Fracture Criteria for Anisotropic Rocks

By

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Most rocks of sedimentary and regional metamorphic origin are characterized by some degree of strength anisotropy. On a large scale, systems of rocks usually contain joints, bedding-plane partings or faults which impart considerable anisotropy to strength and other properties. In the present paper, the general propesties of fracture criteria of anisotropic rocks or systems of rocks are discussed. A General fracture criterion of them can be approximated in the load space or in the principal stress space by the minimum composite surface constructed from all the possible surfaces, each of which is determined as if a single plane of weakness or a crack were contained in the materials or in the systems of materials.

Several fracture criteria in two dimensions based on three types of model are critically reviewed. It is shown that they are best summarized in the Coulomb or the Mohr quadratic type criterion. Two types of fracture surface in the principal stress space are also discussed as a formal extensison of the Coulomb and Mohr type criteria.

These criteria, both the two dimensional and the three dimensional, are in good agreement with experimental results. These criteria, thus, can be used to predict fracture of anisotropic rocks or systems of rocks which exhibit planar anisotropy.

1. Introduction

Most rocks of sedimentary and regional metamorphic origin in the upper crust are characterized by some type of foliation and thus exhibit some degree of strength anisotropy. On a large scale, systems of rocks, even though compositional and other characteristics are sufficiently uniform to be considered reasonably homogeneous, usually contain joints, bedding-plane partings or faults. As a natural consequence of the effects of them systems of rocks can be expected to exhibit some degree of anisotropy in strength and deformational property.

Knowledge of deformation and fracture of rocks and systems of rocks are of basic importance to design the mining excavations or the foundations for civil engineering structures. An extensive research into the effects of anisotropy on rocks and systems of rocks is strongly desired.

The primary object of the present paper is to discuss the general properties of

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fracture criteria of anisotropic rocks or systems of rocks. In Sect. 2, concepts of the fracture initiation surface, the subsequent fracture surface and the final fracture surface are introduced and fundamental properties of them are discussed. In Sect. 3, several two-dimensional fracture criteria of anisotropic rocks and systems of rocks are critically reviewed. Formal extension of the two-dimensional fracture criteria into three-dimensions is made in Sect. 4, and the general fracture criterion of anisotropic rocks or systems of rocks is discussed. In Sect. 5, the proposed fracture criterion is compared with experimental results on anisotropic rocks and models of systems of rocks.

2. Fundamental Properties of Fracture Criteria

It is well recognized that fracture of rocks or of systems of rocks originates from inherent cracks and flaws contained in them due to stress or strain concentrations. Therefore, the problem of determining the stress and strain fields and configurations and distributions of cracks and flaws is of extreme importance. However, the configurations and the distributions of the inherent cracks and flaws in a given body or in a given system are not specified beforehand but must be determined, in general, through complete loading processes. The problem of analytical determination of the stress and strain fields of a given body or a system under a given system of loading is essentially nonlinear and extremely difficult to obtain the complete solution. Fortunately in most practical applications we do not need the complete solution of this kind of probelm, but it is only important to know whether the given body or the system under the given loading conditions has enough carrying capacity or not.

In general the crack and flaw may be orientated in any way relative to the directions of applied loads, so there will be no uniqueness in the shape of the end region of the crack and flaw at which the branching crack initiates. In order to determine the conditions for the initiation of the crack development for any body or system in which fracture initiates in a brittle or quasi-brittle manner, according to Barenblatt[1], there may be assumed the existence of a universal function expressed in terms of theoretical strengths

$$\Phi(N_\sigma,\ T_\sigma,\ S_\sigma,\ N_\varepsilon,\ T_\varepsilon,\ S_\varepsilon)$$

such that

$$\Phi(N_\sigma,\ T_\sigma,\ S_\sigma,\ N_\varepsilon,\ T_\varepsilon,\ S_\varepsilon)\leqq 0 \tag{2.1}$$

at all points on the contours of all cracks and flaws within the body or the system, where N_σ, T_σ, S_σ, N_ε, T_ε, and S_ε represent the characteristic strengths of the materials in tension, in plane shear and in longitudinal shear, respectively, expressed in terms of stress and strain. At points at which $\Phi=0$, the state of stress and strain is limiting in the sense that the attainment of this state at some point on the contour makes the crack or flaw move at that point and any increase in load which would have led to $\Phi>0$ in fact tends to crack development.

If a unique relation between the stress and strain is presumed, the function Φ can be expressed in terms of stress strengths only. Thus, the condition for fracture initiation is expressed

$$\Phi(N_\sigma, T_\sigma, S_\sigma) = 0 . \tag{2. 2}$$

In the most general form of fracture criterion other factors such as gradients and rates of stress and strain and temperature also may be taken into consideration, although they are of secondary importance.

It should be emphasized that Eq. (2.2) defines the conditions for the initiation of crack development at the contour of the inherent crack and flaw, but it does not say anything more. In some special applications Eq. (2.2) is very advantageous, since for any given situation we need only know the values of the stress intensity factors or stress concentration factors and need not undertake a complete analysis of the stress and strain fields. In general, however, as mentioned already, the configurations and distributions of the inherent crack and flaw in a given body or in a system are not specified beforehand, Eq. (2.2) is strongly limited in application.

In order to establish more general criterion for brittle or quasi-brittle fracture as is required in most practical applications, we had better formulate a criterion from the macroscopic point of view, that is to develope a criterion as a function of the applied loads, because the applied loads uniquely determine the stress and strain fields of a given body or of a given system containing the inherent cracks and flaws provided that the body or the system is stable in the sense of Drucker[2]).

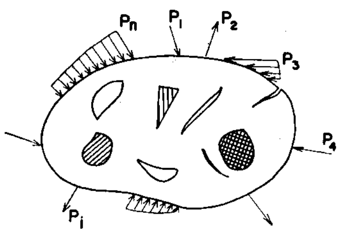

Fig. 1. A body containing systems of initial cracks and flaws subjected to a system of loading

Suppose that a body or a system containing a certain initial crack or flaw system is subjected to a system of loading P_i $(i=1, 2, \cdots, n)$ as shown in Fig. 1. As the loads increase along a certain loading path from zero, stresses and strains increase and continually redistributed on the contours of cracks and flaws and finally at one point, at least, on the contour the limiting condition Eq. (2.2) will be satisfied. At that moment, the condition in terms of applied loads may be expressed as

$$F_I(P_1, P_2, \cdots, P_n) = 0, \tag{2.3}$$

which represents a surface in the n-dimensional load space (P_1, P_2, $\cdots$, P_n) and is called the fracture initiation surface. If the initial state $P_i=0$ ($i=1, 2, \cdots, n$) is taken to correspond to the initial null stress and strain fields, the origin of the coordinates of the load space is contained in the surface. The fracture initiation surface has the following properties; for any state of load (or point) inside this surface and for any loading path wholly contained in this surface, no crack initiation takes place at any point on the contour of the inherent cracks and flaws and no crack develops in the body. For any state represented by a point on or outside of this surface, one crack, at least, initiates and/or develops. Thus, this condition essentially depends on the configurations and distributions of inherent cracks and flaws as well as the property of the external loads applied. In very brittle materials or highly heterogeneous materials such as concrete, the crack initiation is observed to take place at a very low stage of loading[3].

As the state of load transfer to any point outside of the fracture initiation surface, stresses and strains in the body or in the system are continually redistributed and more than one point successively becomes the limiting state corresponding to the condition Eq. (2.2) and different systems of crack initiate and/or develop. At this stage of loading, the subsequent fracture surface may exist. This surface, in general, depends not only on the current state of configurations and distributions of cracks and flaw and on the state of the current load system, but also on the complete history of the loadings.

As the loads increase along a certain loading path, a radial one for example, continually more and more cracks initiate and develop from the inherent cracks and flaws or from current cracks and the region of loss of stability gradually grows in size and in number and finally global instability—the final fracture—of the body or the system will be attained. To the state of loads at which the final fracture takes place a general function is defined

$$F_F(P_1, P_2, \cdots, P_n) = 0, \tag{2.4}$$

which represents a surface in the load space and is called the final fracture surface. This surface is characterized such that for all points inside this surface and for all loading paths wholly contained in this surface, an equilibrium state of the body or the system exists and no state of loading outside of this surface cannot be realized.

