# Study of Equilibrium and Stability of Plasma in Heliotron Magnetic Field

By

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Equilibrium and stability problems of plasma in the Heliotron magnetic field are discussed under the assumption that the plasma behaves as a hydromagnetic fluid and its pressure is much less than magnetic pressure. Both plasma pressure and diamagnetic current satisfying the equilibrium conditions are analyzed in the case where the plasma carries only a diamagnetic $j_\theta$ current and not a $j_z$ current due to an external electric field.

Also by using the curvilinear coordinate system peculiar to Heliotron magnetic field, the stability condition for the flute type instability is discussed according to the energy principle which was investigated by I.B. Bernstein et al..

It is concluded that Heliotron magnetic field provides the equilibrium of plasma inside the separatrix and the plasma in this field is stable against the interchange instability under the proper gradient of plasma pressure.

## 1. Introduction.

The consideration about plasma confinement is a most important problem for a controllable thermonuclear fusion and one of the promising methods which achieve this fusion is the cofining of a high temperature plasma with the aid of external or self-magnetic field.

In history, magnetic fields to confine plasma in a stable condition has been investigated and many interesting configurations such as helical field, Ioffe field, multipole field, have been analyzed.

Also Heliotron magnetic field investigated first by K.Uo has an interesting nature from the confinement points of view.[1,2]

In this paper, first an axisymmetric Heliotron magnetic field produced by the current flowing in the sheet coils which are wound over a straight discharge tube at regular interval is analyzed. Next the theoretical treatment about plasma confinement in the Heliotron magnetic field is discussed.

The stability of a hydromagnetic fluid in static equilibrium can be deter-

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mined by an energy principle formalism which was investaigte by I. Bernstein et. al.[3] The present purpose is to apply this method to a plasma in an axisymmetric Heliotron magnetic field.

When the plasma carries only a diamagnetic $j_\theta$ current and not $j_z$ current due to a external electric field, for instance joule heating electric field, both equilibrium plasma pressure and diamagnetic current can be expressed in the forms respectively;

$$p=\alpha_0\frac{\psi_0}{n}\left(1-\frac{\psi}{\psi^0}\right)^n$$

$$j_\theta=-\alpha_0 r\left(1-\frac{\psi}{\psi_0}\right)^{n-1}$$

where $\psi=rA_\theta$.

The stability condition for the flute type instability due to the $\xi_\psi=f(\theta)/rB$ pertubation is given as follows;

$$\delta W(\psi)=\int d\chi\frac{\partial}{\partial\psi}\left(\frac{1}{B^2}\right)\left[\gamma pB^2\frac{\partial}{\partial\psi}\left(\frac{1}{B^2}\right)+\frac{\partial p}{\partial\psi}\right]>0$$

The equilibrium state satisfying the above condition takes the next inequality $\gamma/n>1.51$, where $\gamma$ is the ratio of the specific heats.

## 2. Analysis of an axisymmetric Heliotron magnetic field.

We consider an axisymmetric Heliotron magnetic field produced by sheet coils which are wound at regular intervals over a straight discharge tube. The sheet coils consist of two different sets of coils; one of them is a set of positive coils carrying the current $I$ and another is a set of negative coils carring the current $-\lambda I$.

Using the cylindrical coordinate system $(r, \theta, z)$, these sets of sheet coils are located at $r=a$ with regular interval $L/2$, shown in Fig. 1.

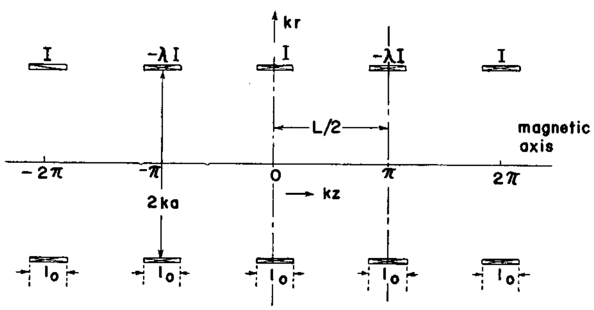


Fig. 1. Set of sheet currents located at $r=a$ with regular interval $L/2$.

In any region without current, the magnetic flux intensity $\boldsymbol{B}$ satisfies

$$\text{rot } \boldsymbol{B}=0. \tag{2-1}$$

Then $\boldsymbol{B}$ can be described as the following equation;

$$B = \text{grad}\ \phi, \tag{2-2}$$

where $\phi$ is a magnetic scalar potential.

