On a Semilinear Dispersive Equation

By

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The semilinear dispersive equation,

$$\partial^2 y/\partial t^2 = \partial p(\partial y/\partial x)/\partial x - \partial^4 y/\partial x^4$$

is investigated in the region $0 \leq x \leq 1$, $0 \leq t$, where the initial and boundary values are

$$\begin{cases} y(0, x) = y_0(x), \quad \partial y(0, x)/\partial t = y_1(x) \quad 0 \leq x \leq 1, \\ y(t, 0) = y(t, 1) = \partial^2 y(t, 0)/\partial x^2 = \partial^2 y(t, 1)/\partial x^2 = 0 \quad 0 \leq t. \end{cases}$$

This is an equation of a vibration of a nonlinear model string-beam. It is shown that there exists a unique smooth solution in the large ($t \geq 0$), and a certain finite difference scheme related to it is investigated.

1. Introduction

In this paper we consider the following semilinear dispersive equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x} p\left(\frac{\partial y}{\partial x}\right) - \frac{\partial^4 y}{\partial x^4} \quad 0 \leq x \leq 1, \quad 0 \leq t, \tag{1}$$

where the initial and bonudary values are

$$\begin{cases} y(0, x) = y_0(x), \quad \partial y(0, x)/\partial t = y_1(x) \quad & 0 \leq x \leq 1 \\ y(t, 0) = y(t, 1) = \partial^2 y(t, 0)/\partial x^2 = \partial^2 y(t, 1)/\partial x^2 = 0 \quad & 0 \leq t. \end{cases} \tag{2}$$

The equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x} p\left(\frac{\partial y}{\partial x}\right) \tag{3}$$

is assumed to be a quasilinear hyperbolic partial differential equation in the region $y_x > \alpha = \text{const}$. Some of the examples are provided by taking

(4) $p = y_x + (y_x)^3$ cf. 1)

(5) $p = y_x / \{1 + (y_x)^2\}^{\frac{1}{2}}$

(6) $p = y_x + (y_x)^2$,

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where subscripts denote differentiations with respect to the variables. The case (5) corresponds to a force given by a nonlinear (i.e., before linearization) potential of a string. This quasilinear hyperbolic partial differential equation has, in general, not been solved concerning the global existence theorem. In 2) the equation of parabolic type

(7) $$y_{tt}=p(y_x)_x+y_{xtx}$$

was investigated concerning the existence, uniqueness and stability of the solution. The term y_{xtx} may be considered as an added dissipation of equation (3). In equation (1) the term y_{xxxx} may be considered as an added dispersion to equation (3), which is compared with the equation considered in 3). The equation (1) can be treated as a nonlinear perturbation to the linear equation:

(8) $$y_{tt}+y_{xxxx}=0,$$

that appears in a vibration of a beam.

It will be shown that the problem (1), (2) has a unique smooth solution in the large for every initial datum (the function space is mentioned later) for cases such as (4) and (5), but in cases such as (6) we get an analogous result, if we restrict the initial datum. For these purposes we shall make use of a theory of some nonlinear perturbation to a linear evolution equation (4–6) and so on), and Sobolev's lemma, for the latter case such as (6) we also use an idea in 7). In § 3 a finite difference scheme is mentioned concerning a solution of the problem (1).

The paper 10) treats an analogous problem by a method of approximation to equation (1) by a sequence of systems of ordinary differential equations, where the equation is assumed to have the property that the potential energy has its absolute minimum such as (4), and the initial condition is given periodically.

2. Existence and Uniqueness

Let us consider a global existence of a classical solution of the problem (1). A general theory of a global existence of a weak solution for semilinear second order evolution equations, which may be treated as a nonlinear perturbation of a linear evolution equation, gives the following. (for example Segal 6)).

B is a selfadjoint operator in a Hilbert space H, J is a locally lipschitzian transformation from the domain D_B of the operator B to H. (metrized relative to the inner product $(x, y)_{D_B}=(Bx, By)_H$), where we assume D_B is complete with respect to the norm $\|\cdot\|_{D_B}=(\cdot,\cdot)_{D_B}^{\frac{1}{2}}$.