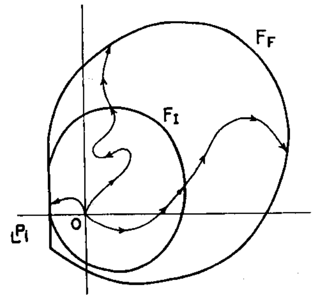

Fig. 2. Fracture surfaces in the load space.

Thus, the surface of final fracture is the most important characteristic of the body or the system for a given system of loading. Fracture surfaces discussed above are schematically shown in Fig. 2.

As a special case of fracture surfaces, though very important in practical applications, we consider fracture surfaces of rocks or systems of them subjected to a system of short-term static loading which produces the macroscopically homogeneous state of stress in the materials or in the system of materials. In this case fracture surfaces of the materials or the systems of materials can be expressed in terms of stress instead of the system of load. Thus, fracture surfaces are expressed as

$$f(\sigma_{ij};\ \theta) = 0 \tag{2.5}$$

in the six dimensional stress space σ_{ij} $(i, j=1, 2, 3)$ or

$$f(\sigma_1,\ \sigma_2,\ \sigma_3;\ \theta) = 0 \tag{2.6}$$

in the three dimensional principal stress space, where θ is a material descripter which characterizes the directional properties of the materials or the systems of materials. If the materials or the systems of materials are assumed isotropic, fracture surfaces become independent of the material descripter θ. The surfaces of fracture initiation, of subsequent fracture and of final fracture, respectively, are defined in a similar way as those defined in terms of loads. If fracture surfaces

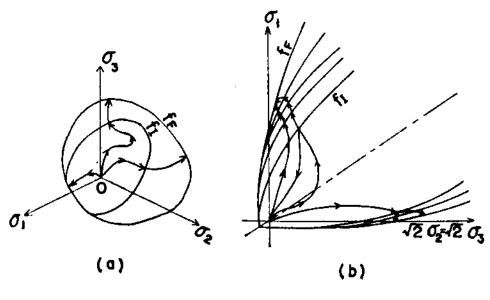

Fig. 3. Fracture susfaces in the stress space

are assumed to be independent of loading path, they are fixed in the stress space as shown schematically in Fig. 3.

Rocks or systems of rocks in general can be considered to have global stability in the sense of Drucker. Thus, all fracture surfaces such as ones for fracture initiation, subsequent fracture and final fracture are convex in the stress space, provided that the elastic or recoverable response of the materials or the systems of materials does not vary appreciably from the initial one even if permanent deformations may take place. In some, rock systems the recoverable responses are observed to vary notably from the initial one under cycles of loading, so the local concavity in fracture surfaces may be expected. In most cases, however, concave curvature may not be detectable since they are restricted to the reciprocal of Young's modulus.

3. Two-dimensional Fracture Criteria for Anisotropic Rocks and Systems of Rocks

On discussing the general fracture criterion of anisotropic rocks and systems of rocks, two-dimensional fracture criterion, especially in the state of plane-deformation, is most fundamental. In this section, several fundamental two-dimensional fracture criteria are formulated based on three types of basic mechanical model; a model containing systems of weak planes and a model containing systems of cracks in otherwise homogeneous and isotropic matrix, and a model containing a system of cracks in an orthotropic matrix.

3.1. Fracture Criteria Based on a Model Containing Systems of Weak Planes

As a simple mechanical model of a two-dimensional anisotropic rock or a system of rocks, Jaeger[4] proposed a "single plane of weakness" model. In extending his idea we consider a model containing n-systems of weak planes, each of which

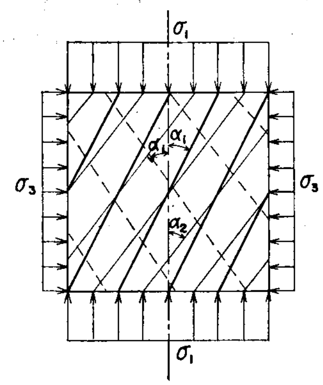

Fig. 4. A model containing systems of weak planes

consists of equally spaced parallel planes of weakness in an otherwise homogeneous and isotropic matrix as shown in Fig. 4.

Let the principal stresses be σ_1 and σ_3 ($\sigma_1 \geqq \sigma_3$, and compressive stresses are positive), and the orientation angle of the i-th system of weak planes measured from the axis of the major principal stress σ_1 be α_i ($|\alpha_i| \leqq 90°$).

The following assumptions are made throughout this subsection;

1) Interactions between the weak planes belonging to different systems are negligibly small and thus the characteristics of a model containing n-systems of weak planes can be approximated by those of a model containing n weak planes, each of which represents the global properties of a system of equally spaced parallel multiplanes of weakness.
2) Strengths of weak planes are far lower than the strength of the matrix and thus fracture of the model takes place exclusively in weak planes.
3) A system of loading applied increases proportionally, i.e., the radial loading paths exclusively traced.

From these assumptions it can be concluded that fracture criteria depend only on the final state of loadings and that fracture surfaces are fixed in the load or stress space.

3.1.1. Application of the Coulomb Criterion

In order to describe the mechanical behaviors of the model mentioned above, in addition to the assumptions (1), (2) and (3), an assumption is made that the fracture of each weak plane obeys the Coluomb criterion. Fracture of the i-th weak plane takes place when the normal stress σ and the shearing stress τ working on the plane satisfy a condition

$$\begin{aligned}\tau &= \sigma \tan \varphi_i + C_i \\ &= \sigma \mu_i + C_i \qquad (i=1, 2, \cdots, n),\end{aligned} \tag{3. 1}$$

where φ_i, $\mu_i = \tan \varphi_i$ and C_i are material constants of the i-th weak plane. The same criterion expressed in terms of principal stresses σ_1 and σ_3 and the orientation angle α_i is as follows

$$\sigma_1 = \frac{\sin 2\alpha_i + \mu_i(1+\cos 2\alpha_i)}{\sin 2\alpha_i - \mu_i(1-\cos 2\alpha_i)}\sigma_3 + \frac{2C_i}{\sin 2\alpha_i - \mu_i(1-\cos 2\alpha_i)}\,. \tag{3. 2}$$

This equation gives a criterion for fracture of the model containing a "single plane of weakness" (i-th plane). This criterion is shown in Fig. 5. The figure shows that the strength of the model depends not only on material constants and the confining pressure, but also on the orientation angle of the weak plane.

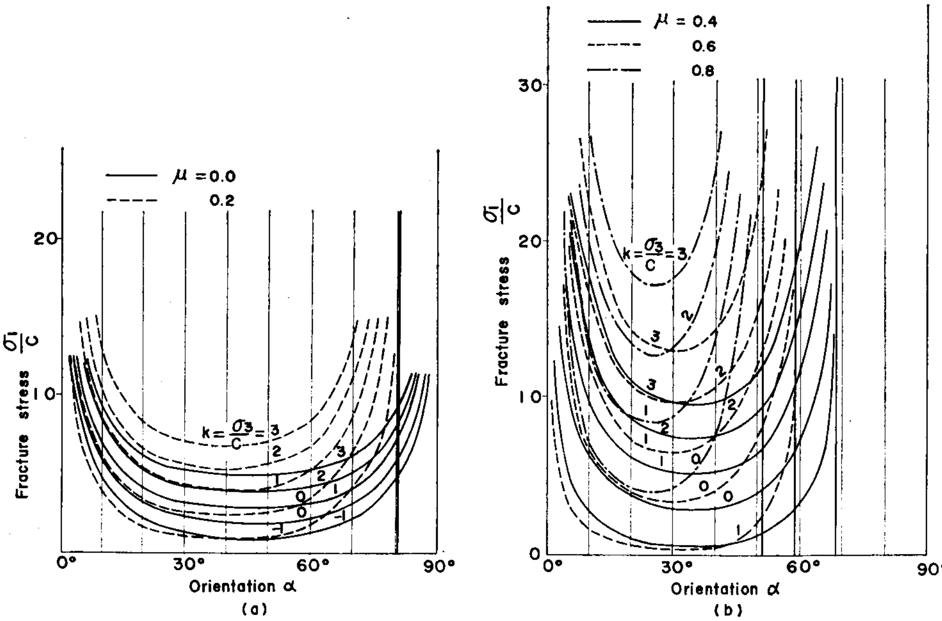

Fig. 5. A fracture criterion for a "single plane of weakness" model (based on the Coulomb criterion)

Since the model to be considered here contains n weak planes with different properties and orientation angles from the axis of the major principal stress, we must take all the effects of these weak planes into consideration. On the assumptions (1)~(3) the effect of each weak plane on the model can be analyzed independently as if it were the only weak plane existing in the model, so the strength of the entire model is determined by the weakest strength, each of which is independently determined for a model containing a single weak plane with a fixed orientation angle from the axis of the major principal stress. It is noted that the plane with weakest material constants does not necessarily determine the strength of the entire model.

