U}^s$. That is,

$$z^{*(s)} = \begin{cases} \infty & \text{if} \quad z_s = \phi \\ \min_{Z_s} \quad z_q & \text{if} \quad z_s \neq \phi \end{cases} \tag{18}$$

Let N_s denote the set of subscripts of the variable x_j to be introduced into the basis at the $s+1$th iteration. Of course, N_s is the subset of the set $(N_s\subseteq N)$. Further let $j(p)$ denote the element of the indices set ${}_pN$.

Now we shall introduce a certain criterion for the choice of the vector to introduce into the basis for the vector solution $\boldsymbol{U}^s$ and for each $j\in N_s$. First, we define

$$v_j^s = \begin{cases} \sum_{i\in j} M_p^{s-}(y_i^s - a_{ij}) + \sum_{i\in j} M_\beta^{s-}(y_i^s - a_{ij}) & \\ & \text{for} \quad ({}_jM_p^{s-} \cup {}_jM_\beta^{s-}) \neq \phi \\ 0 & \text{for} \quad ({}_jM_p^{s-} \cup {}_jM_\beta^{s-}) = \phi \end{cases} \tag{19}$$

where $\quad {}_jM_q^{s-} = \{i(y_i^s - a_{ij}) < 0,\ i \in M_q\},\ q = p \quad \text{or} \quad \beta\ .$

After the s-th iteration, if there exists any negative element y_i^s having the index i which belongs to the indices M_p, we shall define P_s as the set of these indices p. That is,

$$P_s = \{p \mid y_i^s < 0 \qquad \text{for at least one } i \in M_p\}\ . \tag{20}$$

Moreover, we shall define ${}_pN_s$ as

$${}_pN_s = {}_pN \cap N_s\ . \tag{21}$$

Then we know empirically that we can obtain efficiently a feasible solution by choosing the variable x_j to introduce into the basis for the solution vector U^s, where j belongs to the indices set ${}_pN_s$ and p belongs to the set P_s.

This fact means that we examine the feasibility of the solution vector, first of all, for the constraints α. In this case, it is expected to introduce at a time as much variables as possible into the basis. As the criterion of choosing the variables, we select the variable x_j so that j belongs to the indices set ${}_pN_s$ and the value v_j^s is maximum among the set ${}_pN_s$.

In this case, if we select the all of variables x_j of which index j belongs to the set ${}_pN_s$ for $p \in P_s$, then there exists the possibility that the value of the objective function exceeds the value $z^{*(s)}$. Therefore, we select the variables to introduce into the basis from the ones having the small value of p, in convenience, until the value of the objective function does not exceed the value $z^{*(s)}$. Of course, if the set Z_s is ϕ, then we have to select for all the $p \in P_s$.

If $P_s = \phi$, that is, y_i^s nonnegative for all $i \in M_\alpha$ and the solution vector U^s is not a feasible solution, we select the index j which corresponds to the maximum value of v_j^s among the indices set N_s. And we can define the set I_s of these selected indices as follows:

$$I_s = \begin{cases} \{j_0(p) \mid v_{j0}^s = \max v_j^s,\ j = j(p),\ j(p) \in N_s,\ p \leqq p_0,\ p \in P_s & \\ & \text{for} \quad P_s \neq \phi \\ \{j_0(p) \mid v_{j0}^s = \max v_j^s,\ j \in N_s\} & \text{for} \quad P_s = \phi\ , \end{cases} \tag{22}$$

where p_0 is the maximum of p which satisfies the following relation,

$$\sum_{j_0 = j_0(p),\, p \in P_s} c_{j0} < (z^{*(s)} - z_s) \tag{23}$$

Then we have the new indices set J_{s+m} from the set J_s and I_s.

$$J_{s+m} = J_s \cup I_s$$

In this case, the cardinal number of the indices set I_s is m and J_{s+m} yields the new solution vector U^{s+m}. Moreover, the indices set $J_{s+k}(1 \leqq k \leqq m)$ is the union of the set I_s of which cardinal number is k and the set J_s, where the element of the set I_s is ordered such that $j_1 \leqq j_2$ for $j_1 \in (J_{s+k} - J_{s+k-1}), j_2 \in (J_{s+l} - J_{s+l-1})$ where $k \leqq l$.

From the above mentioned idea, we can obtain a feasible solution at only one iteration by using the operation of (22). However, we need m iterations to obtain the same feasible solutions by using the Balas' procedure. In addition, we shall denote by $J_{\bar{s}}$ the J_s in order to show that the value of v_i^s is computed.

Next, we define the indices set N_k^s which corresponds to the variable x_j to introduce into the basis for the solution vector U^k where two solution vector U^k and U^s are given and $k \leqq s$, $J_k \subseteq J_s$.

Also, we define by k_m the maximum k_1 such as $J_{\bar{k}_1} \subseteq J_k$, and by ${}_pN_k^s$ the joint of the sets ${}_pN$ and N_k^s. That is,

$$ {}_pN_k^s = {}_pN \cap N_k^s \tag{24}$$

Here, we select the variables x_j to introduce into the basis for the given solution vector U^k so that the value of $v_j^{k_m}$ can be maximum among the variables having the indices of the indices set ${}_{p1}N_k^s$ and that the value of the objective function can be, at any case smaller than $z^{*(s)}$ where $p_1 \in P_{k_m}$, $p_1 \geqq p(j(p) \in J_{k+1} - J_k)$.

