Stress Distribution Around a Tunnel with an Arbitrary Cross Section Excavated in Anisotropic Elastic Ground

By

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The present paper is concerned with the theoretical determination of stresses and deformations around a tunnel with an arbitrary cross section excavated in anisotropic elastic ground under a three-dimensional stress state. In the first part general methods are proposed for determining these phenomena based on the use of comformal mappings, and in the second are given the numerical results of computation of some typical cross sections.

1. Introduction

In order to understand the earth pressure phenomena, it is of importance to analyze the stress distribution around underground tunnels taking into account the fact that the undisturbed ground is generally in a three-dimensional stress state. It is not easy to clarify the stress distribution under every conditions, since it is affected by the initial stress conditions of ground, shapes of tunnel, mechanical properties of rock and so on.

On the hypothesis that the rock is homogenious and isotropic elastic body, Hiramatsu and Oka[1], and Fairhurst[2] analyzed theoretically the stress distribution around a circular tunnel under a three-dimensional stress state. And further, Hiramatsu and Oka investigated experimentally the stresses around a tunnel with several shapes of cross section using a method of the photoelastic technique[1].

For the purpose of extending the above results, the authors have attempted to analyze theoretically the stress distribution around a tunnel with an arbitrary cross section under a three-dimensional stress state. In the present paper, we make the assumptions that

(a) the rock is homogenious and anisotropic elastic body,

(b) the stresses in the undisturbed ground do not vary along the generator of a tunnel, that is, the three-dimensional stress state is uniform over the

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wide region as compared with the diameter of a tunnel, and

(c) body force is absent.

Assumptions (b) is a reasonable one when a tunnel is excavated in considerable depth far from the ground surface. This assumption also has been adopted by the above investigators.

2. Statement of the Problem

The tunnel is considered to be an opening of arbitrary cross section with its axis coincided with the z-axis of a rectangular cartesian coordinate system (x, y, z). In this case, principal elastic axes of the body assumed that the surrounding material is homogenious and anisotropic elastic body, incline to arbitrary directions against this coordinate system.

The principal stresses $\sigma_1{}^0$, $\sigma_2{}^0$ and $\sigma_3{}^0$ in the undisturbed ground applied at infinity from arbitrary orthogonal directions inclined independently with not only the directions of principal elastic axes but also the axis of a tunnel, can be divided into six components of stress along the axial directions of coordinates (x, y, z) as shown in Fig. 1 (a). Then owing to the assumption that the ground is perfect elastic body, the stress distribution in Fig. 1 (a) can be divided into the stress distributions in Fig. 1 (b) and in Fig. 1 (c). The former is the case where the

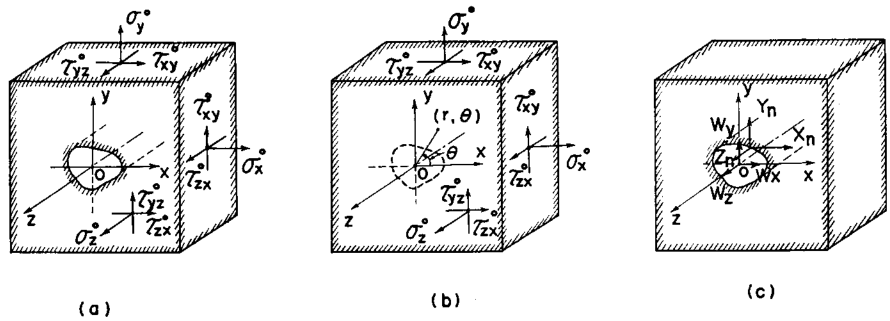


Fig. 1. Anisotropic elastic body with a tunnel under a three-dimensional stress state.

external stresses $\sigma_x{}^0$, $\sigma_y{}^0$, $\sigma_z{}^0$, $\tau_{xy}{}^0$, $\tau_{yz}{}^0$ and $\tau_{zx}{}^0$ equal to the ones in Fig. 1 (a) apply to the same body without the tunnel, and the latter is the case where the stresses X_n, Y_n, Z_n apply on the contour of the tunnel and the concentrated force W_0 acts at the origin of the coordinates (x, y, z). Where X_n, Y_n, Z_n are the stresses equal and opposite sign on the virtual contour in which a tunnel with an arbitrary cross section would be excavated under the stresses of the undisturbed ground, and W_0 is equal to the concentrated force corresponding to the weight of rock mass eliminated by excavation of the tunnel. The stresses X_n, Y_n,

Z_n and the force W_0 can be easily obtained. Therefore, it may be reduced to the problem to determine the stresses and displacements around the tunnel in anisotropic elastic ground, when the stresses X_n, Y_n, Z_n and the concentrated force W_0 apply at the contour of the tunnel.

3. Theoretical Foundations

3.1 Basic Equations for Anisotropic Elastic Body

From the assumptions (b) and (c) of chapter 1 and the coordinate system as shown in Fig.1, equilibrium equations of stress are expressed as*

$$\left.\begin{aligned} &\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0,\\ &\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0,\\ &\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0. \end{aligned}\right\} \tag{3.1}$$

For anisotropic elastic body, there exists generally the following relations between stresses and strains, i.e. the generalized Hooke's law.

$$\left.\begin{aligned} &\varepsilon_x=a_{11}\sigma_x+a_{12}\sigma_y+a_{13}\sigma_z+a_{14}\tau_{yz}+a_{15}\tau_{zx}+a_{16}\tau_{xy},\\ &\varepsilon_y=a_{12}\sigma_x+a_{22}\sigma_y+a_{23}\sigma_z+\cdots\cdots\cdots\cdots\cdots+a_{26}\tau_{xy},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\gamma_{xy}=a_{16}\sigma_x+a_{26}\sigma_y+a_{36}\sigma_z+\cdots\cdots\cdots\cdots+a_{66}\tau_{xy}, \end{aligned}\right\} \tag{3.2}$$

where the coefficients in above relations a_{11}, a_{12}, ... , a_{66} are the elastic constants of anisotropic body.

The equilibrium equations (3.1) may be satisfied by introducing to the two stress functions $F(x, y)$ and $\psi(x, y)$:

$$\left.\begin{aligned} &\sigma_x=\frac{\partial^2 F}{\partial y^2},\quad \sigma_y=\frac{\partial^2 F}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2 F}{\partial x\partial y},\\ &\tau_{zx}=\frac{\partial\psi}{\partial y},\quad \tau_{yz}=-\frac{\partial\psi}{\partial x} \end{aligned}\right\} \tag{3.3}$$

Having to satisfy the compatibility conditions of strains, we obtain the following systems of the basic differential equations for these stress functions,

$$L_4F+L_3\psi=0,\qquad L_3F+L_2\psi=0, \tag{3.4}$$

in which L_2, L_3 and L_4 are the differencial operators of the second, third and fourth orders which have the form:

* Theoretical aspects considered to body forces will appear in reference (10).