Further we get Laplace's equation due to the divergence free of $\boldsymbol{B}$;

$$\nabla^2\phi = 0 \tag{2-3}$$

In an axisymmetric case, $\phi$ is not a function of a coordinate $\theta$ and eq. (2–3) can be rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) + \frac{\partial^2}{\partial z^2}\phi = 0. \tag{2-4}$$

The general solution for a periodic field with regard to the coordinate $z$, may then be written as

$$\phi = Cz + \sum_{n=1}^{\infty}[A_n I_0(\text{nkr}) + B_n K_0(\text{nkr})]\ \sin(\text{nkz}) \tag{2-5}$$

where $k$ is defined as $k=2\pi/L$, $C$, $A_n$ and $B_n$ are arbitrary constants and $I_0(nkr)$, $K_0(nkr)$ are modified Bessel functions. Substituting $\phi$ from eq. (2–5) into eq. (2–2), the magnetic flux intensity can be described as

$$B_r = \sum_{n=1}^{\infty}[A_n I_1(nkr) - B_n K_1(nkr)]nk\ \sin(nkz) \tag{2-6}$$

$$B_z = C + \sum_{n=1}^{\infty}[A_n I_0(nkr) + B_n K_0(nkr)]nk\ \cos(nkz) \tag{2-7}$$

where $C$, $A_n$, and $B_n$ can be determined by the boundary conditions.

First we consider the magnetic field produced by the set of positive coils. For this purpose we expand the current sheets into Fourier series such as

$$j_0(z) = \tfrac{1}{2}a_0 + a_1\ \cos(kz) + a_2\ \cos(2kz) + \cdots \tag{2-8}$$

where

$$\tfrac{1}{2}a_0 = j_0 kl_0/2\pi, \quad a_n = 4j_0\ \sin(nkl)/2n\pi, \quad 21 = 1_0 \tag{2-9}$$

Now an arbitrary constant $A_n$ must be zero for $r>a$ because $I_0(nkr)\to\infty$ as $r\to\infty$ and $B_n$ must be zero for $r<a$ because $K_0(nkr)\to\infty$ as $r\to 0$, The conditions on $\boldsymbol{B}$ are that $B_r$ is continuous at $r=a$ and the jump in $B_z$ is equal to $\mu_0 j_\theta$. Namely the conditions are

$$\begin{aligned} &\boldsymbol{n}\times[\boldsymbol{B}] = \mu_0 je_\theta \\ &\boldsymbol{n}\cdot[\boldsymbol{B}] = 0 \end{aligned} \tag{2-10}$$

at $r=a$, where $[\boldsymbol{B}]$ is the jump in $\boldsymbol{B}$ at the current sheet and $\boldsymbol{n}$ is the unit vector normal to the surface.

Substituting eqs. (2–6), (2–7) and (2–8) into eq. (2–10), we get

$$C = \tfrac{1}{2}\mu_0 a_0 \tag{2-11}$$

$$A_n = \mu_0 a a_n K_1(nka) = \mu_0\frac{a_n}{nk}D_n \tag{2-12}$$

$$-B_n = \mu_0 a a_n I_1(nka) = \mu_0\frac{a_n}{nk}E_n \tag{2-13}$$

Then magnetic field produced by the set of positive coils is obtained as folllows, In the region $r<a$, using the suffix I, the magnetic flux intensity $B_z$, $B_r$ are

$$B_z{}^I=\mu_0 \tfrac{1}{2} a_0+\mu_0 \sum_{n=1}^{\infty} a_n D_n I_0(nkr) \cos(nkz)$$

$$=\mu_0\frac{j_0 l_0}{2\pi}k\left[1+\sum_{n=1}^{\infty}\frac{2\sin(nkl)}{nkl}D_n I_0(nkr)\cos(nkz)\right], \quad (2\text{–}14)$$

$$B_r{}^I=\mu_0\frac{j_0 l_0}{2\pi}k\sum_{n=1}^{\infty}\frac{2\sin(nkl)}{nkl}D_n I_1(nkr)\sin(nkz) \quad (2\text{–}15)$$

In the region $r<a$, using the suffix II,

$$B_z{}^{II}=-\frac{\mu_0 j_0 l_0}{2\pi}k\sum_{n=1}^{\infty}\frac{2\sin(nkl)}{nkl}E_n K_0(nkr)\cos(nkz) \quad (2\text{–}16)$$

$$B_r{}^{II}=\frac{\mu_0 j_0 l_0}{2\pi}k\sum_{n=1}^{\infty}\frac{2\sin(nkl)}{nkl}E_n K_1(nkr)\sin(nkz). \quad (2\text{–}17)$$

Modifying eqs. (2–14), (2–15), (2–16) and (2–17), we also can analyze the magnetic field produced by the set of negative coils. That is transforming the parameters $j_0$ and $kz$ into $-\lambda j_0$ and $kz-\pi$ respectively, we get the magnetic field produced by the set of negative coils.