In this case, if $z_s = \phi$, we select the variable x_j to which the value of $v_j^{k_m}$ is maximum among the variables having the indices of the set N_k^s. Moreover, if $z_s = \phi$ and $P_s \neq \phi$, we select the variable x_j so that the value of $v_j^{k_m}$ can be maximum for $p_1 \in P_{k_m}$, $P_1(\geqq p)$. We define by I_k the set of indices j, selected as mentioned above, that is,

$$I_k = \begin{cases} \{j_1(p) \mid v_{j_1}^{k_m} = \max v_j^{k_m},\ j = j(p_1),\ j(p_1) \in N_k^s,\ p \leqq p_1 \leqq p_0,\ p_1 \in P_{k_m}\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } P_{k_m} \neq \phi \\ \{j_1(p) \mid v_{j_1}^{k_m} = \max v_j^{k_m},\ j \in N_k^s\} \qquad \text{for } P_{k_m} = \phi \end{cases} \tag{25}$$

where p_0 is the maximum p such as

$$\sum_{j_1 = j_1(p), p_1 \geqq p, p_1 \in P_s} c_{j_1} < (z^{*(s)} - z_k) \,. \tag{26}$$

As we described in the previous section, in order to obtain a feasible solution, we must take the value of x_j as one to which $a_{ij}(i \in M)$ is as small as possible. For this purpose, we considered the value of v_{js}; i.e., v_j^s is the sum of negative components of the solution vector U^{s+m}. Therefore, if we can determine the values of $x_j(j \in I_s')$, we must cancel v_j^s. From these facts, the values $v_j^{k_m}$ assigned to a certain solution U^k are successively cancelled in the subsequent iterations according to certain rules. In this case, we define $C_k^s(k \leqq s)$ as the indices set of the index j which corresponds to the cancelled $v_j^{k_m}$ until the solution vector U^s has been obtained. Moreover, we denote by C^s the set of cancelled v_j^k until the solution vector U^s has been obtained for all $k(J_{\bar{k}} \subset J_k)$; that is

$$C^s = \cup_{k, J_{\bar{k}} \subset J_k} C_k^s . \tag{27}$$

We shall now define for the solution U^k the set of those indices $j \in N - C^s$ such that, if the variable x_j were introduced into the basis, the value of the objective function would exceed the value $z^{*(s)}$ as D_s;

$$D_s = \{j \mid j \in N - C^s,\ c_j \geqq z^{*(s)} - z^s\} \tag{28}$$

Further, we shall define the following sets.

$$\bar{N}_s = N - (C^s \cup D_s) , \tag{29}$$

$$w_j^s = \textstyle\sum_{i \in j} M_p^{s-}(y_i^s - a_{ij}), \text{ where } j \in {}_pN . \tag{30}$$

Then, if there is some set $j_1(p)$ which has only one element, among the set of sets $j(p) \in \bar{N}_s$ and $w_{j_1}^s$ is negative ($w_{j_1}^s < 0$) for some $p \in P$, it is not necessary to introduce the variable x_{j_1} into the basis. Since, the new solution U^k which is obtained by the iterations for the solution U^s does not satisfy the constraint p from the definition of w_j^s and moreover ${}_pN \cap N_k = \phi$ because of $N_k \subset N_s$ and $j_1 \notin N_s$.

Let E_s denote the above mentioned indices set $j(w_j^s < 0)$, that is,

$$E_s = \{j \mid j \in ({}_pN \cap \bar{N}_s), \text{ Card } ({}_pN \cap \bar{N}_s) = 1 \rightarrow w_j^s < 0\} , \tag{31}$$

and let N_s define the following indices set.

$$N_s = N - (C^s \cup D_s \cup E_s) = \bar{N} - E_s . \tag{32}$$

It is noted that $N_s = \phi$ for the feasible solution U^s from the definition of the Z_s and D_s.

Similarly to D_s, for the pair of solutions U^k and $U^s(k < s)$, we define D_k^s as the set of indices $j \in (N_{k_m} - C_{k_m}^s)$, such that, if the variable x_j were introduced into the basis, then the value of the objective function would exceed the value $z^{*(s)}$;

$$D_k^s = \{j \mid j \in (N_{k_m} - C_{k_m}),\ c_j \geqq z^{*(s)} - z_k\} \tag{33}$$

Denoting the indices set $\bar{N}_k^s$ as

$$\bar{N}_k^s = N_{k_m} - (C_k^s \cup D_k^s) \tag{34}$$

if there exists a set $j_1(p) \in \bar{N}_k^s$ which has the only one element and $w_{j_1}{}^s<0$ for some $p\in \boldsymbol{P}$, then it is not necessary to introduce the variable x_j into the basis.

Then, we shall define the set of indices j_1 as $E_k{}^s$;

$$E_k{}^s = \{j \mid j\in {}_pN \cap N_k{}^s,\ \text{Card.}\ ({}_pN \cap N_k{}^s) = 1 \rightarrow w_j{}^s<0\} \tag{35}$$

From the above mentioned fact, we define the following indices set $N_k{}^s$;

$$N_k{}^s = N_{k_m} - (C_k{}^s \cup D_k{}^s \cup E_k{}^s) = \bar{N}_k{}^s - E_k{}^s \tag{36}$$

Whenever a solution $\boldsymbol{U}^s$ is obtained, only the improving vectors for that solution are considered for introduction into the basis. Whenever the set of improving vectors for a solution $\boldsymbol{U}^s$ is found to be void, this is to be interpreted as a stop signal, which means that there is no feasible solution $\boldsymbol{U}^k$ such that $J_k \subset J_s$ and $z_k<z^{*(s)}$.