$$
\left.
\begin{aligned}
L_2 &= \beta_{44}\frac{\partial^2}{\partial x^2} - 2\beta_{45}\frac{\partial^2}{\partial x\partial y} + \beta_{55}\frac{\partial^2}{\partial y^2}, \\
L_3 &= -\beta_{24}\frac{\partial^3}{\partial x^3} + (\beta_{25}+\beta_{46})\frac{\partial^3}{\partial x^2\partial y} - (\beta_{14}+\beta_{56})\frac{\partial^3}{\partial x\partial y^2} + \beta_{51}\frac{\partial^3}{\partial y^3}, \\
L_4 &= \beta_{22}\frac{\partial^4}{\partial x^4} - 2\beta_{26}\frac{\partial^4}{\partial x^3\partial y} + (2\beta_{12}+\beta_{66})\frac{\partial^4}{\partial x^2\partial y^2} - 2\beta_{16}\frac{\partial^4}{\partial x\partial y^3} + \beta_{11}\frac{\partial^4}{\partial y^4},
\end{aligned}
\right\} \tag{3.5}
$$

$$
\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}}, \qquad (i,\ j=1,\ 2,\ 4,\ 5,\ 6). \tag{3.6}
$$

Therefore, by solving the equations (3.4), we can culculate the components of stress from equation (3.3), and the displacements from integrations of equation (3.2) and some reductions. Lekhnitskii has given the general expressions for the components of stress and displacement by terms of three analytic functions as follows[3].

$$
\left.
\begin{aligned}
\sigma_x &= 2Re[\mu_1^2\phi_1'(z_1) + \mu_2^2\phi_2'(z_2) + \mu_3^2\lambda_3\phi_3'(z_3)], \\
\sigma_y &= 2Re[\phi_1'(z_1) + \phi_2'(z_2) + \lambda_3\phi_3'(z_3)], \\
\tau_{xy} &= -2Re[\mu_1\phi_1'(z_1) + \mu_2\phi_2'(z_2) + \mu_3\lambda_3\phi_3'(z_3)], \\
\tau_{zx} &= 2Re[\mu_1\lambda_1\phi_1'(z_1) + \mu_2\lambda_2\phi_2'(z_2) + \mu_3\phi_3'(z_3)], \\
\tau_{yz} &= -2Re[\lambda_1\phi_1'(z_1) + \lambda_2\phi_2'(z_2) + \phi_3'(z_3)],
\end{aligned}
\right\} \tag{3.7}
$$

$$
\left.
\begin{aligned}
u &= 2\,Re\sum_{k=1}^{3} p_k\phi_k(z_k) - \omega_3 y + u_0, \\
v &= 2\,Re\sum_{k=1}^{3} q_k\phi_k(z_k) + \omega_3 x + v_0, \\
w &= 2Re\sum_{k=1}^{3} r_k\phi_k(z_k) + w_0.
\end{aligned}
\right\} \tag{3.8}
$$

Where Re is the notation for the real part of the complex expression, $\phi_k(z_k)$ are analytic functions with argument of the complex variables $z_k = x + \mu_k y$ $(k=1, 2, 3)$, and μ_k, λ_k, p_k, q_k, r_k $(k=1, 2, 3)$ are complex constants related to the roots of the characteristic equation of anisotropic elastic body and the elastic constants β_{ij}. The real constants ω_3, u_0, v_0, w_0 characterize rotation and rigid displacements of the body which are not accompanied by deformation. Therefore these can be neglected in the case of our problem.

Moreover, normal stress σ_z along the axial direction of a tunnel can be calculated from the assumption of plane strain of the z-direction as follow.

$$
\sigma_z = -\frac{1}{a_{33}}(a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\tau_{yz} + a_{35}\tau_{zx} + a_{36}\tau_{xy}). \tag{3.9}
$$

3.2 Conformal Mappings and Complex Analytic Functions $\phi_k(z_k)$

We assume that the external stresses X_n, Y_n, Z_n on the contour of a tunnel with an arbitrary cross section are defined as Fig. 2. Then the boundary conditions on the contour of this cross section are given as follows.

$$\left.\begin{aligned}\sigma_x \cos(n, x)+\tau_{xy}\cos(n, y)=X_n,\\ \tau_{xy}\cos(n, x)+\sigma_y \cos(n, y)=Y_n,\\ \tau_{zx}\cos(n, x)+\tau_{yz}\cos(n, y)=Z_n,\end{aligned}\right\} \tag{3.10}$$

in which n is a unit vector directed to the inward normal.

Using a tangential unit vector s in a clockwise direction on the contour, we have

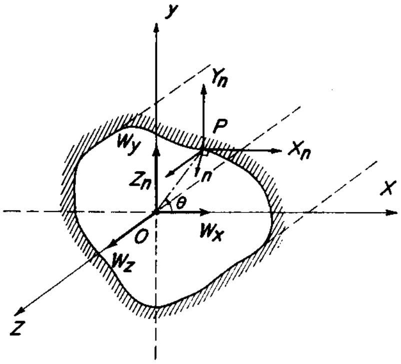


Fig. 2. Cross section of the tunnel with an arbitrary shape.

$$\cos(n, x)=\frac{dx}{dn}\doteq-\frac{dy}{ds}, \qquad \cos(n, y)=\frac{dy}{dn}=\frac{dx}{ds}. \tag{3.11}$$

By substituting equations (3.7), (3.11) in equation (3.10) and integrating with respect to the arc-length s from an arbitrary initial point to the variable point s, we can write these conditions in the following way.

$$\left.\begin{aligned}2Re[\phi_1(z_1)+\phi_2(z_2)+\lambda_3\phi_3(z_3)] &= \int_0^s Y_n ds+C_1,\\ 2Re[\mu_1\phi_1(z_1)+\mu_2\phi_2(z_2)+\mu_3\lambda_3\phi_3(z_3)] &= -\int_0^s X_n ds+C_2,\\ 2Re[\lambda_1\phi_1(z_1)+\lambda_2\phi_2(z_2)+\phi_3(z_3)] &= -\int_0^s Z_n ds+C_3.\end{aligned}\right\} \tag{3.12}$$

Where C_1, C_2, C_3 are constants which can be fixed arbitrarily on the contour which bounds the region; particularly, without loss of generality with respect to our problem we can set these constants equal to zero.

We refer the body under consideration to a coordinate system (x, y, z) where the origin lies at the center of gravity of an arbitrary cross section of a tunnel, and the z-axis is directed along the axis of the tunnel. We consider an infinite anisotropic elastic body with the tunnel, the contour of which is given by equations:

$$\left.\begin{aligned}x_0&=\alpha_0\cos\theta+\sum_{m=1}^{\nu}(\alpha_m\cos m\theta+\beta_m\sin m\theta),\\ y_0&=\alpha_0\sin\theta-\sum_{m=1}^{\nu}(\alpha_m\sin m\theta-\beta_m\cos m\theta).\end{aligned}\right\} \tag{3.13}$$

In which θ is a parameter varing from 0 to 2π in a counter-clockwise direction on the contour, α_m, β_m $(m=1, 2, \ldots\ldots, \nu)$ represent the real constants to be decided by the cross section of a tunnel, and ν is a finite integer with plus sign. For example, it may be set $\alpha_m=\beta_m=0$, $(m=1, 2, \ldots\ldots, \nu)$ for a circular opening and $\beta_1=\alpha_m=\beta_m=0$, $(m=2, 3, \ldots\ldots, \nu)$ for an elliptical one. Moreover, assuming that $\nu=9$, $\beta_m=0$, $(m=1, 2, 3, \ldots, 9)$, Heller and others[4] have calculated the values of the constants α_0, α_m together with the variations of radius of the rounded corner of an opening with several rectangular cross sections.