A fracture criterion as a minimum combination criterion of the n indepentent fracture criteria is schematically shown in Fig. 6. The fracture criterion depends not only on the confining pressure and material constants, but also strongly on the orientation angle of the system of weak planes. The orientation angle which gives the minimum strength may vary with the confining pressure. In the real anisotropic rocks or in systems of rocks, some degree of interactions between weak planes does exist in the process of loading and at the moment of fracture. Thus, the criterion curve may be leveled out to some extent at the transition point from one fracture criterion curve corresponding to a certain weak plane system to another curve corresponding to another weak plane system, and sometimes strength may be lowered to some extent.

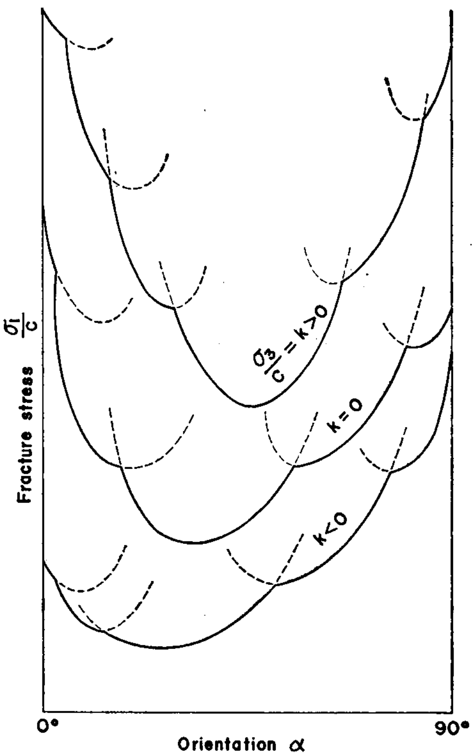

Fig. 6. A fracture criterion for a model containing systems of planes of weakness (based on the Coulomb Criterion)

When various degrees of weak planes with a random orientation are assumed to be contained in a material or in a system of materials, the strength of it does not depend on the orientation angle of weak planes and the criterion of an isotropic and homogeneous material or of the system having macroscopically isotropic properties will be obtained.

3.1.2. Application of the Mohr Quadratic Criterion

If an assumption is made that fracture of weak planes obeys the criterion of Mohr quadratic form, the fracture of the i-th weak plane takes place when the equation

$$\tau^2 = \lambda_i\sigma + D_i \qquad (i=1, 2, \cdots, n) \tag{3. 3}$$

is satisfied, where λ_i and D_i are material constants of the i-th weak plane. This equation is also expressed in terms of principal stresses and the orientation angle of the plane as follows

$$(\sigma_1-\sigma_3)^2\sin^2 2\alpha_i = 2\lambda_i\{\sigma_1(1-\cos 2\alpha_i)+\sigma_3(1+\cos 2\alpha_i)\}+4D_i \tag{3. 4}$$

or

$$\sigma_1 = \frac{1}{\sin^2 2\alpha_i}[\sigma_3\sin^2 2\alpha_i+\lambda_i(1-\cos 2\alpha_i) \pm \{\lambda_i^2(1-\cos 2\alpha_i)^2+4\sin^2 2\alpha_i(\lambda_i\sigma_3+D_i)\}^{1/2}] . \tag{3. 5}$$

This equation gives a criterion for fracture of the model containing a "single plane of weakness." The figure of this equation is similar to that in Fig. 9 to be shown in subsection 3.2.1.

For a model containing many systems of weak planes, for the same reasons as discussed in subsection 3.1.1., the curves of the fracture criterion of the entire material or the system become also similar as those shown in Fig. 6.

3.2. Fracture Criterion Based on A Model Containing Systems of Cracks

In most rocks which are characterized by some type of foliation, preferentially orientated slender cracks or flaws along the foliation plane may be reasonably assumed. In the systems of rocks, on a large scale, also the existence of preferentially orientated cracks, flaws, joint systems, bedding-plane partings or faults are assumed. As an idealized model of these rocks or the systems of rocks, we consider a model containing n-systems of cracks, each of which consists of a parallel

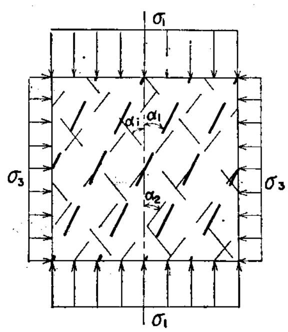

Fig. 7. A model containing systems of cracks

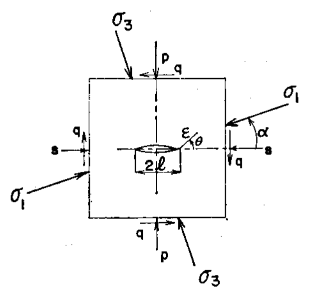

Fig. 8. Stresses acting upon a crack with orientation angle α from the axis of the major principal stress

set of cracks spaced in equal distance in an otherwise homogeneous and isotropic matrix as shown in Fig. 7.

In this subsection the similar assumptions as in subsection 3.1. are made;

1) No interaction between the different systems of cracks as well as in the own system of cracks is assumed.

2) A system of loading applied increases exclusively along radial loading paths. On these assumptions the effect of each crack can be analyzed as if it were the only crack existing in the model. The strength of the entire model can be determined by the same process as discussed in subsection 3.1.1. In what follows, we will discuss the applications of the Griffith theory, the modified Griffith theory and the shear fracture criterion.

3.2.1. Applications of the Griffith and Modified Griffith Theories.

Walsh and Brace[5], and Hoek[6] applied independently the Griffith and the modified Griffith theories in order to explain the anisotropic strength of rocks. They assumed that two types of microcracks exist in the otherwise homogeneous and isotropic rock such that one is very slender and solely alined in the bedding planes and the other is short and distributed in a random manner in the entire rock. In extending their idea, we apply the Griffith and the modified Griffith theories to the model containing *n*-systems of cracks.

Applying the Griffith theory to the i-th crack, the major axis of which is inclined α_i from the axis of the major principal stress σ_1 as shown in Fig. 7, a fracture initiation criterion in the neighborhood of the crack tip is expressed as[7]

$$\begin{aligned} -2K &= \frac{1}{2}\{(\sigma_1+\sigma_3)-(\sigma_1-\sigma_3)\cos 2\alpha_i\} \\ &\qquad \pm\left[\frac{1}{2}\{(\sigma_1^2+\sigma_3^2)-(\sigma_1^2-\sigma_3^2)\cos 2\alpha_i\}\right]^{1/2} \\ &= \frac{1}{2}\{\sigma_1(1-\cos 2\alpha_i)+\sigma_3(1+\cos 2\alpha_i)\} \\ &\qquad \pm\left[\frac{1}{2}\{\sigma_1^2(1-\cos 2\alpha_i)+\sigma_3^2(1+\cos 2\alpha_i)\}\right]^{1/2}, \end{aligned} \qquad (3.\,6)$$

where K is the tensile strength of the matrix. When the applied stresses are compressive, the crack may close and the above expression may not be used. In such a case the modified Griffith theory by McClintock and Walsh[8] may be applicable.

Thus, the fracture criterion is

$$2K = \frac{1}{2}[(\sigma_1-\sigma_3)\sin 2\alpha_i-\mu_i\{\sigma_1(1-\cos 2\alpha_i)+\sigma_3(1+\cos 2\alpha_i)\}], \quad (3.7)$$

where μ_i represents the coefficient of microfriction between the surfaces of the closed crack.

When pore pressure σ_p must be taken into account, we must replace σ_1 and σ_3 in Eq. (3.6) by $\sigma_1-\sigma_p$ and $\sigma_3-\sigma_p$, respectively.