Theorem

If $y_0\in D_B$, $y_1\in H$, then there exists $\varepsilon>0$ such that for $0\leq t\leq\varepsilon$ the equation

(9) $$y(t)=\cos Bty_0+\frac{\sin Bt}{B}y_1+\int_0^t \frac{\sin B(t-s)}{B}J(y(s))ds$$

has a unique continuous solution in D_B. If in addition there exists a nonpositive differentiable functional E on D_B (relative to $\|\cdot\|_{D_B}$) such that $\partial E/\partial y=R_e(J(y),\cdot)$, then the foregoing solution exists globally.

This gives at first a local existence theorem of a weak solution for the problem (1), that is, if we take the operator $B=-d^2/dx^2$ in $H=L^2(0,1)$ with the definition domain $D_B=\{y\in W_2{}^2,\ y(0)=y(1)=0\}\equiv\tilde{W}_2{}^2$, where $W_2{}^l$ is the Sobolev's space of the functions which have square summable l-th order derivatives in (0,1), then the operator B is a selfadjoint operator and the transformation $J(y)$: $y\to p(y_x)_x$ is locally lipschitzian from D_B into H, because

$$\begin{aligned}\|J(y^1)-J(y^2)\|_H&=\|p'(y^1{}_x)y^1{}_{xx}-p'(y^2{}_x)y^2{}_{xx}\|\leq\\ &\leq\max|p'(y^1{}_x)-p'(y^2{}_x)|\cdot\|y^1{}_{xx}\|+\max|p'(y^2{}_x)|\cdot\|y^1{}_{xx}-y^2{}_{xx}\|\leq\\ &\leq C(\|y^1\|_2,\ \|y^2\|_2)\cdot\|y^1-y^2\|_2\\ &=C(\|y^1\|_{D_B},\ \|y^2\|_{D_B})\cdot\|y^1-y^2\|_{D_B},\end{aligned}$$

where we used Sobolev's lemma, i.e., if $y\in\tilde{W}_2{}^2$, then y is bounded continuous with its first derivative and also

(10) $$\begin{gathered}\|y\|_2=\|By\|+|y(0)|+|y(1)|=\|By\|=\|y\|_{D_B}\\ (\|y\|_2\equiv\|y\|_{\tilde{W}_2{}^2},\ \|y\|\equiv\|y\|_{L^2})\end{gathered}$$

In order to obtain global existence of the weak solution of (1) for all data (2) we restrict the function $p(\cdot)$ such that the potential energy

(11) $$P_E=\frac{1}{2}\int_0^1[(y_{xx})^2+2P(y_x)]dx$$

has the absolute minimum at a point $v_0\in\tilde{W}_2{}^2$, where $P(\cdot)$ is an integral of $p(\cdot)$.

Corollary

If the nonlinear term $p(\cdot)$ has the property mentioned above and $y_0\in\tilde{W}_2{}^2$, $y_1\in L^2$, then we have a unique weak solution of the problem (1), which is $y(t)\in\xi_t{}^0(\tilde{W}_2{}^2)\cap\xi_t{}^1(L^2)$ in the large for all t, where $y(t)\in\xi_t{}^k(U)$ denotes the vector valued function such that $y(t)$ is k-times continuously differentiable in a vector space U with respect to t.

Now we proceed to the problem of smoothness of the weak solution obtained above. If we take the space $H=\tilde{W}_2{}^2(\|y\|_H=\|By\|)$ in the theorem and the definition domain of the operator $B=-d^2/dx^2$ as $D_B=\{y\in\tilde{W}_2{}^2,\ By\in\tilde{W}_2{}^2\}\equiv\tilde{W}_2{}^4$, $(\|y\|_4(\equiv\|y\|_{\tilde{W}_2{}^4})=\|B^2y\|+|By(0)|+|By(1)|+|y(0)|+|y(1)|=\|B^2y\|=\|By\|_H=\|y\|_{D_B})$, then the nonlinear transformation $J(y)$ is also locally lipschitzian from D_B into H and so for $y_0\in\tilde{W}_2{}^4$, $y_1\in\tilde{W}_2{}^2$ the weak solution $y(t)$ belongs to $\xi_t{}^0(\tilde{W}_2{}^4)\cap\xi_t{}^1(\tilde{W}_2{}^2)$ locally. But in this case the equation (9) is twice continuously differentiable in the space L^2