5. Solution Algorithm

We start with the feasible solution $\boldsymbol{U}^0$, for which

$$\boldsymbol{X}^0 = 0,\ \boldsymbol{Y}^0 = \boldsymbol{B},\ z_0 = 0,\ J_0 = \phi \tag{37}$$

Suppose that after s iterations we have obtained the solution $\boldsymbol{U}^s$, for which

$$x_j{}^s = \begin{cases} 1 & j\in J_s \\ 0 & j\in (N-J_s) \end{cases} \tag{38}$$

$$y_i{}^s = b_i - \textstyle\sum_{j\in J_s} a_{ij} \qquad (i\in M) \tag{39}$$

$$z_s = \textstyle\sum_{j\in J_s} c_j \tag{40}$$

For the above mentioned situation, the following procedure is then adopted:

Step 1. Check $y_i{}^s$ $(i\in M)$.

1a. If $y_i{}^s \geqq 0 (i\in M)$, set $z_s=z^{*(s)}$ and go to step 4. However, if $s=0$, then $\boldsymbol{U}^0$ is the optimal solution and the algorithm stops.

1b. If there exists i_1 such that $y_{i_1}{}^s<0$, then go to step 2.

Step 2. Check the following equation. If $N_s=\phi$, then we have the same procedure of the step 2a.

$$\textstyle\sum_{j\in N_s} a_{\bar{i}\bar{j}} \leqq y_i{}^s \qquad (i \mid y_i{}^s<0)\,, \tag{41}$$

where $a_{\bar{i}\bar{j}}$ means the negative element of the matrix $\boldsymbol{A}$.

2a. If there exists $i_1\in M_p(p\in \boldsymbol{P})$ for which eq. (41) does not hold, set k $(j_k=j_k(p))$ as k_0 and go to step 3. In this case return the cancelled v_{jk_1} to the original place where $j\in(J_s-J_k)$ and $J_{\bar{k}1}\subset J_s$. However, if $k=s$ and there exists i_1 for which eq. (41) does not hold and which belongs only to M_β, then go to step 4.

2b. If eq. (41) holds for all $i(y_i^s<0)$ as strict inequalities, compute the value of I_s and obtain the following value.

$$J_{s+m} = J_s \cup I_s, \quad m = \text{Card. } (I_s) \tag{42}$$

$$z_{s+l} = z_{s+l-1}+c_{j_{s+l}}, \qquad (l = 2, 3, \cdots, m) \tag{43}$$

$$y_i^{s+l} = y_i^{s+l-1}+a_{ij_{s+l}}, \tag{44}$$

where j_{s+l} means j which belong to $(J_{s+l}-J_{s+l-1})$. Then after the computation of the above described values (42), (43), (44), set J_s as $J_{\bar{s}}$ and cancel v_j^s $(j\in I_s)$ and go to next iteration (say, step 1).

2c. If eq. (41) holds for all i $(y_i^s<0)$, and there exists a set M_s^s such that eq. (41) holds as equalities for $i\in M_s^s$, go to step 6.

Step 3. Check the relation

$$\sum_{j\in N_k} a_{\bar{i}j} \leqq y_i^k \qquad (i\,|\,y_i^k<0) \tag{45}$$

from $k=k_0$ by the increasing order of k.

(If $N_k=\phi$, then we have the same procedure of the step 3 a.)

3a. If there exists $i_1\in M_p(p\in \boldsymbol{P})$ for which eq. (45) does not hold, then set $k=s$ and go to step 4.

3b. If eq. (45) holds for all i $(y_i^s<0)$ as strict inequalities, cancel $v_{jk_1}{}^{k_m}$ and repeat step 3 for k_1 ($J_k \underset{m}{\subset} J_{k_1}$, where $A\underset{m}{\subset}B$ means that that A is the maximum set of A' which is contained in B; $A'\subset B$.)

3c. If eq. (45) holds for all i $(y_i^s<0)$, and there exists a set M_k^s such that eq. (45) hold as equalities for $i\in M_k^s$, go to step 6.

Step 4. Check the relation

$$\sum j\in N_k^s a_{\bar{i}j} \leqq y_i^k \qquad (i\,|\,y_i^k<0) \tag{46}$$

from the k such as $J_k \underset{m}{\subset} J_s$ by the decreasing order of k for $i(y_i^k<0)$.

(If $N_k^s=\phi$, then we have the same procedure of the step 4a.)

4a. If there exists $i_1\in M_p(p\in \boldsymbol{P})$ for which eq. (46) does not hold, return back $v_{k_1}{}^{k_m}(J_k \underset{m}{\subset} J_{k_1})$ to the original place and repeat the step 4 for k_2 $(J_{k_2} \underset{m}{\subset} J_{k_1})$. If ther exists $i\in M_p$ $y_i^{k_1}<0$, $j_{k_1}=j_{k_1}(p)$, put Q_k^s as the set of $i_1\in M_p$ for which eq. (46) does not hold and go to step 5.

If there does not exist k_2, the algorithm stops. Then, if $z_s=\phi$, there is no feasible solution and if $z_s\neq\phi$, then $z_q=z^{*(s)}$ becomes the optimal solution U^q in the stage.