The criterion Eq. (3.6) is shown in Fig. 9 in the $\sigma_1/K-\alpha$ plane with a parameter σ_3/σ_1. This criterion Eq. (3.6) is not different essentially from that of

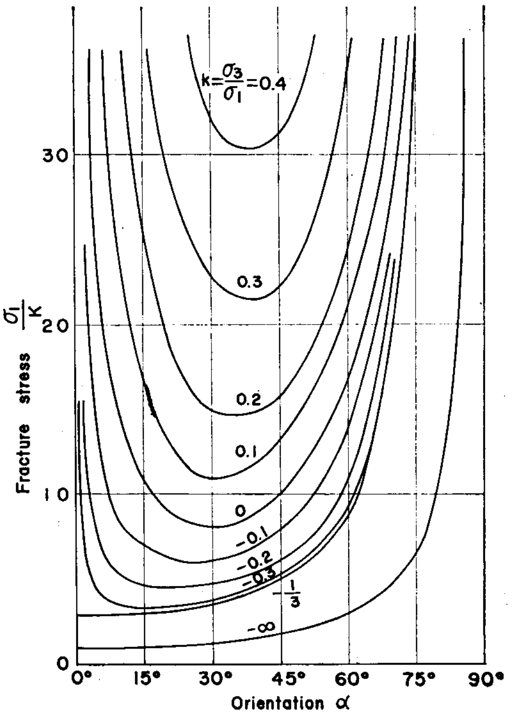

Fig. 9. Fracture initiation criterion from a single open Griffith crack (based on the Griffith theory)

Eq. (3.4). The criterion Eq. (3.7) is also similar to Eq. (3.2) and figures similar to Fig. 5 are obtained.

For a material or a system of materials containing systems of initial cracks or flaws, the fracture initiation criterion of the entire model is determined, as discussed in subsection 3.1.1., by the weakest one of the strengths, each of which is independently determined for a model containing a single crack with a fixed orientation angle from the axis of the major principal stress. The criterion curves of fracture initiation, therefore, become similar to those already shown in Fig. 6. This exposition may be easily approved by considering the equivalency of the Griffith and the modified Griffith theories to the Mohr and the Coulomb criteria, respectively. In fact, replacements of λ_i, D_i and C_i by $4K$, $4K^2$ and $2K$, respectively, in Eqs. (3.5) and (3.2) lead to Eqs. (3.6) and (3.7), respectively.

It must be noted here that these criteria based on the Griffith and the modified Griffith theories give the general conditions for fracture initiation, but not for final fracture. As is well known, if both principal stresses are compressive, branching cracks emenated from the neighborhood of the tip of the initial crack on flaw turn away from the direction in which they start and gradually become alined with the axis of the major compressive stress, and finally cease to propagate because of the decrease of the stress concentration at the tip of the branching crack.

In a material or in a system of materials containing many systems of initial cracks or flaws, branching cracks continually initiate at the limiting tips of initial or the propagating cracks as the applied loads increase. As already discussed in Sect. 2, this process causes the loss of local stability and finally leads to global instability—the final fracture—of the material or the system of materials. In other terms, the subsequent and the final fracture strengths depend not only on the configurations and the distributions of the initial cracks and the otientation angles of the initial cracks from the axis of the major principal stress, but also on subsequent developments of branching cracks and other complex processes of local instability. Thus, for the subsequent and the final fracture no explicit criterion can be expected.

3.2.2. Application of the Shear Fracture Criterion

In formulation of the Griffith and the modified Griffith theories the maximum normal stress fracture criterion was applied to the initial microcrack. To the same crack the shearing stress fracture criterion[9] also may be reasonably applied, since the crack lies in general in bedding-planes which are weaker than the rest of the rock or the system of rocks and since the crack development may be fully contained in these planes. We apply this idea to the model with systems of initial cracks, when both principal stresses are compressive. When both principal stresses are tensile or the tensile stress exceeds the compressive stress, the maximum normal stress fracture criterion is adopted.

Consider a single crack with length $2l$ inclined α_i from the axis of the major compressive stress σ_1 as shown in Fig. 8. Stresses at the tip of the crack expressed in Cartesian coordinates are as follows[10], provided that the crack does not close,

$$\left.\begin{aligned}\sigma_x &= \frac{1}{\sqrt{2\varepsilon}}\left\{k_1\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3}{2}\theta\right)-k_2\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\right\}\\ \sigma_y &= \frac{1}{\sqrt{2\varepsilon}}\left\{k_1\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3}{2}\theta\right)+k_2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\right\}\\ \tau_{xy} &= \frac{1}{\sqrt{2\varepsilon}}\left\{k_1\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}+k_2\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right\},\end{aligned}\right\} \quad (3.8)$$

where ε represents a distance measured from the tip of the crack and θ represents an angle measured anticlockwise from the x-axis as shown in Fig. 10, and coefficients k_1 and k_2 are defined as

$$\left.\begin{aligned}k_1 &= p\sqrt{l} = \frac{\sqrt{l}}{2}\{\sigma_1+\sigma_3-(\sigma_1-\sigma_3)\cos 2\alpha_i\}\\ k_2 &= q\sqrt{l} = -\frac{\sqrt{l}}{2}(\sigma_1-\sigma_3)\sin 2\alpha_i\end{aligned}\right\} \quad (3.9)$$

When stresses are compressive, crack development takes place at the tip of the initial crack by shearing stress. Thus, a criterion of fracture initiation is given by putting $\theta=0$ in Eq. (3.8)

$$(\sigma_1-\sigma_3)\sin 2\alpha_i = 2L, \quad (3.10)$$

where L is the shear strength of the matrix material.

When the applied stresses are tensile, crack initiates at the tip of the initial crack by tensile stress and a criterion is given as follows

$$\sigma_1+\sigma_3-(\sigma_1-\sigma_3)\cos 2\alpha_i = -2K, \quad (3.11)$$

where K is the tensile strength of the matrix material as given in Eq. (3.6).

When we take into consideration the effect of the closure of the crack under the compressive stresses, we must replace p and q in Fig. 8 by $p-\sigma_n$ or σ_c and $q-\sigma_f$, $q+\mu_i\sigma_n$ or $q+\mu_i(p-\sigma_c)$, respectively, by applying the same procedure as used by McClintock and Walsh in deriving the modified Griffith theory[11]. We obtain the following expressions instead of Eq. (3.9)

$$\begin{aligned}k_1 &= \sigma_c\sqrt{l}\\ k_2 &= \{q+\mu_i(p-\sigma_c)\}\sqrt{l} = \frac{\sqrt{l}}{2}\{\mu_i(\sigma_1+\sigma_3-2\sigma_c)\\ &\qquad -(\sigma_1-\sigma_3)(\sin 2\alpha_i+\mu_i\cos 2\alpha_i)\}\end{aligned}, \quad (3.12)$$

where σ_c is the critical stress of p at which crack closes, σ_n is the normal stress on the crack surface, σ_f is the friction stress on crack surface and μ_i is the coefficient of friction the crack surface. Finally a criterion for crack initiation based on the shear fracture hypothesis becomes as follows

$$\mu_i(\sigma_1+\sigma_3-2\sigma_c)-(\sigma_1-\sigma_3)(\sin 2\alpha_i+\mu_i \cos 2\alpha_i) = -2L\,. \qquad (3.13)$$

This criterion is quite similar to the modified Griffith theory, although the physical meanings for crack initiation are quite different in both criteria. Since the stress σ_c can be considered neglibigily small compared to principal stresses, we obtain

$$\mu_i(\sigma_1+\sigma_3)-(\sigma_1-\sigma_3)(\sin 2\alpha_i+\mu_i \cos 2\alpha_i) = -2L\,, \qquad (3.14)$$

which is identical to Eq. (3.7) or Eq. (3.2) when L is replaced by $2K$ and C_i, respectively. Thus, the criterion curves for a model containing a single crack subjected to compressive stresses become similar to those shown in Fig. 5. For a model containing systems of cracks the criterion curves become similar to those shown in Fig. 6. by the same reasons as discussed in subsection 3.1.1. The subsequent and the final fracture criteria coincide with the fracture initiation criterion, since the crack develops in the original slit plane.