with respect to t, because $J(y(t))$ is continuously differentiable in L^2:

$$\begin{aligned}\frac{dJ(y(t))}{dt} &= \lim_{\Delta t\to 0}\{p'(y_x(t+\Delta t))y_{xx}(t+\Delta t)-p'(y_x(t))y_{xx}(t)\}/\Delta t\\ &= \lim_{\Delta t\to 0}\Big\{\frac{p'(y_x(t+\Delta t)-p'(y_x(t))}{\Delta t}y_{xx}(t+\Delta t)+\\ &\quad +p'(y_x(t))\frac{y_{xx}(t+\Delta t)-y_{xx}(t)}{\Delta t}\Big\}\\ &= p''(y_x(t))y_{xx}(t)\dot{y}_x(t)+p'(y_x(t))\dot{y}_{xx}(t).\end{aligned}$$

Thus $y(t)$ belongs to $\xi_t^2(L^2)$ locally. Furthermore a global estimate may be obtained in the following way. The energy conservation of the weak solution,

$$\|y(t)\|_2^2+\|\dot{y}(t)\|^2+\int_0^1 2P(y_x(t))dx=\|y_0\|_2^2+\|y_1\|^2+\int_0^1 2P(y_{0,x})dx$$

together with the equation

$$\dot{y}(t)=-B\sin Bty_0+\cos Bty_1+\int_0^t\cos B(t-s)J(y(s))ds$$

gives

$$\begin{aligned}\|B\dot{y}(t)\| &\le\|B^2y_0\|+\|By_1\|+\|B\int_0^t\cos B(t-s)J(y(s))ds\|\\ &\le\|y_0\|_4+\|y_1\|_2+\|\sin BtJ(y_0)\|+\|\int_0^t\sin B(t-s)\dot{J}(y(s))ds\|\\ &\le\|y_0\|_4+\|y_1\|_2+\|J(y_0)\|+\int_0^t(\|p''(y_x)y_{xx}\dot{y}_x\|+\|p'(y_x)\dot{y}_{xx}\|)ds\\ &\le C(y_0,y_1)+\int_0^t C(\|y\|_2)\|B\dot{y}\|ds,\end{aligned}$$

$$\|B\dot{y}(t)\|\le C(y_0,y_1)\exp\{C(y_0,y_1)t\},$$

that is $y(t)\in\xi_t^1(\tilde{W}_2^2)$ globally. Therefore

$$\begin{aligned}\|\ddot{y}(t)\| &\le\|B^2y_0\|+\|By_1\|+\|J(y(t))\|+\|\int_0^t B\sin B(t-s)J(y(s))ds\|\\ &\le C\exp(Ct),\end{aligned}$$

$$\|B^2y(t)\|\le\|\ddot{y}(t)\|+\|J(y(t))\|\le C\exp(Ct).$$

In order to obtain a smooth solution of the problem (1), we may take

$$\begin{aligned}&H=\tilde{W}_2^4\equiv\{y\in W_2^4,\ \ y(0)=y''(0)=y(1)=y''(1)=0\},\\ &D_B=\{y\in\tilde{W}_2^4,\ \ By\in\tilde{W}_2^4\}\equiv\tilde{W}_2^6,\ \ y_0\in D_B,\ \ y_1\in H\end{aligned}$$

in the above theorem, then we obtain a solution $y(t)\in\xi_t^0(\tilde{W}_2^6)\cap\xi_t^1(\tilde{W}_2^4)$ locally. In this case $J(y(t))\in\xi_t^1(\tilde{W}_2^2)$ because

$$\dot{J}(y(t))=p''(y_x(t))y_{xx}(t)\dot{y}_x(t)+p'(y_x(t))\dot{y}_{xx}(t)\in\xi_t^0(\tilde{W}_2^2)$$