An infinite z_0-plane $(z_0=x+iy)$ with the tunnel as given by equation (3.13) is conformally mapped onto the exterior of a unit circle $|\zeta|=1$ in the ζ-plane. The mapping function is

$$z_0=\omega(\zeta)=\alpha_0\zeta+\sum_{m=1}^{\nu}(\alpha_m+i\beta_m)\zeta^{-m}. \tag{3.14}$$

Similarily, we consider the solution of the problem by mapping conformally the planes z_k $(=x+\mu_k y,\ k=1, 2, 3)$ onto the exterior of unit circles $|\zeta_k|=1$ in the ζ_k-planes, using the complex roots μ_k of the characteristic equation for anisotropic elastic body under consideration. Then the mapping functions may be expressed to the form:

$$\left.\begin{aligned} z_k=\omega_k(\zeta_k)=\frac{1}{2}[\alpha_0\{(1-i\mu_k)\zeta_k+(1+i\mu_k)\bar{\zeta}_k\} \\ +\sum_{m=1}^{\nu}\{(\alpha_m+\mu_k\beta_m)+i(\mu_k\alpha_m-\beta_m)\}\,\bar{\zeta}_k{}^{-m} \\ +\sum_{m=1}^{\nu}\{(\alpha_m+\mu_k\beta_m)-i(\mu_k\alpha_m-\beta_m)\}\,\zeta_k{}^{-m}]. \end{aligned}\right\} \tag{3.15}$$

Each of these functions on the contour of the cross section of a tunnel takes a value equal to $\zeta=\zeta_k\equiv\sigma$ $(=e^{i\theta})$.

Performing the above transformations, the expressions for the complex analytic functions $\phi_k(z_k)$ may be sought. That is, the expressions on the right-hand sides of equation (3.12) as a result of the integration of equation (3.10) are calculated as the forms with trigonometric series for the convenience of the suc ceeding calculations. Then by solving the basic equations (3.4), complex analytic functions $\phi_k(z_k)$ and their first derivartives $\phi_k'(z_k)$ are finally obtained as*

$$\left.\begin{aligned} \phi_k(z_k)=\frac{1}{\Delta}\sum_{m=1}^{\nu}\Gamma_{km}\zeta_k{}^{-m}, \\ \phi_k'(z_k)=-\frac{1}{\Delta\cdot I_k}\cdot\sum_{m=1}^{\nu}m\Gamma_{km}\zeta_k{}^{-(m+1)}. \end{aligned}\right\} \tag{3.16}$$

* Complex analytic functions corresponding to the force W_0 are given in Appendix.

$$\left.\begin{aligned}
&\Gamma_{1m}=(\mu_2-\mu_3\lambda_3\lambda_2)\bar{a}_m+(\lambda_2\lambda_3-1)\bar{b}_m+\lambda_3(\mu_3-\mu_2)\bar{c}_m,\\
&\Gamma_{2m}=(\mu_3\lambda_1\lambda_3-\mu_1)\bar{a}_m+(1-\lambda_1\lambda_3)\bar{b}_m+\lambda_3(\mu_1-\mu_3)\bar{c}_m,\\
&\Gamma_{3m}=(\mu_1\lambda_2-\mu_2\lambda_1)\bar{a}_m+(\lambda_1-\lambda_2)\bar{b}_m+(\mu_2-\mu_1)\bar{c}_m,\\
&\Delta=\mu_2-\mu_1+\lambda_2\lambda_3(\mu_1-\mu_3)+\lambda_1\lambda_3(\mu_3-\mu_2),\\
&I_k=\frac{dz_k}{d\zeta_k},\qquad (k=1,2,3).
\end{aligned}\right\}\tag{3.17}$$

In which $\bar{a}_m$, $\bar{b}_m$, $\bar{c}_m$ are the complex constants determined by the stresses of the undisturbed ground and the shape of a tunnel. When the components of stress $\sigma_x{}^0$, $\sigma_y{}^0$, $\sigma_z{}^0$, $\tau_{xy}{}^0$, $\tau_{yz}{}^0$ and $\tau_{zx}{}^0$ along the axial directions of the coordinates (x, y, z) represent the stresses in the undisturbed ground applied to three-dimensional, these constants are determined by the following expressions.

$$\left.\begin{aligned}
&\bar{a}_1=\frac{1}{2}[\{\sigma_y{}^0(\alpha_0+\alpha_1)-\tau_{xy}{}^0\beta_1\}-i\{\tau_{xy}{}^0(\alpha_0-\alpha_1)-\sigma_y{}^0\beta_1\}],\\
&\bar{b}_1=\frac{1}{2}[-\{\tau_{xy}{}^0(\alpha_0+\alpha_1)-\sigma_x{}^0\beta_1\}+i\{\sigma_x{}^0(\alpha_0-\alpha_1)-\tau_{xy}{}^0\beta_1\}],\\
&\bar{c}_1=\frac{1}{2}[-\{\tau_{yz}{}^0(\alpha_0+\alpha_1)-\tau_{zx}{}^0\beta_1\}+i\{\tau_{zx}{}^0(\alpha_0-\alpha_1)-\tau_{yz}{}^0\beta_1\}],\\
&\bar{a}_m=\frac{1}{2}\{(\sigma_y{}^0\alpha_m-\tau_{xy}{}^0\beta_m)+i(\tau_{xy}{}^0\alpha_m+\sigma_y{}^0\beta_m)\},\\
&\bar{b}_m=\frac{1}{2}\{(\tau_{xy}{}^0\alpha_m-\sigma_x{}^0\beta_m)-i(\sigma_x{}^0\alpha_m+\tau_{xy}{}^0\beta_m)\},\\
&\bar{c}_m=\frac{1}{2}\{(\tau_{yz}{}^0\alpha_m-\tau_{zx}{}^0\beta_m)+i(\tau_{zx}{}^0\alpha_m+\tau_{yz}{}^0\beta_m)\}.\\
&\qquad (m=2,3,\ldots\ldots,\nu)
\end{aligned}\right\}\tag{3.18}$$

According to above expressions, we can find theoretically the components of stress and displacement along the axial directions of the coordinates (x, y, z). Hence, components of stress and displacement in a system of orthogonal curvilinear coordinates (α, θ, z) are obtained by the form[5]:

$$\left.\begin{aligned}
&\sigma_\alpha+\sigma_\theta=\sigma_x+\sigma_y,\\
&\sigma_\alpha-\sigma_\theta+2i\tau_{\alpha\theta}=\frac{\bar{\zeta}}{\zeta}\cdot\frac{d\bar{z}_0}{d\bar{\zeta}}\cdot\frac{d\zeta}{dz_0}(\sigma_x-\sigma_y+2i\tau_{xy}),\\
&u_\alpha+iu_\theta=\left[\frac{\bar{\zeta}}{\zeta}\cdot\frac{d\bar{z}_0}{d\bar{\zeta}}\cdot\frac{d\zeta}{dz_0}\right]^{\frac{1}{2}}(u+iv).
\end{aligned}\right\}\tag{3.19}$$

Where σ_α and u_α are normal stress and displacement along the normal to the contour of the tunnel, σ_θ and u_θ along the tangential to the contour, and $\tau_{\alpha\theta}$ is shearing stress. The symbol of upper bar represents the conjugate complex variable of the complex one without the symbol of upper bar. Components of stress and displacement on the right-hand sides of these equations can be obtained by equations (3.7) and (3.8).