3.3. Fracture Criterion Based on A Model Containing A System of Cracks in An Orthotropic Matrix.

As an extension to the material discussed in the previous subsection, we consider a model which is made in such a way that a system of cracks is preferentially orientated in the direction of one of the principal planes of an orthotropic matrix material.

Let the axes of Carterisan coordinates be taken to coincide with principal axes of the orthotropic matrix. Stresses in the neighborhood of the tip of a crack with length $2l$ and inclined α from the major principal stress are as follows[12]

$$\left.\begin{aligned}
\sigma_x &= \frac{1}{\sqrt{2\varepsilon}}\Big[k_1 Re\Big\{\frac{s_1 s_2}{s_1-s_2}\Big(\frac{s_2}{\sqrt{\cos\theta+s_2\sin\theta}}-\frac{s_1}{\sqrt{\cos\theta+s_1\sin\theta}}\Big)\Big\} \\
&\qquad +k_2 Re\Big\{\frac{1}{s_1-s_2}\Big(\frac{s_2^{\,2}}{\sqrt{\cos\theta+s_2\sin\theta}}-\frac{s_1^{\,2}}{\sqrt{\cos\theta+s_1\sin\theta}}\Big)\Big\}\Big] \\
\sigma_y &= \frac{1}{\sqrt{2\varepsilon}}\Big[k_1 Re\Big\{\frac{1}{s_1-s_2}\Big(\frac{s_1}{\sqrt{\cos\theta+s_2\sin\theta}}-\frac{s_2}{\sqrt{\cos\theta+s_1\sin\theta}}\Big)\Big\}\Big] \\
&\qquad +k_2 Re\Big\{\frac{1}{s_1-s_2}\Big(\frac{1}{\sqrt{\cos\theta+s_1\sin\theta}}-\frac{1}{\sqrt{\cos\theta+s_1\sin\theta}}\Big)\Big\}\Big]
\end{aligned}\right\} \qquad (3.15)$$

$$\tau_{xy}=\frac{1}{\sqrt{2\varepsilon}}\left[k_1 Re\left\{\frac{s_1 s_2}{s_1-s_2}\left(\frac{1}{\sqrt{\cos\theta+s_1\sin\theta}}-\frac{1}{\sqrt{\cos\theta+s_2\sin\theta}}\right)\right\}\right.$$
$$\left.+k_2 Re\left\{\frac{1}{s_1-s_2}\left(\frac{s_1}{\sqrt{\cos\theta+s_1\sin\theta}}-\frac{s_2}{\sqrt{\cos\theta+s_2\sin\theta}}\right)\right\}\right],$$

where Re means the real part, k_1 and k_2 are given by Eq. (3.9), complex parameters s_1 and s_2 are the roots of the characteristic equations of the orthotropic elastic material

$$a_{11}s_i^4+(2a_{12}+a_{66})s_i^2+a_{22}=0\,,\qquad i=1,\,2 \tag{3.16}$$

The coefficients a_{11}, a_{12}, etc., can be associated with the principal elastic constants E_x, E_y, etc., as follows;

$$a_{11}=\frac{1}{E_x}\,,\quad a_{22}=\frac{1}{E_y}\,,\quad a_{12}=\frac{-\nu_{xy}}{E_x}=\frac{-\nu_{xy}}{E_y}\,,\quad a_{66}=\frac{1}{G_{xy}} \tag{3.17}$$

where E_x, E_y are the Young's moduli, ν_{xy}, ν_{yx}, the Poisson's ratios and G_{xy} the shear modulus. When new notations are defined as

$$\alpha_0=\sqrt{\frac{E_x}{E_y}}\,,\quad \beta_0=\frac{E_y}{2G_{xy}}-\nu_{xy};\ \alpha_0>\beta_0$$

the roots of Eq. (3.16) are

$$s_1s_2=-\alpha_0\,,\quad s_1+s_2=i\sqrt{2(\alpha_0+\beta_0)}\,,\quad s_1-s_2=\sqrt{2(\alpha_0-\beta_0)}\,. \tag{3.18}$$

By the same reasons as discussed in the previous subsection, we use the same fracture hypothesis, i.e., the shear fracture criterion in compression and the maximum normal stress fracture criterion in tension. The exact same procedure as in the previous subsection leads to the following criteria

$$(\sigma_1-\sigma_3)\sin 2\alpha_i=2L' \tag{3.19}$$

in compression and

$$(\sigma_1+\sigma_3)-(\sigma_1-\sigma_3)\cos 2\alpha_i=-2K'\,. \tag{3.20}$$

in tension, where L' and K' represent the strength of the matrix material in shear and in tension, respectively.

When the effect of the closure of the initial crack must be taken into consideration, by the same procedure as used in the previous subsection, Eq. (3.19) is modified to

$$\mu_i(\sigma_1+\sigma_3-2\sigma_c)-(\sigma_1-\sigma_3)(\sin 2\alpha_i+\mu_i\cos 2\alpha_i)=-2L'\,. \tag{3.21}$$

These equations Eqs. (3.19), (3.20) and (3.21) are similar to Eqs. (3.10), (3.11) and (3.13), respectively. Thus, the similar criteria for both models with a single crack and a system of cracks can be expected as those obtained in the previous sub-section.

In summary of this section, fracture criteria of two-dimensional anisotropic rocks or systems of rocks became quite similar to each other although they were in their forms derived on the basis of different models, and the criteria may be best summarized in the Coulomb or Mohr quadratic type criteria. In the next section, the criteria of Coulomb or Mohr will be formally extended into three-dimensions.

4. Fracture Surfaces of Anisotropic Rocks and Systems of Rocks

The general fracture criteria of anisotropic rocks and systems of rocks subjected to a system of short-term static loading, as discussed in Sect. 2, can be expressed by convex surfaces in the principal stress space, provided that there exists an unique relation between stresses and strains. The shape of the fracture surface may be best visualized by its cross section expressed on an arbitrary equipressure plane. In order to express it in a simpler form, change Cartesian coordinates $(\sigma_1, \sigma_2, \sigma_3)$ to new ones $(\sigma_1', \sigma_2', \sigma_3')$ according to

$$\left.\begin{aligned} \sigma_1 &= -\frac{1}{\sqrt{6}}\sigma_1' - \frac{1}{\sqrt{2}}\sigma_2' + \frac{1}{\sqrt{3}}\sigma_3' \\ \sigma_2 &= \sqrt{\frac{2}{3}}\sigma_1' + \frac{1}{\sqrt{3}}\sigma_3' \\ \sigma_3 &= -\frac{1}{\sqrt{6}}\sigma_1' + \frac{1}{\sqrt{2}}\sigma_2' + \frac{1}{\sqrt{3}}\sigma_3'. \end{aligned}\right\} \tag{4.1}$$

The equipressure plane is expressed by σ_3=const. i.e., $\sigma_1''\sigma_2''$-plane parallel to $\sigma_1'\sigma_2'$-plane (Fig. 10).

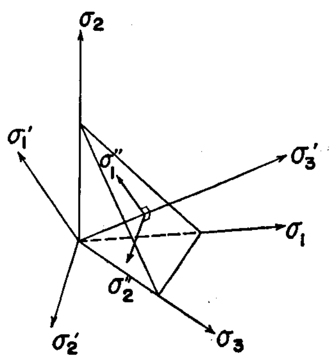

Fig. 10. Transformation of coordinates.

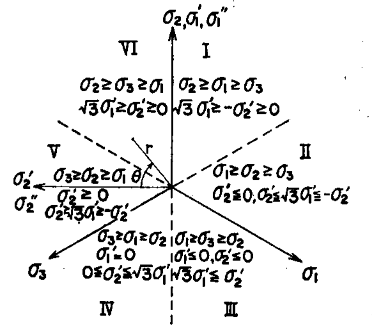

Fig. 11. Six regions of an equipressure plane

The equipressure plane is divided into six regions according to the ordering of principal stresses as shown in Fig. 11. The length of a vector originated from the hydrostatic pressure axis to any point on the equipressure plane and the distance from the origin to the equipressure plane are respectively given by[13]

$$r = \sqrt{(\sigma_1'')^2+(\sigma_2'')^2} = \frac{1}{\sqrt{3}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$
$$= \sqrt{3}\,\tau_{oct} = \sqrt{\frac{2}{3}(I_1^2-3I_2)}\,, \qquad (4.2)$$

and

$$d=\frac{1}{\sqrt{3}}(\sigma_1+\sigma_2+\sigma_3) = \sqrt{3}\,\sigma_{oct} = \frac{I_1}{\sqrt{3}}\,, \qquad (4.3)$$

where $I_1=\sigma_1+\sigma_2+_3$, $I_2=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1$.
The angle measured clockwise from the σ_2'' axis is given by

$$\theta = \tan^{-1}\frac{\sigma_1''}{\sigma_2''} = \tan^{-1}\left[-\frac{1}{\sqrt{3}}\,\frac{2\sigma_2-\sigma_1-\sigma_3}{\sigma_1-\sigma_3}\right] = \tan^{-1}\left(\frac{-\lambda}{\sqrt{3}}\right), \qquad (4.4)$$

where λ is the Lode parameter. The general fracture criterion, thus, can be represented by a surface in cylindrical coordinates (r, θ, d) or $\left(\sqrt{3}\,\tau_{oct},\ \tan^{-1}\left(\frac{-\lambda}{\sqrt{3}}\right),\ \sqrt{3}\,\tau_{oct}\right)$.