4b. If eq. (46) holds for all $i \in M(y_i^k<0)$ as strict inequalities, compute the value of I_k and obtain the following values.

$$J_{s+m} = J_k \cup I_k, \quad \text{Card. } (I_k) = m \tag{47}$$

$$z_{s+1} = z_k + c_{j_{s+1}} \tag{48}$$

$$z_{s+l} = z_{s+l-1} + c_{j_{s+l}} \qquad (l = 2, 3, \cdots, m) \tag{49}$$

$$y_i^{s+1} = y_i^k - a_{ij_{s+l}} \tag{50}$$

$$y_i^{s+l} = y_i^{s+l-1} - a_{ij_{s+l}}. \tag{51}$$

Then, after the computation of the above described values, cancel $v_j^{k_m}$ for all $j \in \bar{J}_k$ and go to the next iteration.

4c. If eq. (46) holds for all $i(y_i^s<0)$, and there exists a set M_k^s such that eq. (46) hold as equalities for $i \in M_k^s$, go to step 6.

Step 5. Check the relation (52) for i $(y_i^{k_1}<0)$ and the relation (53) for $i \in Q_k^s$.

If $X=\phi$, then we have the same procedure as that of step 5a. If Card. $(X)=1$ and $w_{j_1}^{k_1}<0$ and $w_j^k<0$ for $j_1 \in X$, we have the same procedure of that of step 5a. If $w_{j_1}^{k_1} \geqq 0$ and $w_{j_1}^k \geqq 0$, we have the procedure of step 5b. Where, $X={}_pN \cap (N_{k_m} \cup D_{k_m}) - \{j_{k_1}\}$

$$\textstyle\sum_{j \in X} a_{\bar{i}j} \leqq y_i^{k_1} \qquad (i \mid i \in M_p,\ y_i^{k_1}<0) \tag{52}$$

$$\textstyle\sum_{j \in X} a_{\bar{i}j} \leqq y_i^k \qquad (i \in Q_k^s) \tag{53}$$

5a. If there exist $i_1 \in M_p$ and $i_2 \in Q_k^s$ simultaneously for which eq. (52) and eq. (53) do not hold respectively, then we repeat step 4 for k_4, where $J_{k_4} \underset{m}{\subseteq} J_{k_3} \subset J_k$, $J_{k_3} = J_{k_3}(p)$. If there does not exist k_4, we have the same procedure as that of steps 4b, where there does not exist k_2.

5b. If eq. (52) and eq. (53) hold for all $i(y_i^{k_1}<0)$ and $i \in Q_k^s$ respectively, we repeat the procedure of step 4 for k_5 $(J_{k_5} \underset{m}{\subseteq} J_k)$. If there does not exist k_5 we have the same procedure as that of step 4b.

Step 6. Check the relation

$$\textstyle\sum j \in F_k^s c_j < z^{*(s)} - z_k\,, \tag{54}$$

where F_k^s means the set of indices j for which $a_{ij}<0$ at least one $i \in M_k^s$. Note that $j \in N_k$ if from step 2c or 3c, and $j \in N_k^s$ if from step 4c.

6a. If eq. (54) holds, set

$$J_{s+1} = J_k \cup F_k^s$$
$$y_i^{s+1} = y_i^k - \textstyle\sum j \in F_k^s a_{ij}$$

and cancel v^{k_m} for all $j \in F_k^s$ and go to the next iteration.

6b. If eq. (54) does not hold, return back $v_{jk_1}{}^{k_m}$ $(J_k \underset{m}{\subset} J_{k_1})$ to the original place and repeat the procedure of step 4 for $k_2(J_{k_2} \underset{m}{\subset} J_k)$. If there does not exist k_2, we have the same procedure as that of step 4b.

Remark. In the following, we shall describe the procedure of computations of I_k and $\bar{I}_k$.

1. Computation of I_k

Step 1. Obtain the set P_s.

Step 2. If $P_s = \phi$, then compute $v_j^s (j \in N_s)$ and obtain the maximum $v_{j_0}{}^s$ among $v_j^s (j \in N_s)$ and set $I_s = \{j_0\}$.

Step 3. If $P_s \neq \phi$, then compute v_j^s for $j = j(p)$, $p \in P_s$ and obtain the maximum $v_{j_p}{}^s$ for $p \in P_0$ among $v_j^s (j = j(p))$.

3a. If $z_s = \phi$, then set $I_s = \{j_p \mid p \in P_s\}$.

3b. If $z_s \neq \phi$, then set $I_s = \{j_p \mid p < p_1, p \in P_s\}$ where p_1 is the maximum p which satisfies eq. (55).

$$\sum_{p \in P_s} c_{j_p} < z^{*(s)} - z_s \tag{55}$$

2. Computation of $\bar{I}_k$

Step 1. Obtain the set P_{k_m}.

Step 2. If $P_{k_m} = \phi$, then obtain the maximum $v_{j_0}{}^{k_m}$ of $v_j^{k_m} (j \in N_k^s)$ and set $\bar{I}_k = \{j_0\}$.

Step 3. If $P_{k_m} \neq \phi$, then obtain the maximum $v_{i_p}{}^s$ of v_j^s $(j = j(p))$ for all $p \in P_s$, $p \geqq p_1$, $J_{k_m} J_{k_1}$, $j_{k_1} = j_k(p)$.