Moreover, for shearing stress $\tau_{\theta z}$ acting along the out-of-plane and displacement w_z of the z-direction, we can determine as

$$\left.\begin{aligned} \tau_{\theta z} &= -\tau_{zx}\sin\theta' + \tau_{yz}\cos\theta', \\ w_z &= w\tan\theta', \\ \theta' &= -\tan^{-1}\left(\frac{dx}{dy}\right). \end{aligned}\right\} \tag{3.20}$$

4. Stress Analysis

4.1 Transformations of Elastic Constants

The elastic constants which are appeared in expressions of the generalzied Hooke's law (3.2) of an anisotropic body depend on the direction of the axes of the coordinate system. If the direction of the axes changes, then the elastic constants vary. Hence, we must consider the relations between the elastic constants expressed in one coordinate system and the corresponding constants in another arbitrary coordinate system.

Let the elastic constants for the system (x''', y''', z''') be known and consider the anisotropic elastic body being inclined to the following (see also Fig. 3) against the axis of the tunnel.

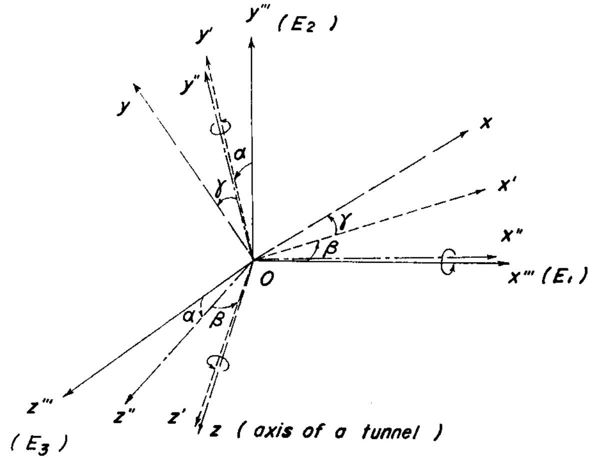


Fig. 3. Angles of rotation α, β and γ from the three principal elastic axes.

$$\left.\begin{aligned} &(1)\quad \alpha : \text{angle of rotation around the } x'''\text{-axis} \rightarrow \text{coordinates } (x'', y'', z'') \\ &(2)\quad \beta : \text{angle of rotation around the } y''\text{-axis} \rightarrow \text{coordinates } (x', y', z') \\ &(3)\quad \gamma : \text{angle of rotation around the } z'\text{-axis} \rightarrow \text{coordinates } (x, y, z) \end{aligned}\right\} \tag{4.1}$$

In this case, the position of the new coordinate system (x, y, z) with respect to the first one (x''', y''', z'''), is defined by the Table 1.

Table 1 Relations between the rectangular cartesian coordinates (x, y, z) and (x''', y''', z''').

	x	y	z
x'''	$\cos\beta\cos\gamma$	$-\cos\beta\sin\gamma$	$\sin\beta$
y'''	$\sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma$	$\cos\alpha\cos\gamma-\sin\alpha\sin\beta\sin\gamma$	$-\sin\alpha\cos\beta$
z'''	$-\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma$	$\sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma$	$\cos\alpha\cos\beta$

The rotations of coordinate axes being such that the angles of rotation α, β and γ take independently any values from 0 to 2π, we can easily obtain the elastic constants in the new system (x, y, z) for all inclinations of axes of elastic moduli[6]. Then by solving the basic equations of the anisotropic elastic body in this coordinate system, we can calculate the stresses and displacements around a tunnel with an arbitrary cross section under a three-dimensional stress state.

Because we do not have enough space to describe both stresses and displacements, we will introduce only a few typical examples being concerned with stresses*.

4.2 Influence Coefficients of Stress

Let us consider two systems of rectangular cartesian coordinates (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$, with the same origin O. In which the $\bar{x}$, $\bar{y}$, and $\bar{z}$-axes are taken in the directions of the principal stresses $\sigma_1{}^0$, $\sigma_2{}^0$ and $\sigma_3{}^0$ in the undisturbed ground. And let the direction cosines of x, y, z-axes be l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3, respectively. In this case, components of stress in reference coordinates (x, y, z) are given as

$$\left.\begin{aligned}
\sigma_x{}^0 &= l_1{}^2\sigma_1{}^0 + l_2{}^2\sigma_2{}^0 + l_3{}^2\sigma_3{}^0,\\
\sigma_y{}^0 &= m_1{}^2\sigma_1{}^0 + m_2{}^2\sigma_2{}^0 + m_3{}^2\sigma_3{}^0,\\
\sigma_z{}^0 &= n_1{}^2\sigma_1{}^0 + n_2{}^2\sigma_2{}^0 + n_3{}^2\sigma_3{}^0,\\
\tau_{xy}{}^0 &= l_1m_1\sigma_1{}^0 + l_2m_2\sigma_2{}^0 + l_3m_3\sigma_3{}^0,\\
\tau_{yz}{}^0 &= m_1n_1\sigma_1{}^0 + m_2n_2\sigma_2{}^0 + m_3n_3\sigma_3{}^0,\\
\tau_{zx}{}^0 &= n_1l_1\sigma_1{}^0 + n_2l_2\sigma_2{}^0 + n_3l_3\sigma_3{}^0.
\end{aligned}\right\} \tag{4.2}$$

And further, let the components of stress at any point P along the wall surface of the tunnel be defined by $\sigma_x{}^0$, $\sigma_y{}^0$, ... , $\tau_{zx}{}^0$ as follows.

$$\left.\begin{aligned}
\sigma_\theta &= A_x\sigma_x{}^0 + A_y\sigma_y{}^0 + A_z\sigma_z{}^0 + A_{xy}\tau_{xy}{}^0 + A_{yz}\tau_{yz}{}^0 + A_{zx}\tau_{zx}{}^0,\\
\sigma_z &= B_x\sigma_x{}^0 + B_y\sigma_y{}^0 + B_z\sigma_z{}^0 + B_{xy}\tau_{xy}{}^0 + B_{yz}\tau_{yz}{}^0 + B_{zx}\tau_{zx}{}^0,\\
\tau_{\theta z} &= C_x\sigma_x{}^0 + C_y\sigma_y{}^0 + C_z\sigma_z{}^0 + C_{xy}\tau_{xy}{}^0 + C_{yz}\tau_{yz}{}^0 + C_{zx}\tau_{zx}{}^0.
\end{aligned}\right\} \tag{4.3}$$

In which A_x, A_y, ... , C_{yz} and C_{zx} are constants to be determined by the shape of a tunnel, mechanical properties of ground under consideration and the position of a point P. For example, coefficients A_x, B_x and C_x represent respectively the values

* Numerical examples of the displacements around a tunnel with a circular cross section under a three-dimensional stress state can be seen in reference (7).

of stresses σ_θ, $\tau_{\theta z}$ and σ_z at the point P, i.e. the influence coefficients of stress, when the unit normal load $\sigma_x{}^0=1.0$ along the x-axis are applied at infinity, and so on. If the elastic constants of the ground are known in any way[8], these coefficients can be calculated by above described theoretical solution, utilizing high-speed digital computer.