In most cases foliation in anisotropic rocks and joints, bedding-plane partings and faults in the systems of rocks are planar. Fracture can be considered to take place in these planes, so the procedure as used in subsection 3.1. can be extended into three-dimensions without any essential modifications.

In order to determine a fracture surface in three-dimensional stress space, we consider a model containing systems of weak planes as discussed in subsection 3.1. The other types of model may be also applicable, but are excluded here since the analytical results based on them are not essentially different from those of this model.

In this section, the same assumptions (1), (2) and (3) as in subsection 3.1. are also made. On the assumptions (1) and (2), the strength of the entire model is determined by the weakest one of the strengths, each of which is independently determined for a model containing a single weak plane with a fixed orientation angle from the axis of the major principal stress and on the assumption (3) the fracture surface can be considered to be fixed in the principal stress space. Therefore, the fracture surface of the entire model is obtained as a minimum composite surface which contains all the common regions enclosed by surfaces independently obtained as if only one single plane of weakness were contained in the model.

4.1 Fracture Surface Based on A Model Containing Systems of Weak Planes Parallel to the Axis of the Principal Stress.

Let us consider a model containing systems of weak planes parallel to the axis of the principal stress σ_2 as shown in Fig. 12. Since the intermediate principal stress in this case may be presumed not to show any essential influence on the

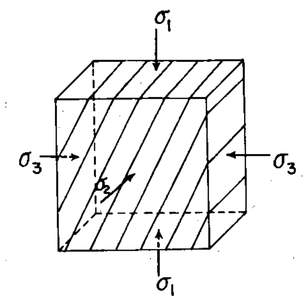

Fig. 12. A model containing systems of weak planes parallel to the axis of the principal stress

fracture of the model, the same fracture hypothesis used in two-dimensional analysis can be directly applied even in the three-dimensional.

4.1.1. Extension of the Coluomb Criterion

A fracture criterion of a model containing a single plane of weakness which obeys the Coulomb criterion is expressed by Eq. (3.2). The expression of this equation in new coordinates $(\sigma_1', \sigma_2', \sigma_3')$ is

$$\sigma_1' = \sqrt{3}\,\frac{\sin 2\alpha_i+\mu_i \cos 2\alpha_i}{\mu_i}\,\sigma_2'+\sqrt{2}\,\sigma_3'+\sqrt{6}\,\frac{C_i}{\mu_i}, \qquad (4.5)$$

where μ_i and C_i are material constants of the weak plane and α_i represents the orientation angle of the plane from the axis of the principal stress. This equation is valid for $\sigma_1 \geqq \sigma_3$, irrespective of any value of σ_2, i.e., in the regions I, II and III in Fig. 13. On the equipressure plane $\sigma_3'=\frac{I_1}{\sqrt{3}}=\text{const.}$, the equation is expressed by a straight line passing through two points

$$\left(\sqrt{\frac{2}{3}}I_1+\sqrt{6}\,\frac{C_i}{\mu_i},\ 0\right) \quad \text{and} \quad \left[0,\ -\left(\frac{\sqrt{2}}{3}I_1+\sqrt{2}\,\frac{C_i}{\mu_i}\right)\frac{\mu_i}{\sin 2\alpha_i+\mu_i \cos 2\alpha_i}\right).$$

For the matrix material with the material constants $\mu_0=\tan\varphi_0$, C_0 and $\alpha_0=\frac{\pi}{4}-\frac{\varphi_0}{2}$, the fracture criterion is expressed by replacing the subscript i in Eq. (4.5) by subscript 0[14])

$$\sigma_1' = \sqrt{3}\cdot\frac{\sigma_2'}{\mu_0}(\cos\varphi_0+\mu_0\sin\varphi_0)+\sqrt{2}\,\sigma_3'+\sqrt{6}\,\frac{C_0}{\mu_0}$$
$$=\sqrt{3}\,\frac{\sqrt{1+\mu_0^2}}{\mu_0}\sigma_2'+\sqrt{2}\,\sigma_1'+\sqrt{6}\,\frac{C_0}{\mu_0}. \tag{4.6}$$

This equation is valid in the region $\sigma_1 \geqq \sigma_2 \geqq \sigma_3$, i.e., in region II of Fig. 11. However, due to the isotropy of the matrix, this equation can be applicable for the entire space by merely changing the subscript 1, 2 and 3 in a cyclic permutation. On the equipressure plane this equation is expressed by a straight line passing through two points

$$\left(\sqrt{\frac{2}{3}}I_1+\sqrt{6}\,\frac{C_0}{\mu_0},\,0\right) \quad \text{and} \quad \left[0,\,-\left(\frac{\sqrt{2}}{3}I_1+\sqrt{2}\,\frac{C_0}{\mu_0}\right)\frac{\mu_0}{\sqrt{1+\mu_0^2}}\right].$$

The fracture surface, therefore, can be expressed by the minimum composite surface constructed from Eqs. (4.5) and (4.6). The section of the surface cut by the equipressure plane is also expressed by the minimum composite curve composed from Eqs. (4.5) and (4.6).

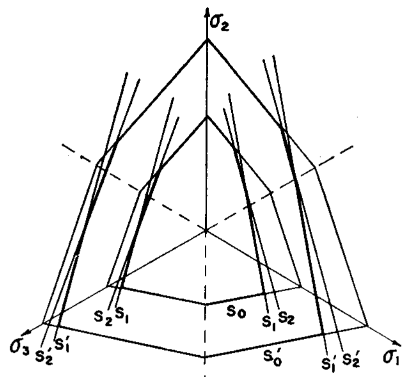

Fig. 13. Cross sections of fracture surface cut by equipressure planes (based on the extended Coulomb criterion)

For a model containing many systems of weak planes the fracture surface for the entire model can be determined as the minimum composite surface constructed from surfaces given by Eq. (4.6) and Eq. (4.5) with $i=1, 2, \cdots$ as shown in Fig. 13. The section of the surface cut by the equipressure plane is also expressed by the minimum composite curve composed from curves Eq. (4.6) and Eq. (4.5) with $i=1, 2, \cdots$.

4.1.2. Extehsion of the Mohr Quadratic Criterion

When fracture of the weak plane of the model and the matrix material obeys the Mohr quadratic criterion, we obtain the following expression for the model with a single weak plane from Eq. (3.5) merely by transforming coordinates according to Eq. (4.1)

$$\sigma_1' = -\sqrt{\frac{3}{2}}\frac{\sin^2 2\alpha_i}{\lambda_i}(\sigma_2')^2+\frac{1}{\sqrt{3}}\sin\alpha_i\sigma_2'+\sqrt{2}\sigma_3+\sqrt{6}\frac{D_i}{\lambda_i}, \quad (4.7)$$

where λ_i and D_i are material constants of i-th weak plane, and α_i represents the orientation angle of the plane from the axis of the major principale stress.

This equation is valid for $\sigma_1 \geqq \sigma_2$, irrespective of σ_2, i.e., region I, II and III of Fig. 11 as discussed in the previous section.

The fracture criterion of the matrix is obtained in the same manner as explained in the previous section[15]

$$\sigma_1' = -\sqrt{\frac{3}{2}}\frac{\sigma_2'^2}{\lambda_0}+\sqrt{2}\sigma_3'+\frac{1}{2\lambda}\sqrt{\frac{3}{2}}(4D_0-\lambda_0^2) \quad (4.8)$$

where, $\lambda_0 = \tan\varphi_0$, and $\alpha_0 = \frac{\pi}{4}-\frac{\varphi_0}{2}$ are used.