3a. If $z_s = \phi$, then set $I_s = \{j_p \mid p \geqq p_1, p \in P_s\}$.

3b. If $z_s \neq \phi$, then, set $I_s = \{j_p \mid p_1 \leqq p \leqq p_0, p \in P_0$. Where p_0 is the maximum p which satisfies eq. (56).

$$\textstyle\sum_{p_1 \leqq p, p \in P_s} c_{j_p} < z^{*(s)} - z_k \tag{56}$$

6. Some Remarks on the efficiency of the algorithm

Since the solution algorithm which is described in the previous section is an extension of Balas' algorithm, the algorithm is essentially same as the additive algorithm. However, we can delete many iterations by applying the proposed algorithm for the weighted distribution problem comparing with additive algorithm.

The characteristics of our algorithm are the following three points:

(i) Introduction of more than one variables into the basis at a time

(ii) Institution of the step 5

(iii) Institution of the set E_s and the set E_k^s.

The above three points are deviced to check the feasibility for the constraint α in order to utilize the properties of the given problem as much as possible.

In the case of (i), though there exists the over introduction of the variables into the basis, we can modify these over introduction by step 3. In the case of (ii), the merit can be described by the following fact; that is, if we have the process of step 4 for k_4 $(J_{k_4} \underset{m}{\subset} J_{k_3} \subset J_k)$ after the decision of step 5a for a solution U^k, we can delete $n'(\leqq n-1)$ s times of iterations for step 4, and by this fact, we can delete $n'!$ iterations at best. In the case of (iii), this algorithm is efficient for the case, where we have the decision of step 1b and $P_s \neq 0$ for a solution U_j^s that is, we can delete a few interations by considering the elements of the set N_s for the step.

7. Illustrative Esamples

(Problem) Find X such as
Minimize

$$5x_1+7x_2+x_3+2x_4+8x_5+3x_6+x_7+5x_8$$

Subject to

$$\begin{aligned}
&-3x_1-5x_2+2x_3-5x_4 &&\leqq -7\\
&-2x_1+3x_2-2x_3+4x_4 &&\leqq 3\\
&\qquad -5x_5+5x_6-6x_7-x_8 &&\leqq -5\\
&\qquad -4x_5-2x_6-5x_7-2x_8 &&\leqq -6\\
&-3x_1-2x_2-4x_3-4x_4-7x_5-6x_6-3x_7-5x_8 &&\leqq -8\\
&-3x_1+x_2-4x_3+x_4-x_5-2x_6+3x_7-3x_8 &&\leqq -3\\
&x_1, x_2, \cdots, x_8 \geqq 0
\end{aligned}$$

(Solution) First, we obtain the following fundamental set.

$$P=\{1, 2\},\ {}_1N=\{1, 2, 3, 4\},\ {}_2N=\{5, 6, 7, 8\},\ N={}_1N\cup{}_2N$$

$$M_1=\{1, 2\},\ M_2=\{3, 4\},\ M_\alpha=M_1\cup M_2,\ M_\beta=\{5, 6\},$$

$$M=M_\alpha\cup M_\beta \text{ and also } J_0=\phi,\ z_0=0,$$

$$y_1^0=b_1=-7,\ y_2^0=b_2=3,\ y_3^0=-5,\ y_4^0=-6,\ y_5^0=-8,\ y_6^0=-3.$$

(iteration 1)

step 1. $y_i^0<0$ for $i=1, 3, 4, 5, 6$, then this is the state of the step 1b.

step 2. $C^0=D_0=E_0=\phi$, $N_0=N-(C^0\cup D_0\cup E_0)=\{1(1), 2(1), 3(1), 4(1), 5(2), 6(2), 7(2), 8(2)\}$.

$\sum_{j\in N_0} a_{\overline{1}j}=-3-5-5=-13<y_1^0=-7$

$\sum_{j\in N_0} a_{\overline{3}j}=-5-6-1=-12<y_3^0=-5$

$\sum_{j\in N_0} a_{\overline{4}j}=-13<y_4^0=-6$

$\sum_{j\in N_0} a_{\overline{5}j}=-34<y_5^0=-8$

$\sum_{j\in N_0} a_{\overline{6}j}=-13<y_6^0=-3$

then this is the state of step 2b.

$P_0=\{1, 2\}$

$v_1=\sum i\in{}_1M_1{}^0-(y_i-a_{i1})+\sum_{i\in{}_1M_\beta}-(y_i^0-a_{i1})=-4-5=-9$

$v_2=\sum i\in{}_2M_1{}^0-(y_i^0-a_{i2})+\sum_{i\in{}_2M_\beta}-(y_i^0-a_{i2})=-2-6-4=-12$

$v_3^0=-13,\ v_4^0=-11,\ v_5^0=-5,\ v_6^0=-17,\ v_7^0=-12,\ v_8^0=-11$

Henceforth, we have max $v_j^0=v_1^0=-9$ for $j=j(1)$ $(\in N_0)$ and max $v_j^0=v_5^0=-5$ for $j=j(2)$ $(\in N_0)$. Then we have

$I_0=\{1(1), 5(2)\}$, $J_2=J_0\cup I_0=\{1(1), 5(2)\}$, Card. $(I_0)=2$,

$z_1=0+c_1=5,\ z_2=z+c_5=5+8=13$

$y_1^1=y_1^0-a_{11}=-4,\ y_2^1=3-a_{21}=5,\ y_3^1=-5,\ y_4^1=-6,\ y_5^1=-5,$

$y_6^1=0,\ y_1^2=y_1^1=-4,\ y_2^2=y_2^1=5,\ y_3^2=0,\ y_4^2=2,\ y_5^2=2,\ y_6^2=1.$

Putting J_0 as J_0- and cancel v_1^0 and v_5^0 and go to iteration 2.