Then we get the axisymmetric Heliotron magnetic field by adding both the field of positive coils and the field of negative coils. For $r>a$,

$$B_z{}^I=\frac{\mu_0 j_0 l_0}{2\pi}k\left[(1-\lambda)+2\sum_{n=1}^{\infty}\frac{\sin(nkl)}{nkl}D_n(1-\lambda\cos n\pi)\times I_0(nkr)\cos(nkz)\right] \quad (2\text{–}18)$$

$$B_r{}^I=\frac{\mu_0 j_0 l_0}{2\pi}k\left[2\sum_{n=1}^{\infty}\frac{\sin(nkl)}{nkl}D_n(1-\lambda\cos(n\pi))\ L_1(nkr)\sin(nkz)\right. \quad (2\text{–}19)$$

For $r<a$,

$$B_z{}^{II}=\frac{\mu_0 j_0 l_0}{2\pi}k\left[2\sum_{n=1}^{\infty}\frac{\sin(nkl)}{nkl}(1-\lambda\cos n\pi)E_n K_0(nkr)\cos(nkz)\right. \quad (2\text{–}20)$$

$$B_r{}^{II}=\frac{\mu_0 j_0 l_0}{2\pi}k\left[2\sum_{n=1}^{\infty}\frac{\sin(nkl)}{nkl}(1-\lambda\cos n\pi)\ E_n K_1(nkr)\sin(nkz)\right. \quad (2\text{–}21)$$

In the following discussion, we use simple equations for $r<a$, i.e.

$$\begin{aligned} B_z&=B_0[1+\alpha I_0(kr)\cos kz]\\ B_r&=B_0\alpha I_1(kr)\sin kz\\ \phi&=B_0[z+\alpha k I_0(kr)\sin kz]\\ \psi&=rA_\theta=B_0 r\left[(\tfrac{1}{2})r+\frac{\alpha}{k}I_1(kr)\cos kz\right]\end{aligned} \quad (2\text{–}22)$$

where

$$B_0=\frac{\mu_0 j_0 l_0}{2}k(1-\lambda)$$

$$\alpha=\frac{2(1+\lambda)}{1-\lambda}\frac{\sin(kl)}{kl}D_1(ka).$$

Examples of the Heliotorn magnetic field are easily calculated by using the above formula. Fig. 2 shows the magnetic lines of force with $\alpha=0.787$, $\alpha=0.2$, which are produced by the circular point current.

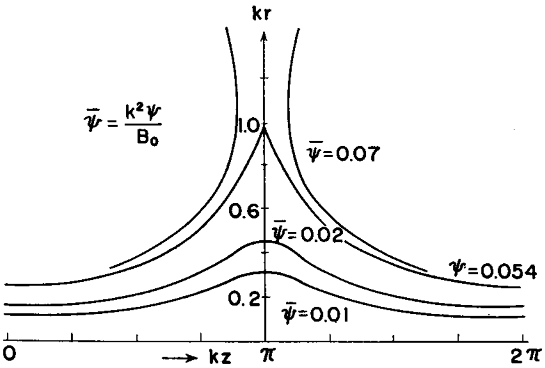


Fig. 2(a). Axisymmetric Heliotron magnetic field with $\alpha=0.789$ and $kd=1.0$.

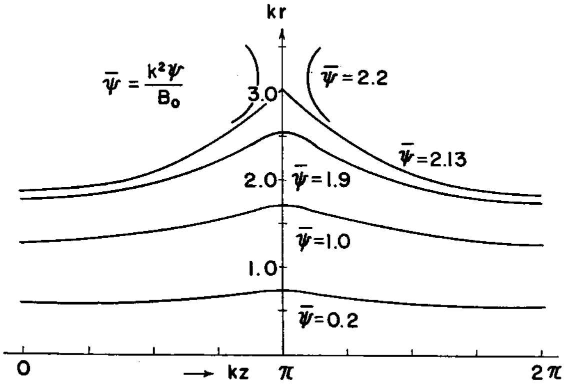


Fig. 2(b). Axisymmetric Heliotron magnetic field with $\alpha=0.2$ and $kd=3.03$.

## 3. Low-$\beta$ plasma equilibrium in Heliotron field.

Next we study the problem of a plasma equilibrium in the Heliotron magnetic field.