This equation is valid in the region $\sigma_1 \geqq \sigma_2 \geqq \sigma_3$, i.e., region II of Fig. 11. However, due to the isotropy, this equation can be applied in the entire space by a cyclic replacement of the subscripts 1, 2 and 3.

The fracture criterion for a model containing many systems of weak planes in the otherwise homogeneous matrix can be obtained as the minimum composite surface constructed from the surfaces given by Eq. (4.8) and (4.7) for $i=1, 2, \cdots$, by the same process as mentioned in the previous subsection. The cross sections of the surface cut by the equipressure planes are expressed schematically in Fig. 14.

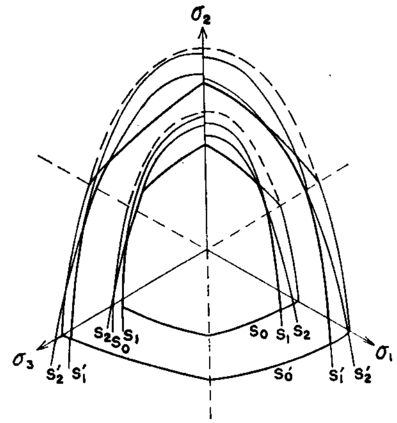

Fig. 14. Cross sections of fracture surface cut by equipressure planes (based on the extended Mohr criterion)

4.2. Fracture Surfaces Based on the Model Containing Systems of Weak Planes with arbitary Orientation

In a most general case of three-dimensions, let us consider a fracture surface for a model whose systems of weak planes are inclined arbitarily from the axis of the principal stress. Let the direction of the principal stress σ_1, σ_2 and σ_3 coincide with the Cartesian coordinates x_1, x_2 and x_3, respectively and directional cosines of the k-th plane be denoted ${}^k\nu_1$, ${}^k\nu_2$ and ${}^k\nu_3$. Normal and shearing stress N and S, working on the plane are given, respectively, by

$$N = \sigma_i \, {}^k\nu_i^2 \tag{4. 9}$$

$$S^2 = \sigma_i^2 \, {}^k\nu_i^2 - (\sigma_i \, {}^k\nu_i)^2 \, , \tag{4. 10}$$

where the summation convention rule is used.

If the weak plane obeys the Coulomb criterion

$$S = N\mu_k + C_k \, , \tag{4. 11}$$

where μ_k and C_k are material constants of the k-th weak plane, the expression in terms of the principal stress becomes, omitting the superscript and subscript k for the sake of simplicity,

$$\begin{aligned} &\sigma_1^2[1-\nu_1^2(1+\mu^2)]\nu_1^2+\sigma_2^2[1-\nu_2^2(1+\mu^2)]\nu_2^2+\sigma_3^2[1-\nu_3^2(1+\mu^2)]\nu_3^2 \\ &\quad -2(1+\mu^2)(\sigma_1\sigma_2\nu_1^2\nu_2^2+\sigma_2\sigma_3\nu_2^2\nu_3^2+\sigma_3\sigma_1\nu_3^2\nu_1^2) \\ &\quad -2\mu C(\sigma_1\nu_1^2+\sigma_2\nu_2^2+\sigma_3\nu_3^2)-C^2 = 0 \end{aligned} \tag{4. 12}$$

with conditions

$$\nu_i\nu_j = \delta_{ij} \, , \tag{4. 13}$$

where δ_{ij} represents Kronecker's delta.

If the weak plane obeys the Mohr quadratic criterion

$$S^2 = N\lambda_k + D_k \, , \tag{4. 14}$$

then the corresponding equation in terms of the principal stress are

$$\begin{aligned} &\sigma_1^2(1-\nu_1^?)\nu_1^2+\sigma_2^2(1-\nu_2^2)\nu_2^2+\sigma_3^3(1-\nu_3^2)\nu_3^2 \\ &\quad -2(\sigma_1\sigma_2\nu_1^2\nu_2^2+\sigma_2\sigma_3\nu_2^2\nu_3^2+\sigma_3\sigma_1\nu_3^2\nu_1^2) \\ &\quad -\lambda(\sigma_1\nu_1^2+\sigma_2\nu_2^2+\sigma_3\nu_3^2)-D = 0 \end{aligned} \tag{4. 15}$$

with conditions (4.13), where superscripts and subscript k are omitted.

Therefore the model cotaining many systems of weak planes inclined arbitarily to the axis of the principal stress, fracture surface is given by the minimum surface composed from surfaces given by Eq. (4.6) and Eq. (4.12) for $i=1, 2, \cdots$, or Eq. (4.8) and Eq. (4.13) for $i=1, 2, \cdots$, provided the assumption of no interaction between the weak planes is reasonable.

Fracture surface for more elaborate models can also be obtained similarly.

5. Comparison with Experiments

In order to illustrate the applicability of the fracture criteria (curves or surfaces) discussed in this paper and to check the validity of these criteria in predicting real strengths of anisotropic rocks or the systems of rocks, comparison was made with three types of experiment, i.e. anisotrpoic rocks and two types of model of systems of rocks.

The collected results of the experiments on anisotropic rocks were plotted in Fig. 15. The data of the model with a system of slit made of mortar with mix-proportion $c{:}w{:}s=1.0{:}0.6{:}2.0$ are shown in Fig. 16. In Fig. 17 are shown the results for the other type of model with parallel multiple weak layers, in which the matrix was made of cement mortar with mix proportion $c{:}w{:}s=1.0{:}0.6{:}2.0$ and the layer was made of cement mortar added with flyash with mix proportion c:w:fa=1.0:1.8:2.0 and the compressive strength of the layer is one-tenth that of the matrix.

Examination of the experimental results of anisotropic rocks reveals that the the compressive strength is most strongly influenced by the orientation of the bedding planes from the axis of the major principal stress. The lowest strength is obtained when the inclination angle is about 30° at low confining pressure.

General form of the curcves drawn in Fig. 15 does not vary appreciably even in a wide range of confining pressure.

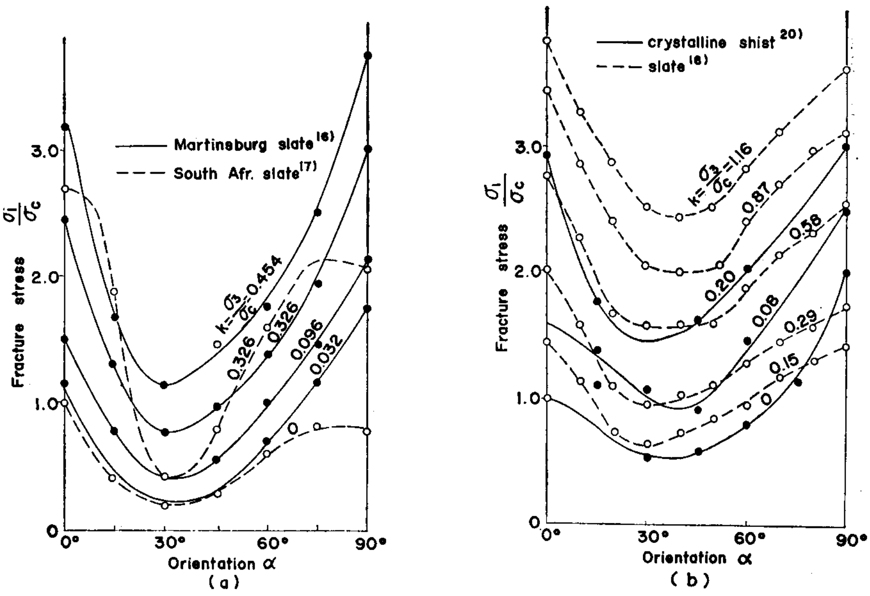

Fig. 15. Collected results of experiments on anisotropic rocks

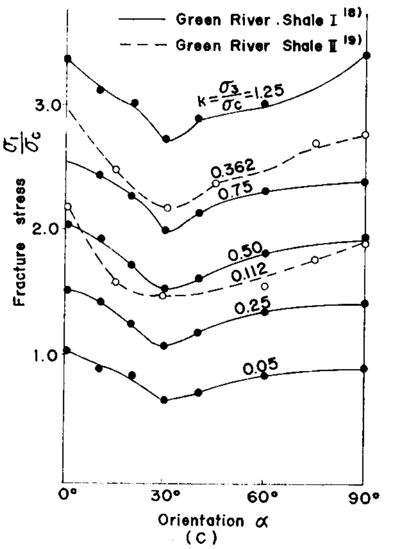

Fig. 15 (continued)