(iteration 2)

step 1. $y_i^2<0$ for $i=1, 4$ (1b)

step 2. $C^2=\{1, 5\}$, $D_2=\phi$, $E_2=\phi$. Then we have

$N_2=\{2(1), 3(1), 4(1), 6(2), 7(2), 8(2)\}$.

$\sum_{j\in N_2} a_{\overline{1}j}=-10<y_1^2=-4,$

$\sum_{j\in N_2} a_{\overline{4}j}=-9<y_4^2=-2$ (2b)

$P_2=\{1, 2\}$

$v_2^2=0,\ v_4^2=0,\ v_6^2=-5,\ v_7^2=-2,\ v_8^2=0$

For $j=j(1)$, max $v_j^2=v_2^2=v_4^2=0$. However, we take v_4^2 for max v_j^2 because $c_4=2<c_2<4$.

For $j=j(2)$, max $v_j^2=v_8^2=0$.

Therefore, we have

$I_2=\{4(1), 8(2)\}$, $J_{2+2}=J_4=\{(1), 5(2), 4(1), 8(2)\}$.

In this case,

$$z_3=15,\ z_4=20$$

$y_1^3=1,\ y_2^3=1,\ y_3^3=0,\ y_4^3=-2,\ y_5^3=6,\ y_6^3=0$
$y_1^4=1,\ y_2^4=1,\ y_3^4=1,\ y_4^4=0,\ y_5^4=11,\ y_6^4=3.$
Put J_2- instead of J_2 and cancel v_4^2 and v_8^2.

(iteration 3)

step 1. For all $y_i^4 \geqq 0$ (1a), we put $z_4=z^{*(4)}$ and go to step 4.

step 4. Since $C_3^4=\{4, 8\}$, $c_2=7>z^{*(4)}-z_3=5$ for $k=3$, we have $D_3^4=\{2\}$.
$\bar{N}_3^4 = N_2-(C_3^4 \cup D_3^4) = \{3(1), 6(2), 7(2)\}$.
Since Card. $\{j(1)\}=1$, $w_3^4=-1<0$, we have $E_3^4=\{3\}$.
Therefore,

$$N_3^4 = \{6(2),\ 7(2)\}\,.$$

$\sum_{j \in N_3^4} a_{\overline{4j}} = -2-5=-7<y_4^3=-2$ (4b).
For $j=j(2)$, max $v_j^2=v_7^2=-2$, $c_7<z^{*(4)}-z_3$.
Therefore, $I_3=\{7(2)\}$, $J_5=\{1(1), 5(2), 4(1), 7(2)\}$
$=J_3 \cup I_3$.
In this case,

$$z_5 = 16$$

$y_1^5=1,\ y_2^5=1,\ y_3^5=6,\ y_4^5=3,\ y_5^5=9,\ y_6^5=-3$
and cancel v_7^2.

(iteration 4)

step 1. $y_i^5<0$, for $i=5$ (1b)

step 2. $C^5=\{1, 4, 5, 7, 8\}$, $D_5=\{2\}$
$E_5=\{3\}$ $(w_3^5=-1<0)$.
Therefore,

$$N_5 = N-(C^5 \cup D_5 \cup E_5)=\{6(2)\}\,.$$

$\sum_{j \in N_5} a_{\overline{6j}} = -2<y_6^5=-3$ (2a).
Since $i(=6) \in M_\beta$, go to step 4.

step 4. Since $C_3^5=\{4, 7, 8\}$, $D_3^5=\{2\}$, $E_3^5=\{6\}$
for $k=3$ $(J_3 \underset{m}{\subset} J_5)$, we have $N_3^5=\{3(1)\}$.
$\sum_{j \in N_3} a_{\overline{4j}}=0>y_4^3=-2$ (4a).
Since $y_i^5 \geqq 0$ for $i \in M_2$, return back v_7^2 to the original place and repeat step 4 for $k=2$.
Then we have
$C_2^5=\{4\}$, $D_2^5=\{2\}$, $E_2^5=\{3\}$ and $N_2^5=\{6(2), 7(2), 8(2)\}$.
$\sum_{j \in N_2^5} a_{\overline{1j}}=0>y_1^2=-4$ (4a)

Since $y_i^3<0$ for $i\in M_1$, return back v_4^2 to the original place and repeat step 4 for $k=1$.
Then,
$C_1^5=\{1, 5\}$, $D_1^5=E_1^5=\phi$, and $N_1^5=\{2(1), 3(1), 4(1), 6(2), 7(2), 8(2)\}$.
$\sum_{j\in N_1}a_{\bar{i}\bar{j}}<y_i^1$ for $i=1, 3, 4, 5$ (4b)
$\max v_j^0=v_8^0=-11$ (for $j=j(2)$), $c_8<z^{*(5)}-z_1$.
Therefore, we have
$y_1^6=-4$, $y_2^6=5$, $y_3^6=-4$, $y_4^6=-4$, $y_5^6=0$, $y_6^6=3$ and cancel v_8^0.