The equilibrium equations in the ideal case, are expressed by the magnetostatic equations;

$$\boldsymbol{j}\times\boldsymbol{B}=\text{grad } p \tag{3-1}$$

$$\text{rot } \boldsymbol{B}=\mu_0 \boldsymbol{j} \tag{3-2}$$

$$\text{div } \boldsymbol{B}=0 \tag{3-3}$$

By successive iteration, we seek low-$\beta$ solution in which $2\mu p/B^2$ is a small quantity. The zeroth-order solution of $\beta$ is the Heliotron field given by the following equations

$$\text{rot } \boldsymbol{B}_0=0, \qquad \text{div } \boldsymbol{B}_0=0 \tag{3-4}$$

Namely $\boldsymbol{B}_0$ implies eqs. (2–18) to (2–21).

The first order solution is given by

$$j\times B=\text{gran } p,$$
$$\text{rot } B=\mu_0 j, \tag{3-5}$$
$$\text{div } B=0$$

Adding to the above equations, the next equation must exist for the charge separation not to occur, i.e.

$$\text{div } j=0 \tag{3-6}$$

The first necessary condition on $p$ is

$$B \text{ grad } p=0 \quad \text{or} \quad \frac{\partial}{\partial s}p=0 \tag{3-7}$$

namely $p$ is constant along a field line.

The next necessary condition is

$$\int \frac{\nabla B_0\cdot(\text{grad } p\times B}{B_0^4}ds=0 \tag{3-8}$$

where the integration is taken along any closed line of force. In the case of scalar pressure, eqs. (3–7) and (3–8) are the necessary and sufficient conditions for the equilibrium.

Introducing $U=-\int dl/B$, we can rewrite eq. (3–8) as

$$\text{grad } p\times\text{grad } U=0. \tag{3-9}$$

Therefore in equilibrium, the gradient of the pressure is everywhere perepndicular to the $U$=const surface.[4]

Now apart from the general outline of equilibrium, we discuss a simple but important equilibrium in the Heliotron field in which the plasma carries only an azimuthual $j_\theta$ current. In this case the current density is

$$j=j_\theta(r, z) \tag{3-10}$$

Taking the component of eq. (3–5), we obtain the next relations

$$\frac{1}{r}\frac{\partial}{\partial\theta}B_z-\frac{\partial}{\partial z}B_\theta=\mu_0 j_r$$
$$\frac{\partial}{\partial z}B_r-\frac{\partial}{\partial r}B_z=\mu_0 j_\theta$$
$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta)-\frac{\partial}{\partial\theta}B_r\right]=\mu_0 j_z$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r)+\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta+\frac{\partial}{\partial z}B_z=0$$
$$\frac{\partial}{\partial r}p=j_\theta B_{z0} \tag{3-11}$$
$$\frac{1}{r}\frac{\partial}{\partial\theta}p=0$$
$$\frac{\partial}{\partial z}p=-j_\theta B_{r0}.$$

Assuming all quantities independent of $\theta$, the above relations can be rewriten as

$$\frac{\partial}{\partial z}B_r-\frac{\partial}{\partial r}B_z=\mu_0 j_\theta$$

$$\frac{\partial}{\partial r}p=j_\theta B_{z0} \tag{3-11$'$}$$

$$\frac{\partial}{\partial z}p=-j_\theta B_{r0}$$

According to $\boldsymbol{B}_0\cdot\text{grad }p=0$, we can arrive at

$$\frac{\partial}{\partial r}p=-\zeta r B_{z0},\qquad \frac{\partial}{\partial z}p=\zeta r B_{r0} \tag{3-12}$$

where $\zeta$ is determined by the div $\boldsymbol{B}=0$. Substituting $B$ from (3–12) into the second equation of (3–4), we get

$$\frac{\partial}{\partial z}p\frac{\partial}{\partial r}\zeta-\frac{\partial}{\partial r}p\frac{\partial}{\partial z}\zeta=0 \tag{3-13}$$

then

$$\zeta=\zeta(p) \tag{3-14}$$

Using the flux function $\psi$ described in the previous section, the next relations are obtained

$$\frac{\partial}{\partial r}p=-\zeta\frac{\partial}{\partial r}\psi,\qquad \frac{\partial}{\partial z}p=-\zeta\frac{\partial}{\partial z}\psi \tag{3-15}$$

then

$$\zeta=\zeta(\psi). \tag{3-16}$$

If we take $\zeta$ such that plasma pressure $p$ becomes zero at $\psi=\psi_0$, $\zeta$ is given by

$$\zeta=\zeta_0\left(1-\frac{\psi}{\psi_0}\right)^{n-1} \tag{3-17}$$

where $\psi_0$ shows the magnetic lines of force through the neutral lines.