Thus in the analogous way as before

$$||B^2\dot{y}(t)|| = || -B^3 \sin Bty + B^2 \cos Bty +$$
$$+ B \sin BtJ(y_0) + \int_0^t B \sin B(t-s)\frac{dJ(y(s))}{ds}ds||$$
$$\leq ||B^3y_0|| + ||B^2y_1|| + C||B^2y_0|| + \int_0^t ||B\dot{J}(y(s))||ds,$$
$$B\dot{J}(y(s)) = p''''(y_x)(y_{xx})^3\dot{y}_x + 3p'''(y_x)y_{xx}y_{xxx}\dot{y}_x +$$
$$+ 3p'''(y_x)(y_{xx})^2\dot{y}_{xx} + 3p''(y_x)y_{xx}\dot{y}_{xxx} +$$
$$+ 3p''(y_x)y_{xxx}\dot{y}_{xx} + p''(y_x)\dot{y}_x y_{xxxx} + p'(y_x)\dot{y}_{xxxx},$$
$$||B\dot{J}(y(s))|| \leq C(\max\{|y_x|, |y_{xx}|, |y_{xxx}|, |\dot{y}_x|\},$$
$$||\dot{y}_{xx}||, ||y_{xxxx}||)(1 + ||\dot{y}_{xxxx}||) = C(y_0, y_1) \cdot ||B^2\dot{y}(s)||,$$

where we used $y(t) \in \xi_t^0(\tilde{W}_2^4) \cap \xi_t^1(\tilde{W}_2^2)$ globally. Therefore

$$||B^2\dot{y}(t)|| \leq ||B^3y_0|| + ||B^2y_1|| + C||B^2y_0|| +$$
$$+ \int_0^t C(y_0, y_1)||B^2\dot{y}(s)||ds,$$
$$||B^2\dot{y}(t)|| \leq C(y_0, y_1)\exp C(y_0, y_1)t,$$

then

$$||B\dot{J}(y(t))|| \leq C||B^2\dot{y}(t)|| \leq C \exp Ct,$$
$$||B^3y(t)|| \leq ||B^3y_0|| + ||B^2y_1|| + ||B^2\int_0^t \sin B(t-s)J(y(s))ds||$$
$$\leq C + ||\int_0^t B\cos B(t-s)\dot{J}(y(s))ds|| \leq C\exp Ct,$$
$$||B\ddot{y}(t)|| \leq ||B^3y(t)|| + ||BJ||$$
$$\leq C\exp Ct + C||B^2y|| \leq C\exp Ct.$$

Theorem 1

If $y_0 \in \tilde{W}_2^6$, $y_1 \in \tilde{W}_2^4$ and the nonlinear term is such that the potential energy P_E has the alsolute minimum at some function $v_0 \in \tilde{W}_2^2$, then a solution of the problem (1) exists uniquely and belongs to

$$\xi_t^0(\tilde{W}_2^6) \cap \xi_t^1(\tilde{W}_2^4) \cap \xi_t^2(\tilde{W}_2^2)$$

in the large, which is also a smooth solution because of sobolev's lemma.

Now we consider the case in which the potential energy has no absolute minimum, but has a local minimum at a point $0 \in \tilde{W}_2^2$. Following Sattinger 7) we define depth of a potential well by

(12) $$d = \inf_{y \in \tilde{W}_2^2} P_E(\lambda_0 y),$$

where $\lambda_0 = \lambda_0(y)$ is the value such that the function $P_E(\lambda y)$ begins to decrease with respect to $\lambda \geq 0$ for fixed $y \in \tilde{W}_2^2$. We call the potential well W:

$$W = \{y \in \tilde{W}_2^2,\ \ P_E(\lambda y) < d,\ \ 0 \leq \lambda \leq 1\}.$$

Theorem 2

If $y^0 \in \tilde{W}_2^6$, $y_1 \in \tilde{W}_2^4$ *and*

$$K_E + P_E = \frac{1}{2}\int_0^1 [(y^1)^2 + (y^0_{xx})^2 + 2P(y^0_x)]\,dx < d,$$

then a solution of equation (1) exists uniquely and belongs to

$$\xi_t^0(\tilde{W}_2^6) \cap \xi_t^2(\tilde{W}_2^2) \quad \text{in the large.}$$

Proof.