(iteration 5)

step 1. $y_i^6<0$ for $i=1, 3, 4$ (1b)

step 2. $C^6=\{1, 5, 8\}$, $D^6=E_6=\phi$ and
$N_6=\{2(1), 3(1), 4(1), 6(2), 7(2)\}$
$\sum_{j\in N_6}a_{\bar{i}\bar{j}}<y_i^6$ for $i=1, 3, 4$.
$P_6=\{1, 2\}$, $v_2^6=0$, $v_3^6=-6$, $v_4^6=0$, $v_6^6=-3$, $v_7^6=0$.
$\max v_j^6=v_2^6=v_4^6=0$ (in this case, we take v_4^6 as $\max v_j^6$, since $c_4<c_2$) for $j=j(1)$
$\max v_j^6=v_7^6=0$ for $j=j(2)$.
Moreover, $c_4+c_7<z^{*(6)}-z_6$. Then we have
$I_6=\{4(1), 7(2)\}$, $J_8=\{1(1), 8(2), 4(1), 7(2)\}$
$z_7=12$, $z_8=18$
$y_1^7=1$, $y_2^7=1$, $y_3^7=-4$, $y_4^7=-4$, $y_5^7=4$, $y_6^7=2$
$y_1^8=1$, $y_2^8=1$, $y_3^8=2$, $y_4^8=1$, $y_5^8=7$, $y_6^8=-1$.
Put J_6 as J_6- and cancel v_4^6 and v_7^6.

(iteration 6)

step 1. $y_i^8<0$ for $i=6$ (1b)

step 2. $C^8=\{1, 4, 5, 7, 8\}$, $D_8=\{2\}$, $E_8=\{3, 6\}$ and $N_8=\phi$ (2a)

step 4. $C_7^8=\{4, 7\}$, $D_7^8=\phi$, $E_7^8=\{6\}$, and $N_7^8=\{2(1), 3(1)\}$ for $k=7$.
$\sum j\in N_9^8 a_{\bar{i}\bar{j}} y_i^7$ for $i=3, 4$. (4a)
Return back v_7^6 to the original place.
$C_6^8=\{4\}$, $D_6^8=E_6=\phi$
and then
$N_6^8=\{2(1), 3(1), 6(2), 7(2)\}$ for $k=6$.
$\sum j\in N_6^8 a_{\bar{i}\bar{j}}<y_i^6$ for $i=1, 3, 4$ (4b)
$\max v_i^6=v_2^6$ for $j=j(1)$
$\max v_2^6=v_7^6$ for $j=j(2)$.
Then we have $I_6=\{2(1), 7(2)\}$, $J_{10}=\{1(1), 8(2), 2(1), 7(2)\}$,

$z_9=17$, $z_{10}=18$,
$y_1^9=1$, $y_2^9=2$, $y_3^9=-4$, $y_4^9=-4$, $y_5^9=2$, $y_6^9=2$
$y_1^{10}=1$, $y_2^{10}=2$, $y_3^{10}=2$, $y_4^{10}=1$, $y_5^{10}=5$, $y_6^{10}=-1$.

(iteration 7)

step 1. $y_i^{10}<0$ for $i=6$ (1b)

step 2. $C^{10}=\{1, 2, 4, 5, 7, 8\}$, $D_{10}=\{6\}$, $E_{10}=\{3\}$
and then
$N_{10}=\phi$ (2a)

step 4. $C_9^{10}=\{2, 4, 7\}$, $D_9^{10}=\{6\}$, $E_9^{10}=\{3\}$
and then
$N_9^{10}=\phi$ for $k=9$ (4a)
$C_6^{10}=\{2, 4\}$, $D_6^{10}=\phi$, $E_6^{10}=\{3\}$
and then
$N_6^{10}=\{6(2), 7(2)\}$ for $k=6$.
$\sum j\in N_6^{10}\ a_{\bar{i}\bar{j}}>y_1$ (4a)
$C_1^{10}=\{1, 5, 8\}$, $D_1^{10}=\phi$
and then
$N_1^{10}=\{2(1), 3(1), 4(1), 6(2), 7(2)\}$ for $k=1$.
$\sum j\in N_1^{10}\ a_{\bar{i}\bar{j}}<y_i^1$ for $i=1, 3, 4, 5$ (4b)
$\max v_j^0=v_7^0$ for $j=j(2)$
$I_1=\{7(2)\}$, $J_{11}=\{1(1), 7(2)\}$, $z_{11}=6$
$y_1^{11}=-4, y_2^{11}=5, y_3^{11}=1, y_4^{11}=-1, y_5^{11}=-2, y_6^{11}=-3$.