If these influence coefficients of stress are obtained, then we can find the stresses in an undisturbed ground such as the manner similar to that of Hiramatsu and Oka[9].

We will show in next chapter the numerical examples of the influence coefficients of stress for a tunnel with several square cross sections*.

5. Numerical Examples

5.1 Calculations of the Influence Coefficients of Stress

For the sake of simlicity, let us concider the cross-anisotropic elastic ground such that the elastic moduli and Poisson's ratios are given respectively by $E_1=E_2$, $E_3=E_1/3$; $\nu_{23}=\nu_{31}=0.15$, $\nu_{12}=0.25$, and the moduli of rigidity are defined by the following formula:

$$\frac{1}{G_{ij}}=\frac{1}{E_i}+\frac{1}{E_j}+\frac{2\nu_{ij}}{E_i}, \qquad (i, j=1, 2, 3) \tag{5.1}$$

These assumptions do not lose in generality for our problem as an anisotropic elastic body. In this case, we will assume that the principal elastic axes of the body coincide firstly with the coordinate axes of the rectangular cartesian system (x''', y''', z''') before doing the rotations of coordinates.

5.1.1 Effects of a Corner Radius of Tunnel on the Influence Coefficients of Stress

Being related to the stress distribution around an opening, it has been well known that the stress concentration factors (i.e. the influence coefficients of stress) under an in-plane load change remarkably according to the variations of rounded corners $\rho=r_0/a$ (a: width, r_0: corner radius) of the opening. Here, we calculate the influence coefficients of stress under the in-plane loads $\sigma_x{}^0$, $\sigma_y{}^0$, $\tau_{xy}{}^0$ and the out-of-plane loads $\tau_{yz}{}^0$, $\tau_{zx}{}^0$ respectively, using the coefficients of mapping function of square openings given by Heller and others[4]. For the sake of simplicity, the ground will be treated as an isotropic elastic body (Poisson's ratio $\nu_0=0.25$) in this section. The influence coefficients of stress for square cross sections are shown in Fig. 4. In this figure, the magnitudes of the influence coefficients of stress are plotted on the

* Numerical results for a circular or a rectangular cross section ($a/h=2.0$, h : height of a tunnel) have been given in reference (7) or (11).

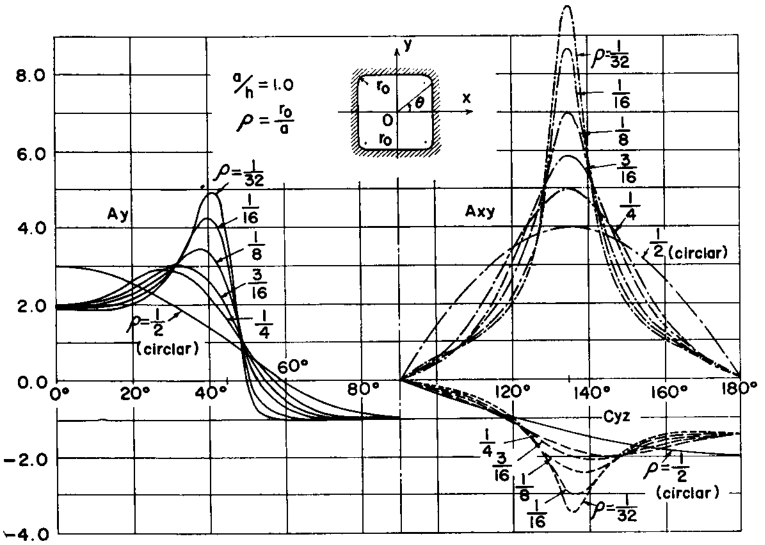


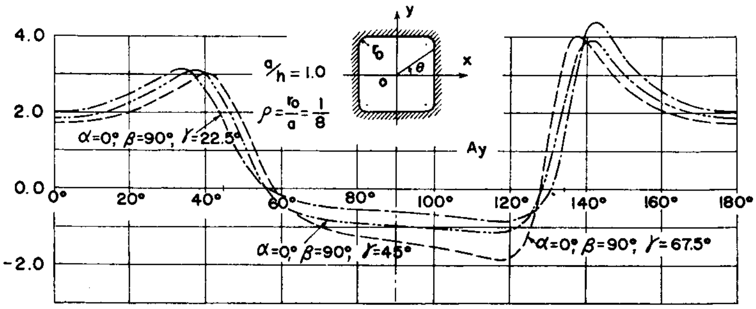
Fig. 4. Influence coefficients of stress around a tunnel having a square cross section with various rounded corners.

development of the wall surface of a tunnel. Although showing only the range of θ from 0 to $\pi/2$, coefficient A_y is in symmetry with respect to the line equal to $\pi/2$, and coefficients A_{xy}, C_{yz} are in antisymmetry with respect to this line. Coefficients A_x and C_{zx} for the square cross sections coincide with A_y and C_{yz} respectively shifted the angle $\pi/2$ from the point $\theta=0$.

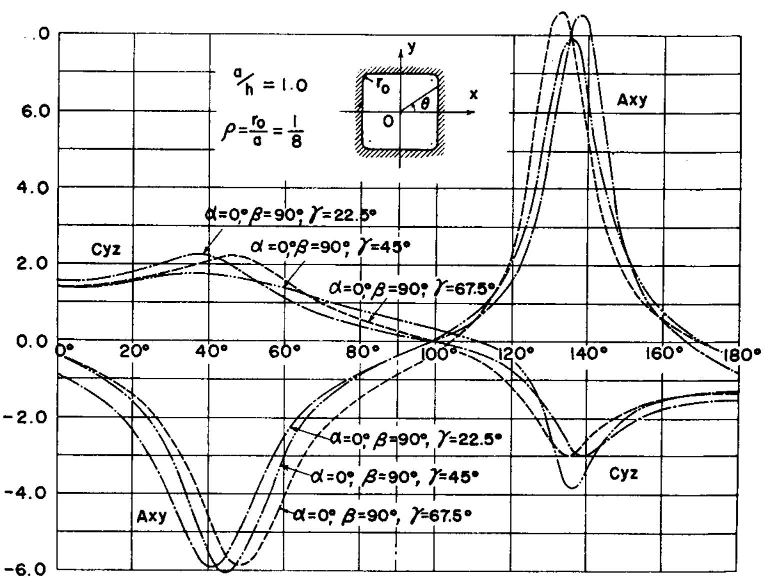
As is known in this figure, care must be taken that these coefficients are greatly affected by the radius of rounded corner of a tunnel.