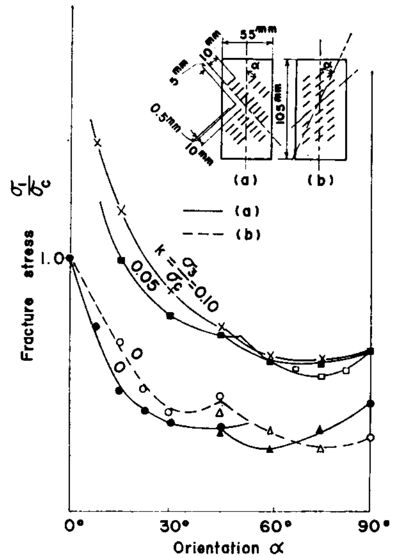

Fig. 16. Results of tests on models with a system of slits

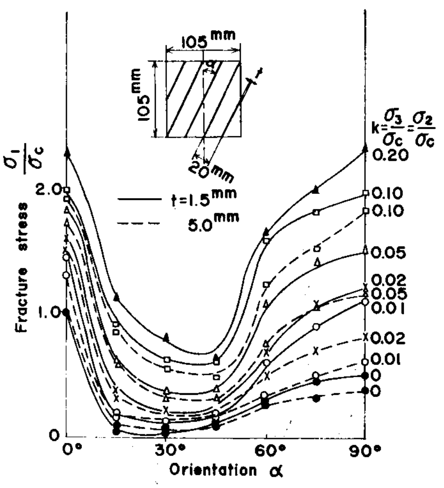

Fig. 17. Results of tests on models with parallel multiple weak planes

The results for various slates and a crystalline schist show similar curves, although the degree of the increase of strength due to the confining pressure is quite different. The effect of anisotropy is not as strong in Green River shale as in the slates and the crystalline schist.

The results for the anisotropic rocks tend to confirm the general form of the the predicted criteria in Sect. 3.

The results on the multi-layered model also show a qualitative agreement with the general form of the predicted criteria. It might be thought that results of the model with a asystem of slit disagreed with the predicted criteria. However, if two systems of weak planes are assumed such that one coincides with the real slit plane and another is perpendicular to the slit plane (virtual plane), the criteria discussed in Sect. 3, are in good agreement with the results.

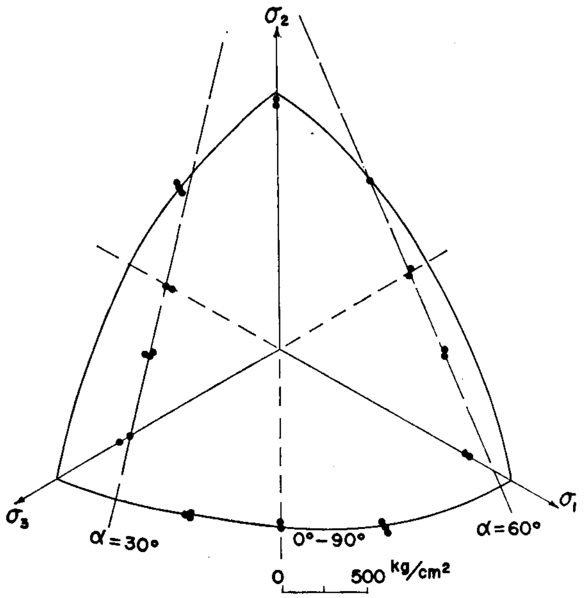

Fig. 18. Experimental data on crystalline schist plotted on an equipressure plane (Akai, Yamamoto and Arioka[20]).

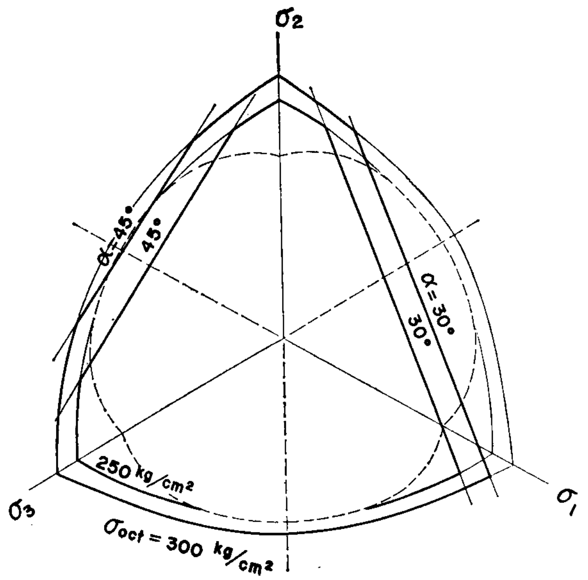

Fig. 19. Results of models with a single plane of weakness expressed on the equipressure planes (Ishida[21]).

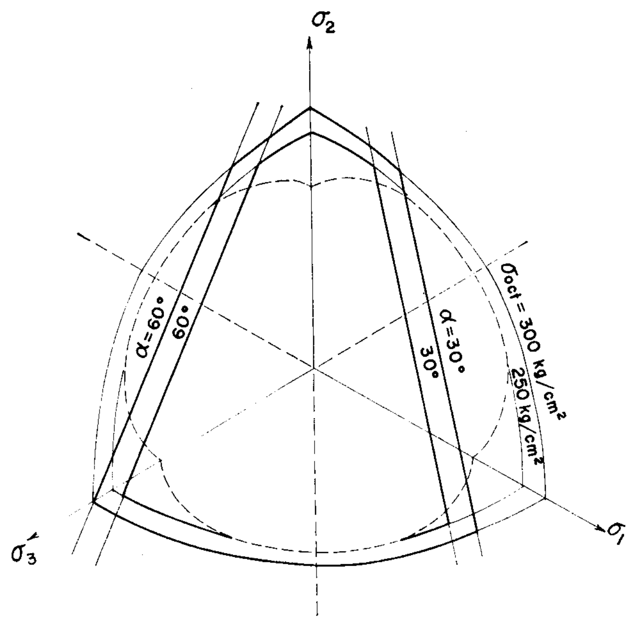

Fig. 20. Results of models with multiple weak layers expressed on the equipressure planes.

In summary, the fracture criteria discussed in Sect. 3, are enought to facilitate the interpretation and rationalization of the experimental results.

The general form of the fracture surface may be complicated and cannot be visualized easily, so the sectional curves of the surface cut by the equipressure planes are compared with the experimental ones.

The results of experiment on a crystalline shists are plotted in Fig. 18. The results on models with a single plane of weakness and with multiple weak layers are shown in Figs. 19 and 20, respectively. The former model was made of cement mortar with mix proportion c:w:s = 1.0:0.6:2.0 and a single plane of partings. In the latter model the matrix was made of cement mortar and the layer was made of cement and fly ash, as already described above in this section.

The fracture criterion curves shown in Figs. 13 and 14 are compared with the results in Figs. 18, 19 and 20, the similarity between the predicted curves and observed ones are remarkable.

Although the experimental results are limited, the fracture surface discussed in Sect. 4 may be enough to facilitate the interpretation and rationalization of the experimental results.

6. Conclusions

It has been shown that the general fracture criterion for the anisotropic rocks or the systems of rocks can be approximated in the load space or in the principal stress space by a minimum composite surface constructed from all the associated surfaces with the anisotropic model as if it contained a single plane or system of weakness.

Several fracture criteria in two-dimensions based on the fundamental three types of model have been discussed and show that they are best summarized in the Coulomb or the Mohr quadratic type criteria. Two types of fracture surface in the principal stress space have been also discussed as a natural extention of the two dimensional Coulomb or Mohr quadratic type criterion.

These criteria, both the two dimensional and the three dimensional, are in good agreement with the experimental results. Thus, these criteria can be used to predict fracture of anisotropic rocks or the systems of rocks which exhibits a high degree of planar anisotropy.

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