(iteration 8)

step 1. $y_i^{11}<0$ for $i=1, 4, 5, 6$ (1b)

step 2. $C^{11}=\{1, 5, 7, 8\}$, $D_{11}=\phi$, $E_{11}=\{6\}$
and then
$N_{11}=\{2(1), 3(1), 4(1)\}$
$\sum_{j\in N_{11}} a_{\bar{i}\bar{j}}>y_i^{11}$ for $i=4$ (4a)

step 4. Taking $k=1$, $N_1^{11}=\{2(1), 3(1), 4(1)\}$ and
$\sum j\in N_1^{11}\ a_{\bar{i}\bar{j}}>y_i$ for $i_1=3, 4$. (4a)
Since $y_i^{11}<0$ for $i(=3)\in M_2$, go to step 5 by taking $Q_1^{11}=\{3, 4\}$.

step 5. $X={}_2N_0\cup({}_2N\cap D_0)-\{7\}=\{5, 6, 8\}$,
$\sum_{j\in X} a_{\bar{i}\bar{j}}<y_i^{11}$ for $i=4$ (5b)

step 6. Taking $k=0$,
$N_0^{11}=\{2(1), 3(1), 4(1), 5(1), 6(1), 7(1), 8(1)\}$
$\sum j\in N_0^{11}\ a_{\bar{i}\bar{j}}<y_i^0$ for $i=1, 3, 4, 5, 6$ (4b)

$\max v_j^0 = v_4^0$ for $j=j(1)$
$\max v_j^0 = v_5^0$ for $j=j(2)$
$I_0^0 = \{4(1), 5(2)\}$, $J_{13} = \{4(1), 5(2)\}$, $z_{12}=2$, $z_{13}=7$.
$y_1^{12}=-2, y_2^{12}=-1, y_3^{12}=-5, y_4^{12}=-6, y_5^{12}=-4, y_6^{12}=-4,$
$y_1^{13}=-2, y_2^{13}=-1, y_3^{13}=0, y_4^{13}=-2, y_5^{13}=3, y_6^{13}=-3.$

(iteration 9)

step 1. $y_i^{13}<0$ for $i=1, 4, 6$ (1b)

step 2. $N_{13}=\{2(1), 3(1), 6(2), 7(2), 8(2)\}$
$\sum_{j\in N_{13}} a_{\bar{i}j} < y_i$ for $i=1, 4, 6$.
$v_2^{13}=-8,\ v_3^{13}=-4,\ v_6^{13}=-7,\ v_7^{13}=-6,\ v_8^{13}=0$
$\max v_j^{13}=v_3^{13}$ for $j=j(1)$
$\max v_j^{13}=v_8^{13}$ for $j=j(2)$
$I_{13}=\{3(1), 8(2)\}$, $J_{15}=\{4(1), 5(2), 3(1), 8(2)\}$
$z_{14}=8,\ z_{15}=13,$
$y_1^{14}=-4, y_2^{14}=1, y_3^{14}=0, y_4^{14}=-2, y_5^{14}=7, y_6^{14}=1$
$y_1^{15}=-4, y_2^{15}=1, y_3^{15}=1, y_4^{15}=0, y_5^{15}=12, y_6^{15}=4.$

(iteration 10)

step 1. $y_i^{15}<0$ for $i=1$ (1b)

step 2. $N_{15}=\{6(2), 7(2)\}$, $\sum_{j\in N_{15}} a_{\bar{i}j} > y_1^{15}$, $i_1=1\in M$ (2a)

step 3. $N_{14}=\{6(2), 7(2)\}$
$\sum_{j\in N_{14}} a_{\bar{i}j} > y_i^{14}$ for $i_1=1$ (3a)

step 4. $N_{13}^{15}=2(1), 6(2), 7(2), 8(2)$
$\sum j\in N_{13}^{15}\ a_{\bar{i}j} > y_i^{13}$ for $i_1=1, 2$ (4a)
Since $i_1\in M_1$ and $y_i^{13}<0$ for $i(=1)\in M_1$,
put $Q_{13}^{15}=\{1, 2\}$ and go to step 5.

step 5. $X=\{2\}$, $w_2^4<0$, $w_2^{13}<0$ (5a)

step 4. Taking $k=0$, $N_0^{15}=\{2(1), 3(1), 5(2), 6(2), 7(2), 8(2)\}$
$\sum j\in N_0^{15}\ a_{\bar{i}j} > y_i^0$ for $i=1$ stop.

Comparison of the number of the iteration of Balas' algorithm and the proposed one

I. Balas' algorithm

Number of iteration	Solution	Number of iteration	Solution
1	{5}	6	{5, 1, 4, 7, 3}
2	{5, 1}	7	{5, 1, 4, 6}
3	{5, 1, 4}	8	{5, 8}

Number of iterations	Solution	Number of iterations	Solution
4	{5, 1, 4, 8}	9	{5, 2}
5	{5, 1, 4, 7}	10	{5, 2, 3}
11	{5, 2, 3, 7}	22	{7}
12	{5, 2, 3, 6}	23	{7, 1}
13	{8}	24	{7, 1, 4}
14	{8, 7}	25	{7, 1, 4, 3}
15	{8, 7, 1}	26	{7, 1, 4, 3, 2}
16	{8, 7, 1, 4}	27	{7, 1, 4, 3, 2, 6}
17	{8, 7, 1, 2}	28	{7, 1, 2}
18	{8, 7, 1, 2, 3}	29	{7, 1, 2, 3}
19	{8, 7, 2}	30	{7, 3}
20	{8, 7, 2, 3}	31	{7, 3, 2}
21	{8, 7, 2, 3, 4}		

II. Proposed algorithm

Number of iterations	Solution	Number of iterations	Solution
1	{1, 5}	6	{1, 8, 2, 7}
2	{1, 5, 4, 8}	7	{1, 7}
3	{1, 5, 4, 7}	8	{4, 5}
4	{1, 8}	9	{4, 5, 3, 8}
5	{1, 8, 4, 7}		

References

1) Egon Balas; An additive Algorithm For Solving Linear Programs With Zero One Variables, JORSA (1955).