5.1.2 Influence Coefficients of Stress in Anisotropic Elastic Ground

Under the assumption that the ground has the elastic constants to be given as in the beginning of this chapter, the influence coefficients of stress, in the case where the principal elastic axes of the body incline to in-plane or out-of-plane for the axis of the tunnel with a square cross section $\left(\text{corner radius } r_0=\frac{1}{8}a\right)$ are shown in Fig. 5~Fig. 7. (a), (b) of Fig. 5 show the influence coefficients of stress in which the axis of the principal elastic modulus E_1 (or E_2) rotates in-plane with respect to the z-axis, that is, in the case which corresponds to an orthotropic elastic plate. Similarly to the case of previous section, coefficients A_x and C_{zx} for this cross section coincide with A_y and C_{yz} respectively shifted the angle $\pi/2$ from the point $\theta=0$. (a), (b) of Fig. 6 show the influence coefficients of stress in which the left half of each figure is equal to the case when the principal axes of elastic moduli E_1 and E_2 rotate out-of-plane with respect to the y-axis, and the right half to the case where the principal axes of

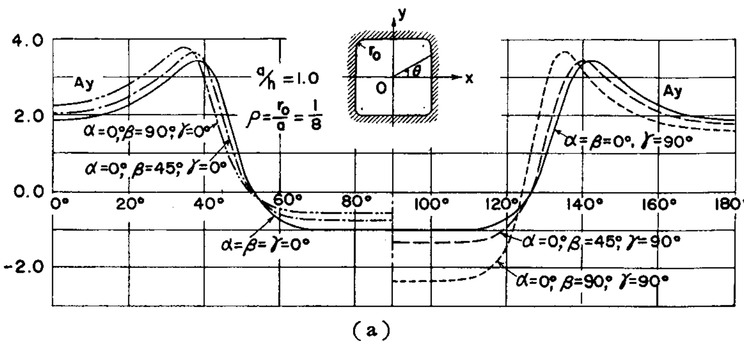


(a)

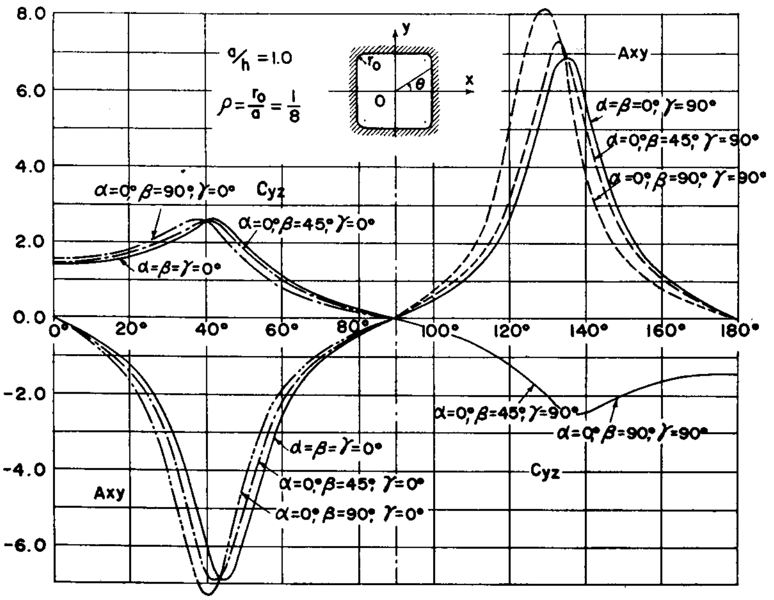


(b)

Fig. 5. Influence coefficients of stress around a tunnel having a square cross section with a radius of the corner $r_0 = \frac{1}{8}a$, when the axes of the principal elastic moduli E_2 and E_3 are rotated in-plane (i. e. around the z-axis).

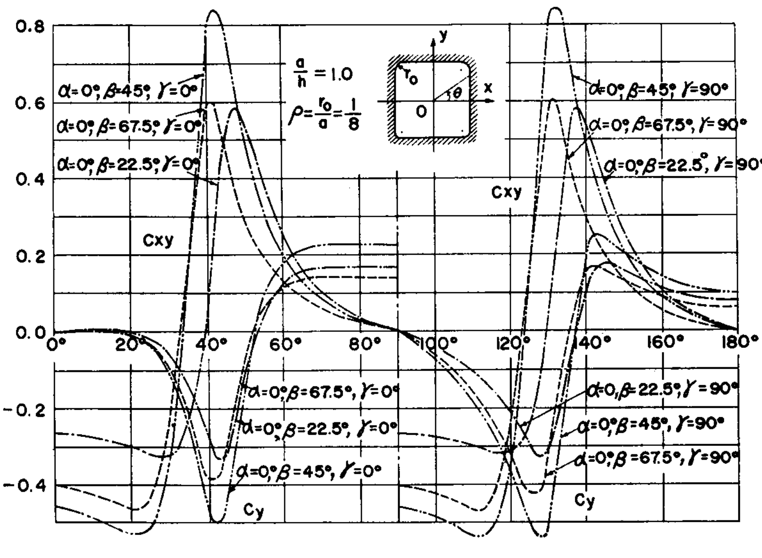


(a)

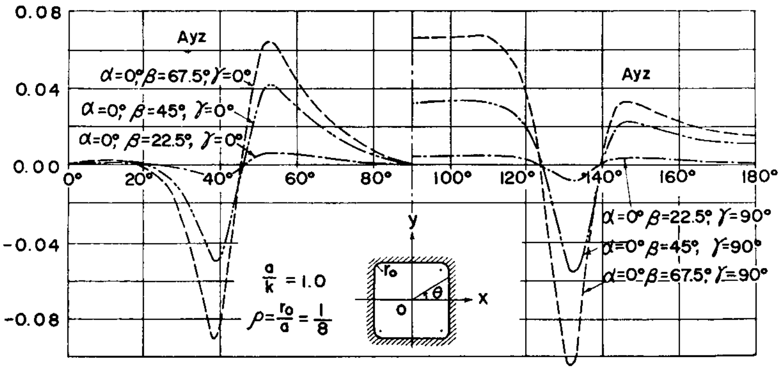


(b)

Fig. 6. Influence coefficients of stress around a tunnel having a square cross section with a radius of the corner $r_0 = \frac{1}{8}a$, when the axes of the principal elastic moduli E_1 and E_3 are rotated out-of-plane (i. e. around the x-and y-axes).



(a)



(b)

Fig. 7. Influence coefficients of stress C_y, C_{xy} and A_{yz} around a tunnel of a square cross section with a radius of the corner $r_0=\frac{1}{8}a$.

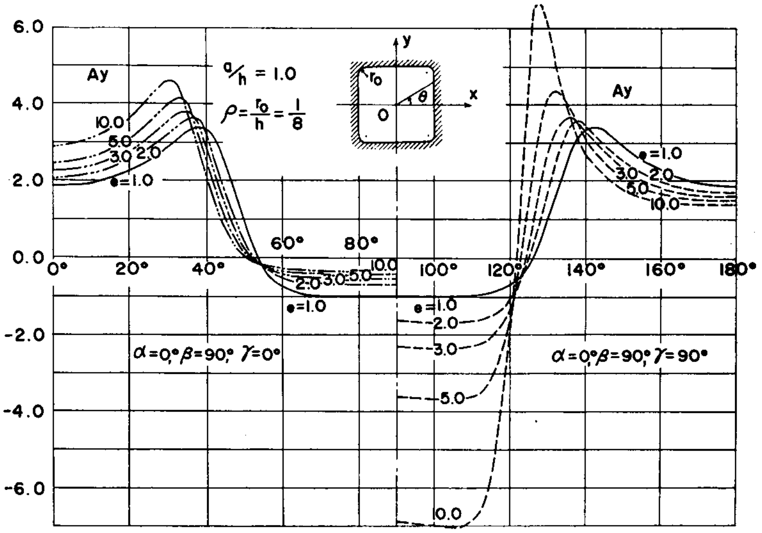
elastic moduli E_2 and E_3 rotate out-of-plane with respect to the x-axis.