Therefore scalar pressure $p$ and current density $j_\theta$ are given by

$$p=\zeta_0\frac{\psi_0}{n}\left(1-\frac{\psi}{\psi_0}\right)^{n} \tag{3-18}$$

$$j_\theta=-\zeta_0 r\left(1-\frac{\psi}{\psi_0}\right)^{n-1} \tag{3-19}$$

The first order perturbation in $B_z$ field satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}B_z\right)+\frac{\partial^2}{\partial z^2}B_z=-\mu_0\frac{1}{r}\frac{\partial}{\partial r}(rj_\theta) \tag{3-20}$$

where $j_\theta$ is that of eq. (3–19).

In the case of $n=2$, the first order quantities, $j_\theta$, $p$, $B_z$ are given respectively

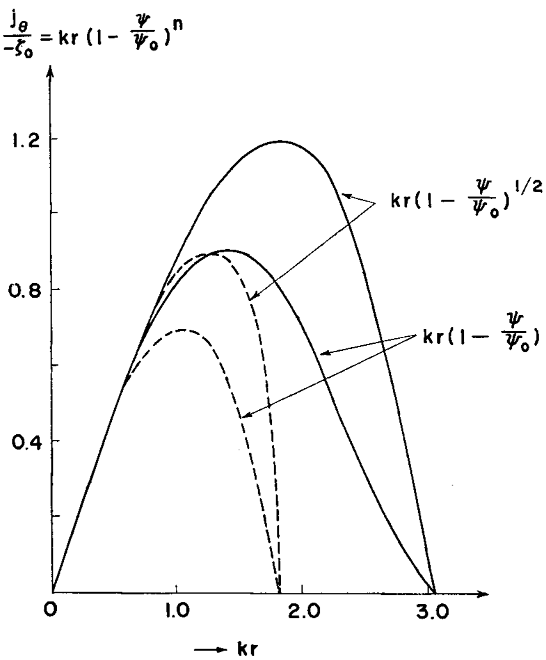


Fig. 3(a). Profile of an azimuthal current which satisfies an equilibrium state. Solid lines show the state just under the negative coil, while dotted lines show the state under the positive coil.

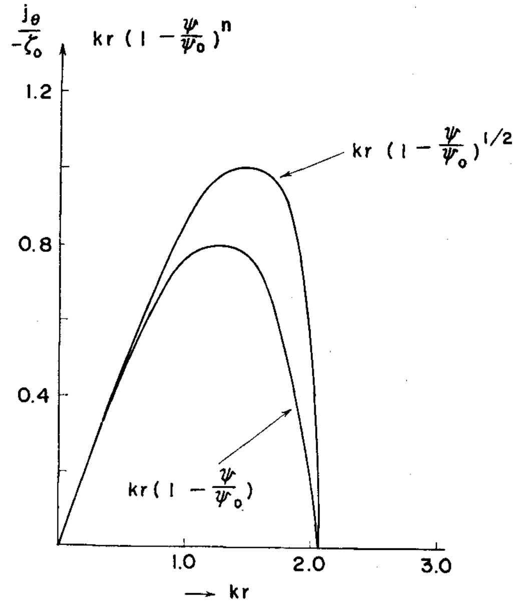


Fig. 3(b). Profile of an azimuthal current which satisfies an equilibrium state. Curves are plotted in the middle position of coils.