A weak solution of equation (1) exists locally by the theorem. If we assume that the solution goes out of the potential well W, then there exists a t_0 such that $P_E(y(t_0)) = d$, which contradicts the energy conservation of the weak solution of equation (1), that is,

$$K_E + P_E \Big|_{t=t_0} = K_E + P_E \Big|_{t=0} < d.$$

The smoothness of the weak solution can be seen in the same way.

3. Finite Difference Scheme

Let $k = \Delta t$ and $h = \Delta x$ be the time and space mesh length, respectively. We use the notation:

$$y^m{}_n = y(m, n) = y(mk, nh)$$
$$y_t(m, n) = (y^{m+1}{}_n - y^m{}_n)/k,\; y_{\bar{t}}(m, n) = y_t(m-1, n)$$
$$y_x(m, n) = (y^m{}_{n+1} - y^m{}_n)/h, \quad y_{\bar{x}}(m, n) = y_x(m, n-1).$$

Hereafter for the sake of simplicity, we discuss case (4) only (the other cases (5) and (6) may be treated analogously):

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x} + \left(\frac{\partial y}{\partial x}\right)^3\right) \qquad 0 \le x \le 1,\; 0 \le t \le {}^{\forall}T, \tag{14}$$

where the initial and boundary values are the same as (2). We replace $\partial^2 y/\partial t^2$ by the ordinary difference scheme $y_{t\bar{t}}\ (m, n)$, but for other derivatives we use the follwoing implicit scheme:

$$\frac{\partial^4 y}{\partial x^4} \sim \frac{1}{2}\{y_{x\bar{x}x\bar{x}}(m+1, n) + y_{x\bar{x}x\bar{x}}(m-1, n)\}$$

$$\frac{\partial^2 y}{\partial x^2} \sim \frac{1}{2}\{y_{x\bar{x}}(m+1, n) + y_{x\bar{x}}(m-1, n)\}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)^3 \sim \frac{1}{4}\Big\{y_x(m+1, n)^3 + y_x(m+1, n)^2 y_x(m-1, n) + $$
$$+ y_x(m+1, n) y_x(m-1, n)^2 + y_x(m-1, n)^3\Big\}_{\bar{x}}\ .$$

By these replacements we obtain the finite difference scheme:

$$
(15)\quad \begin{aligned} &y_{t\bar{t}}(m, n)+\frac{1}{2}\left\{y_{x\bar{x}x\bar{x}}(m+1, n)+y_{x\bar{x}x\bar{x}}(m-1, n)\right\}=\frac{1}{2}\left\{y_{x\bar{x}}(m+1, n)+y_{x\bar{x}}(m-1, n)\right\} \\ &+\frac{1}{4}\{y_x(m+1,n)^3+y_x(m+1,n)^2y_x(m-1,n)+y_x(m+1,n)y_x(m-1,n)^2+y_x(m-1,n)^3\}_{\bar{x}}\,. \end{aligned}
$$

In order to obtain an energy estimate we multiply (15) by $\{y_t(m, n)+y_{\bar{t}}(m, n)\}kh$ and sum up over all m, n such that $0\le m\le T/K=M,\ 0\le n\le 1/h=N.$
Summing up by parts and using the boundary condition $y(m, 0)=y_{x\bar{x}}(m, 0)=y(m, N)=y_{x\bar{x}}(m, N)=0$, we get