Influence coefficients C_y, C_{xy} and A_{yz} in the case of square cross section shown in Fig. 7 (a), (b) may vanish when the ground under consideration is an isotropic elastic body, and the axes of the principal elastic moduli of the ground coincide with the axes of the coordinates (x, y, z) or rotate in-plane with respect to the z-axis. These coefficients as shown in Fig. 7 (a), (b) however take the finite values in the general case of anisotropic elastic body.

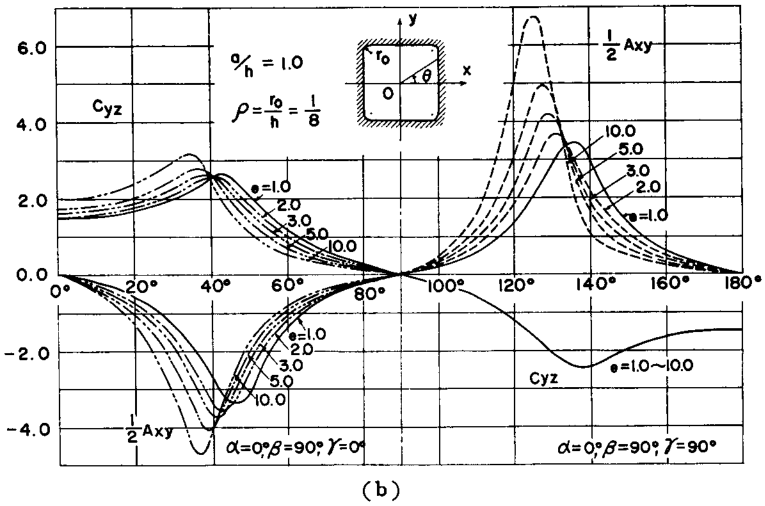
5.1.3 Effects of Ratio of Elastic Moduli on the Influence Coefficients of Stress

We treated several examples in the previous sections assuming that the ratio of elastic moduli e $(=E_1/E_3=E_2/E_3)$ is equal to 3.0. When the ratios of elastic moduli are equal to 1.0 (i.e. isotropic elastic body), 2.0, 3.0, 5.0 and 10.0, variations of the influence coefficients of stress are such as shown in Fig. 8 (a), (b). In this case, we assume that the moduli of rigidity are defined by formula (5.1) and Poisson's ratio are equal to the same values as the case with ratio $e=3.0$ as treated in previous section. It may be understood from these figures that the influence coefficients change remarkably along with the variations of the ratio e.

We did not treat in the above examples the normal stress σ_z along the z-direction, therefore the influence coefficients B_x, B_y, ..., B_{zy}. These coefficients however may be easily calculated by equation (3.9) from the known components of stress σ_x, σ_y, τ_{xy}, τ_{yz} and τ_{zx} around the tunnel.



(a)

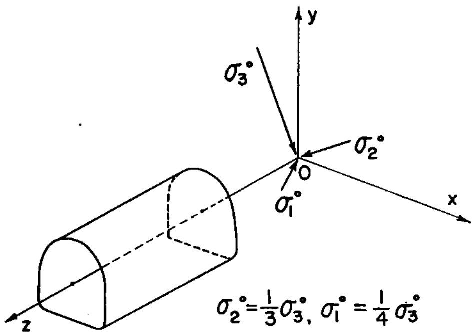


(b)

Fig. 8. Influence coefficients of stress around a tunnel having a square cross section with a radius of the corner $r_0=\frac{1}{8}a$, when the ratio of the principal elastic mudli $e=E_1/E_3$ $(=E_2/E_3)$ varies.

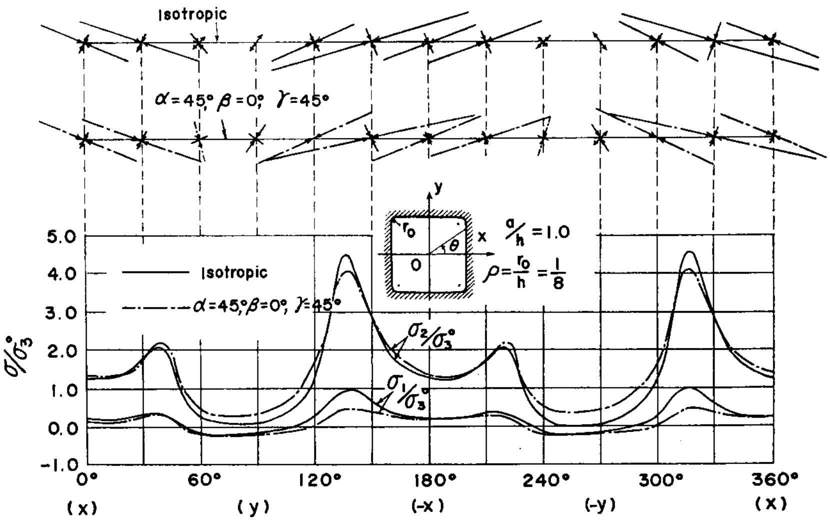
5.2 Stress Distribution around a Tunnel under a Three-Dimensional Stress State

By superposing suitably the results of the influence coefficients of stress as shown in previous sections, we can obtain the stress distribution around the tunnel with a square cross section under a three-dimensional stress state. Now assuming that the principal stresses σ_1^0, σ_2^0 and σ_3^0 in the undisturbed ground with prescribed elastic constants in section 5.1.1 are as shown in Fig.9, the distribution of principal stresses and their directions on the wall of the tunnel with a square or a typical cross section are illustrated in Fig.10 (a), (b). In these (a), (b) figures, the magnitudes and directions of principal stresses are plotted on the development of the wall surface. Solid lines in these figures indicate the distribution of principal stresses and their directions for the isotropic elastic body (Poisson's ratio $\nu_0=0.25$), and broken lines



	$\bar{x}$	$\bar{y}$	$\bar{z}$
x	64°14′	32°43′	71°15′
y	119°37′	94°20′	30°00′
z	41°09′	122°22′	67°27′

Fig. 9. An example of state of stress in the undisturbed ground (After Hiramatsu and Oka).