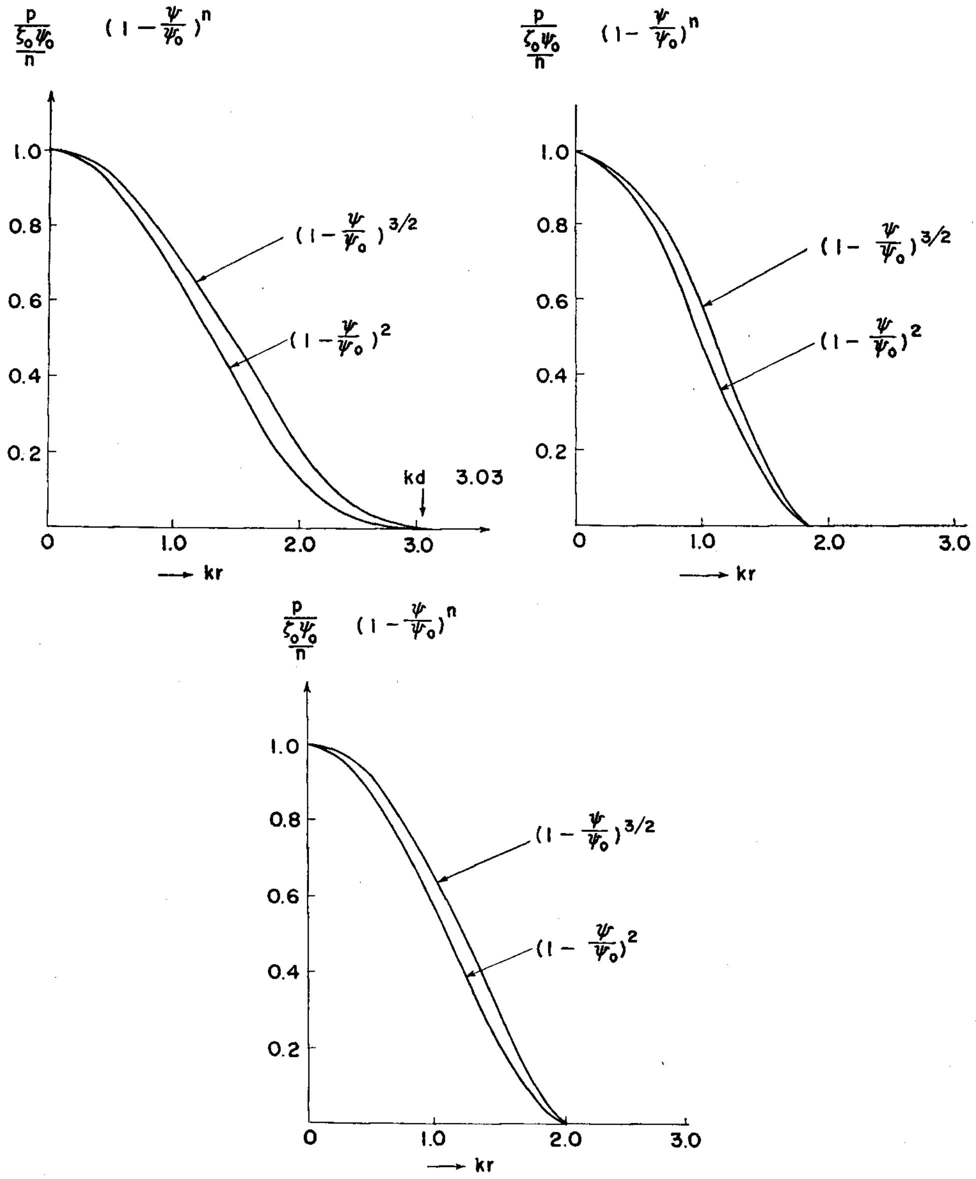


Fig. 4. Radial distribution of plasma pressure which satisfies an equilibrium state.
( a ) It is calculated under the negative coil,
( b ) calculated under the positive coil,
( c ) calculated in the middle position of coils.

by

$$j_\theta = -\zeta_0 r \left(1 - \frac{\psi}{\psi_0}\right),$$

$$p=\zeta_0\frac{\psi_0}{2}\left(1-\frac{\psi}{\psi_0}\right)^2$$

$$\frac{k^2}{\zeta_0}B_z=\tfrac{1}{2}(kr)^2-\frac{1}{8}\frac{(kr)^4}{\psi_0k^2}B_0+F+G(1-\alpha I_0\cos kz)$$

$$+\frac{\alpha}{6}\frac{B_0}{\psi_0k^2}[-4krI_1+2(kr)^2I_0+(kr)^3I_1]\cos kz \tag{3-21}$$

where

$$-F=\tfrac{1}{2}(kd)^2-\frac{1}{8}B_0\frac{(kd)^4}{\psi_0k}+\frac{\alpha}{6}\frac{B_0}{\psi_0k}[-4kdI_1(kd)+2(kd)^2I_0(kd)+(kd)^3I_1(kd)]$$

$$-G=F+\tfrac{1}{2}\frac{\psi_0k^2}{B_0}$$

In the case of $n=2/3$, the first order quantites $j_\theta$ and $p$ are given by

$$j_\theta=-\zeta_0r\left(1-\frac{\psi}{\psi_0}\right)^{1/2}$$

$$p=\zeta_0\frac{2}{3}\psi_0\left(1-\frac{\psi}{\psi_0}\right)^{3/2} \tag{3-22}$$

The stability of these equilibrium solutions will be examined in the next section. The examples of equilibrium situation for given flux function $\psi$ are shown in Figs. 3 and 4. The magnetic lines of force used in above calculation are also shown in Fig. 2.