$$
(16)\quad \begin{aligned} &\sum_{n=0}^{N-1}[y_{\bar{t}}(M+1, n)^2+\frac{1}{2}\{y_{x\bar{x}}(M+1, n)^2+y_{x\bar{x}}(M, n)^2+ \\ &+\frac{1}{2}\{y_x(M+1, n)^2+y_x(M, n)^2\}+\frac{1}{4}\{y_x(M+1, n)^4+y_x(M, n)^4\}]h \\ &=\sum_{n=0}^{N-1}[y_t(0, n)^2+\frac{1}{2}\{y_{x\bar{x}}(1, n)^2+y_{x\bar{x}}(0, n)^2\}+ \\ &+\frac{1}{2}\{y_x(1, n)^2+y_x(0, n)^2\}+\frac{1}{4}\{y_x(1, n)^4+y_x(0, n)^4]\}h. \end{aligned}
$$

This a priori estimate is an energy conservation and gives a solvability of the above implicit difference scheme by an interation for small k/h^2, using $|y^m{}_n|\le C$ (independent of k, h) by Sobolev's lemma for the finite difference. 8). Furtheremore from (16) it follows that $\{y^m{}_n\}_{0<h\le h_0}$ has the following properties:

$$
(17)\quad \begin{aligned} &\|y_t(m, \cdot)\|_{L^2}=\left\{\sum_{n=0}^{N-1}y_t(m, n)^2h\right\}^{\frac{1}{2}}\le C, \\ &\|y_x(m, \cdot)^2\|_{L^2},\ \|y_{x\bar{x}}(m, \cdot)\|_{L^2}\le C, \end{aligned}
$$

where C is a constant independent of $h>0,\ t\ge 0$.
By means of this estimate, Sobolev's lemma for the finite difference and the L^1-compactness argument in 9) the difference scheme (15) gives a weak solution of the problem (14) as a limit of an appropriate subsequence of (15) $(h\to 0)$. The limit (the obtained weak solution of (14)) has the estimate:

$$
(18)\quad \begin{cases} y \text{ is bounded, continuous in } G,\ y(t, 0)=y(t, 1)=0, \\ y(t, \cdot) \text{ is Lipschitz continuous in } L^2[0, 1] \text{ with respect to } t, \\ y_x(t, \cdot) \text{ is continuous in } L^2[0, 1] \text{ with respect to } t, \\ y_t(t, x)\in L^2(G),\ \ y_x\in L^\infty(G),\ \ y_{xx}\in L^2(G), \ \text{ where } G=[0, T]\times[0, 1]. \end{cases}
$$

Next we remark the finite difference scheme:

$$
(19)\quad \begin{aligned} &y_{t\bar{t}}(m, n)=\frac{1}{2}\{y_{x\bar{x}}(m+1, n)+y_{x\bar{x}}(m-1, n)\}+\frac{1}{4}\{y_x(m+1, n)^3+ \\ &+y_x(m+1, n)^2y_x(m-1, n)+y_x(m+1, n)y_x(m-1, n)^2+y_x(m-1, n)^3\}_{\bar{x}}\,. \end{aligned}
$$

It does not contain the term corresponding to y_{xxxx}, and therefore corresponds formally to the hyperbolic equation (3) with p given by (4). For the scheme (19) with the same boundary condition as (2), an energy conservation also holds, namely,

$$\begin{aligned}(20)\qquad &\sum_{n=0}^{N-1}[y_{\bar{t}}(M+1,n)^2+\frac{1}{2}\{y_x(M+1,n)^2+y_x(M,n)^2\}+\\ &+\frac{1}{4}\{y_x(M+1,n)^4+y_x(M,n)^4\}]h=\\ &=\sum_{n=0}^{N-1}[y_t(0,n)^2+\frac{1}{2}\{y_x(1,n)^2+y_x(0,n)^2\}+\frac{1}{4}\{y_x(1,n)^4+y_x(0,n)^4\}]h\end{aligned}$$

It follows from this energy conservation that the difference scheme (19) does not generally approximate the weak solution of the hyperbolic equation (3), because equation (3) has not, in general, the energy conservation of the form (20). The difference scheme (19) approximates rather the system of the ordinary differential equations which was considered by Fermi-Pasta-Ulam (cf. 3)).

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