(a)

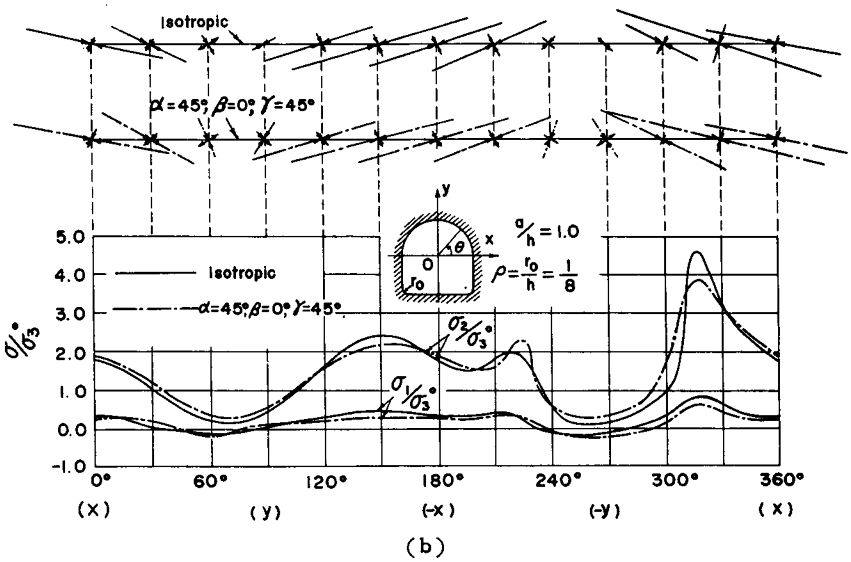


Fig. 10. The distributions of principal stresses and their directions around a tunnel with typical cross sections excavated in elastic ground (isotropic and cross-anisotropic ($\alpha=45°$, $\beta=0°$, $\gamma=45°$)) whose stress state corresponds to that of Fig. 9.

indicate for a typical example with $\alpha=45°$, $\beta=0°$, $\gamma=45°$ as in a case where the principal elastic axes of the cross-anisotropic body under consideration have general inclined angles. The shape of cross section as Fig. 10 (b) was obtained by equation (4.2) in which the integer ν set equal to 24.

6. Concluding Remarks

Three-dimensional stress analysis of ground with a tunnel has been carried out up to the present under the assumption that the ground is homogenious and isotropic elastic body. The present paper treats theoretically the stresses and displacements around a tunnel under a three-dimensional state of stress, assuming that the ground is homogenious and anisotropic elastic body.

By the adoption of the mapping function with coefficients defined by a finite Fourier expansions, theoretical stress analysis can be carried out for a tunnel with an arbitrary cross section as well as a circular or an elliptical one.

Acknowledgement

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Appendix

As a concentrated force W_0 equal to the weight of rock mass eliminated by excavation of a tunnel are applied at the origin O to the vertically upward direction, the corrected term for complex analytic functions $\phi_k(z_k)$ corresponding to the force W_0 must be added. Axial directions of the rectangular cartesian coordinate system (x, y, z) do not always coincide with a vertical direction. Thus, when the relations between the coordinate system (x, y, z) with the z-axis as the center line of a tunnel and the force W_0 are given as shown in Fig. A-1, components of the force along the axial directions of the coordinates can be obtained as follows.

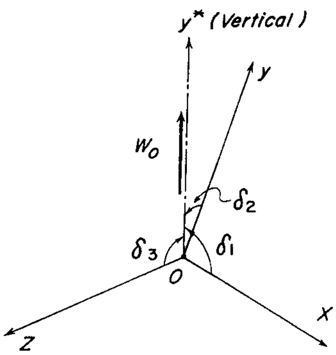

Fig. A-1.

$$\left.\begin{aligned} W_x &= W_0 \cos\delta_1, \\ W_y &= W_0 \cos\delta_2, \\ W_z &= W_0 \cos\delta_3. \end{aligned}\right\} \tag{A.1}$$

Where W_0 is equal to the weight of rock mass eliminated by excavation of a tunnel as given by

$$W_0 = -w_0 \int_c y_0 dx_0 = w_0 \pi \left\{ \alpha_0{}^2 - \sum_{j=1}^{\nu} j(\alpha_j{}^2 + \beta_j{}^2) \right\}, \tag{A.2}$$

and the relation among the angles δ_1, δ_2 and δ_3 is formed by

$$\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1. \tag{A.3}$$

In which w_0 is the weight of rock mass per unit volume.

In the case of such force, the complimentary analytic function $\phi_k{}^*(z_k)$ must be added to the function $\phi_k(z_k)$ as follows.

$$\phi_k{}^*(z_k) = \Gamma_k \ln\zeta_k, \qquad (k=1, 2, 3). \tag{A.4}$$

Where the complex coefficients Γ_k are determined by the simultaneous equations modified to the equations given by Lekhnitskii.

$$\left.\begin{aligned} \Gamma_1+\Gamma_2+\lambda_3\Gamma_3-\bar{\Gamma}_1-\bar{\Gamma}_2-\bar{\lambda}_3\bar{\Gamma}_3 &= \frac{W_y}{2\pi i}, \\ \mu_1\Gamma_1+\mu_2\Gamma_2+\mu_3\lambda_3\Gamma_3-\bar{\mu}_1\bar{\Gamma}_1-\bar{\mu}_2\bar{\Gamma}_2-\bar{\mu}_3\bar{\lambda}_3\bar{\Gamma}_3 &= -\frac{W_x}{2\pi i}, \\ \lambda_1\Gamma_1+\lambda_2\Gamma_2+\Gamma_3-\bar{\lambda}_1\bar{\Gamma}_1-\bar{\lambda}_2\bar{\Gamma}_2-\bar{\Gamma}_3 &= \frac{W_z}{2\pi i}, \\ p_1\Gamma_1+p_2\Gamma_2+p_3\Gamma_3-\bar{p}_1\bar{\Gamma}_1-\bar{p}_2\bar{\Gamma}_2-\bar{p}_3\bar{\Gamma}_3 &= 0, \\ q_1\Gamma_1+q_2\Gamma_2+q_3\Gamma_3-\bar{q}_1\bar{\Gamma}_1-\bar{q}_2\bar{\Gamma}_2-\bar{q}_3\bar{\Gamma}_3 &= 0, \\ r_1\Gamma_1+r_2\Gamma_2+r_3\Gamma_3-\bar{r}_1\bar{\Gamma}_1-\bar{r}_2\bar{\Gamma}_2-\bar{r}_3\bar{\Gamma}_3 &= 0, \end{aligned}\right\} \tag{A.5}$$

the values of Γ_1, Γ_2 and Γ_3 can be easily calculated by the above equations. First derivatives $\phi_k{}^{*\prime}(z_k)$ of the complimentary functions are given as

$$\phi_k{}^{*\prime}(z_k) = \frac{1}{I_k}\Gamma_k, \qquad (k=1, 2, 3). \tag{A.6}$$

From this, we can solve the problem such that the concentrated force W_0 acts at a point O in an infinite anisotropic elastic body. Thus the steresses and displacements due to the weight of rock mass eliminated by excavation of the tunnel are determined as the following manners.

(1) From above mentioned theory, calculate the stresses $\sigma_\alpha{}^*$, $\sigma_\theta{}^*$, $\tau_{\alpha\theta}{}^*$, $\tau_{\alpha z}{}^*$, $\tau_{\theta z}{}^*$ on the virtual contour in which the tunnel would be excavated in the infinite body.

(2) Calculate the stresses and displacements around the tunnel under when the stresses $-\sigma_\alpha{}^*$, $-\tau_{\alpha\theta}{}^*$, $-\tau_{\alpha z}{}^*$ apply on the contour of the tunnel. These can be obtained by the manner similar to this paper.

(3) Superpose the stresses and displacements obtained by (1) and (2).