## 4. Stability of the low-$\beta$ plasma equilibrium in the Heliotron field

The stability of a hydromagnetic field in static equilibrium can be determined by an energy principle formalism which was investigated by I.B. Bernstein et. al. This principle shows if $\delta W$ can be made negative, then the system of our interest is unstable, where $\delta W$ is given by

$$\delta W=1/2\int d\tau\,[Q^2/\mu_0-\boldsymbol{j}\cdot\boldsymbol{Q}\times\xi+p(\text{div}\,\xi)^2+(\text{div}\,\xi)(\xi\cdot\text{grad}\,p)] \tag{4-1}$$

where

$$\boldsymbol{Q}=\text{rot}(\xi\times\boldsymbol{B})$$

In order to calculate the sign of $\delta W$, we introduce the curvilinear coordinate system $(\psi, \theta, \chi)$. In this system the fluxfunction $\psi$ and the magnetic potential $\psi$ are defined respectively as,

$$\psi=rA_\theta=B_0r\left[1/2r+\alpha\frac{I_1(kr)}{k}\cos(kz)\right]$$

$$\chi=B_0\left[z+\alpha\frac{I_0(kr)}{k}\sin(kz)\right] \tag{4-2}$$

According to the theory of the curvilinear coordinate, the volume element, gradient operator and divergence are given respectively by

$$d\tau=h_\psi h_\theta h_\chi d\psi d\theta d\chi=\frac{1}{B^2}d\psi\, d\theta d\chi, \tag{4-3}$$

$$\begin{aligned}\text{grad} &= e_\phi \frac{1}{h_\phi}\frac{\partial}{\partial\phi} + e_\theta \frac{1}{h_\theta}\frac{\partial}{\partial\theta} + e_\chi \frac{1}{h_\chi}\frac{\partial}{\partial\chi} \\ &= rBe_\phi \frac{\partial}{\partial\phi} + \frac{1}{r} e_\theta \frac{\partial}{\partial\theta} + Be_\chi \frac{\partial}{\partial\chi},\end{aligned} \tag{4-4}$$

$$\text{div } A = \frac{1}{B^2}\left[\frac{\partial}{\partial\phi}\left(A_\phi \frac{r}{B}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{rB}A_\theta\right) + \frac{\partial}{\partial\chi}\left(\frac{1}{B}A\chi\right)\right] \tag{4-5}$$

where

$$e_\phi = \text{grad } \phi/|\text{grad } \phi|,\ e_\theta = \text{grad } \theta/|\text{grad}\theta|.\ e_\chi = \text{grad } \chi/|\text{grad } \chi|$$

In this coordinate system, the necessary and sufficient conditions given by eqs. (3–7) and (3–8) can be rewritten as

$$\partial p/\partial\chi = 0 \tag{4-6}$$

$$\int\left(\frac{\partial}{\partial\phi} B \frac{\partial}{\partial\theta} p/B^2\right) d\chi = 0 \tag{4-7}$$

Then we consider only the equilibrium such that

$$\partial p/\partial\chi = 0 \text{ and } \partial p/\partial\theta = 0 \tag{4-8}$$

Now we try to minimize $\delta W$ by means of estimating each term. The terms of grad $p$ and $Q$ are the first order quantities in respect to $\xi$. The terms of $\boldsymbol{j}\cdot\boldsymbol{Q}\times\xi$, $p(\text{div }\xi)$ and $(\text{div }\xi)$ $(\xi$ grad $p)$ are the third order quantities in respect to $\xi$.

Examination of the above integration shows that the term $|Q|^2$ is the second order in $\xi$. The displacement $\xi$ which makes $Q$ to be zero is of interest to us. Physically these displacements are those which do not change the Heliotron magnetic field; this is the so called interchange mode.

Hence in the following we can determine the stability such as

$$\boldsymbol{Q} = \text{rot } (\xi\times B) = 0 \tag{4-9}$$

or

$$\begin{aligned} Q &= e_\phi \frac{B}{r}\frac{\partial}{\partial\chi}(r\xi_\phi B) + e_\theta r B^2 \frac{\partial}{\partial\chi}\left(\frac{\xi_\theta}{r}\right) \\ &\quad + e_\chi B\left[\frac{\partial}{\partial\phi}(r\xi_\phi B) - \frac{\partial}{\partial\theta}\left(\frac{\xi_\theta}{r}\right)\right] = 0\end{aligned} \tag{4-10}$$

Taking the components of eq. (4–10), we get

$$\begin{aligned} &\frac{\partial}{\partial\chi}(r\xi_\phi B) = 0, \qquad \frac{\partial}{\partial\chi}\left(\frac{1}{r}\xi_\theta\right) = 0 \\ &\frac{\partial}{\partial\phi}(r\xi_\phi B) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\xi_\theta\right) = 0\end{aligned} \tag{4-11}$$

On substituting eq. (4–11) into eq. (4–1), we find

$$\delta W = \tfrac{1}{2}\int d\tau\ [\gamma p(\text{div }\xi)^2 + (\text{div }\xi)\ (\xi\cdot\text{grad } p)] \tag{4-12}$$

where

$$\begin{aligned} &\gamma p(\text{div }\xi)^2 + (\text{div }\xi)\ (\xi\cdot\text{grad } p) \\ &= \text{div }\xi\left[\gamma p B^2\left\{\frac{\partial}{\partial\phi}\left(\xi_\phi\frac{r}{B}\right) + \frac{\partial}{\partial\theta}\left(\xi_\theta\frac{1}{rB^2}\right) + \frac{\partial}{\partial\chi}\left(\xi_\chi\frac{1}{B}\right)\right\}\right.\end{aligned}$$

$$+\xi_\phi rB\frac{\partial}{\partial\psi}p\Big] \tag{4-13}$$

Now we consider the simple but important disturbance which is $\xi_\theta=\xi_\xi=0$. Physically this disturbance is only perpendicular to the Heliotron megnetic lines of force, namely the disturbance is

$$r\xi_\phi B=f(\theta) \tag{4-14}$$

Using eq. (4-17), the change of potential energy due to perturbation is given by

$$\delta W(\psi)=\int d\chi\Big[\gamma pB^2\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)+\frac{\partial}{\partial\psi}p\Big]\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big) \tag{4-15}$$

Then stability condition of equilibrium state is

$$\delta W(\psi)=\int d\chi\Big[\gamma pB^2\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)+\frac{\partial}{\partial\psi}p\Big]\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)>0 \tag{4-16}$$

We introduce F such as

$$F=\Big[\gamma pB^2\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)+\frac{\partial}{\partial\psi}p\Big]\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big) \tag{4-17}$$

Within separatrix F takes positive or negative value along the Heliotron magnetic lines of force, because $\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)$ becomes negative under the positive coils but positive under the negative coils.

In general the plasma confined in the magnetic field decreases in the outer direction, then $\frac{\partial}{\partial\psi}p$ takes a negative value. Near the axis of negative coils $\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)$ has positive value and so $F$ becomes positive only if the next inequality is satisfied;

$$\gamma pB^2\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)+\frac{\partial}{\partial\psi}p>0 \tag{4-18}$$

Now we examine whether the equilibrium obtained in previous section is stable or not. On substituting eq. (3-18) into eq. (4-18) we get

$$\frac{\gamma}{n}\psi_0\Big(1-\frac{\psi}{\psi_0}\Big)B^2\frac{\partial}{\partial\psi}\Big(\frac{1}{B^2}\Big)-1>0 \tag{4-19}$$

where $\gamma$ is the ratio of specific heats, and $\psi_0$ is the flux function through the neutral lines of the Heliotron field. In the case of $kr=0$ and $\cos(kz)=1$, we can easily calculate the value of $\gamma/n$ so as to satisfy eq. (4-19); $\gamma/n<1.51$ and also for $kr=2$, we get $\gamma/n>1.87$.

Therefore we have a low-$\beta$ equilibrium state which satisfies the stability condition.

## 5. Conclusion.

Equilibrium and stability problems of plasma in the Heliotron magnetic field are discussed under the assumption that the plasma behaves such that the hydromagnetic fluid and pressure is much less than magnetic pressure. When the plasma carries only an azimuthal $j_\theta$ current, both an equilibrium plasma pressure and azimuthal current can be expressed respectively in the forms,

$$p=\zeta_0\frac{\psi_0}{n}\left(1-\frac{\psi}{\psi_0}\right)^n$$

$$j_\theta=-\zeta_0 r\left(1-\frac{\psi}{\psi_0}\right)^{n-1}$$

where $\psi=rA_\theta$, $A_\theta$ is vector potential and $\psi_0$ is flux function through neutral lines.

So far as stability problem is concerned, the stability condition for the flute type instability due to the $\xi_\psi=f(\theta)rB$, $\xi_\theta=\xi_\chi=0$ perturbation is given as follows;

$$\delta W(\psi)=\int d\chi\frac{\partial}{\partial\psi}\left(\frac{1}{B^2}\right)\left[\gamma pB^2\frac{\partial}{\partial\psi}\left(\frac{1}{B^2}\right)+\frac{\partial}{\partial\psi}p\right]>0$$

The more severe condition is

$$\gamma pB^2\frac{\partial}{\partial\psi}\left(\frac{1}{B^2}\right)+\frac{\partial}{\partial\psi}p>0$$

for all $\chi$.

The equilibrium state satisfying the above condition takes the next inequality; $\gamma/n>1.51$ for $kr=0$.

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