Influence of Couple Stresses on Stress Distributions in Rectangular Specimens Compressed Between Rigid Rough Platens

By

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The present paper is concerned with the influence of couple stresses and the Poisson's ratios on the stress distributions in rectangular specimens compressed between rigid rough platens. An exact solution was obtained by the Fourier expansion method.

The conclusions are as follows.

1) As the material parameter for bending rigidity in the couple stress theory approaches to zero, stresses in the specimen become closer to those in the classical theory.

2) The influence of couple stresses is limited near the boundaries of the specimen and fades out rapidly as it goes away from the boundaries.

3) The larger the material parameter for bending rigidity becomes, the more uniform stresses are expected to develop in the specimen.

4) The magnitude of shear stress acting on the perpendicular plane to the specimen axis is in general larger than that of shear stress acting on the plane parallel to the specimen axis. The shear stress in the classical theory approximately falls between the above two.

5) The Poisson's ratio has predominant influence on the stress distributions throughout the specimen.

6) The larger the Poisson's ratio is, the larger the magnitude of stresses becomes. The smaller the Poisson's ratio is, the more uniform stress distributions are expected.

7) As the Poisson's ratio increases, the apparent Young's modulus decreases. The larger the material parameter for bending rigidity is, the more rapidly the apparent Young's modulus decreases.

8) The influence of the Poisson's ratio and couple stresses on the apparent Young's modulus becomes less dominant as the height to width ratio of the specimen increases.

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1. Introduction

In the investigation of actual mechanical behaviour of rock-like materials, it is desirable to develop the homogeneous state of stress or strain throughout the testing specimen. The direct pressure loading by air or oil is suitable for the purpose. However, it may be impracticable to construct such type of loading equipment with ample capacity for testing the moderate size of specimen of rock-like materials. In practical tests, we are forced to use loading equipment which transmits the force, actually the displacement, to the specimen by solid material, i. e. loading platen, even if the influence of the end constraint of the specimen due to friction between the specimen and the platen on the stress distributions, that is inhomogeneity of the state of stress, is inevitable even in the so-called uniaxial compression test. Therefore, it is of fundamental importance to know the actual stress distribution in the specimen in order to investigate the mechanical behaviour, especially the strength and the mechanism of failure.

The state of stresses in the specimen may be affected by it's own material properties such as the Poisson's ratio and internal structures in addition to the boundary constraints by the loading equipment.

In the previous papers[1)], a thorough discussion was developed on the influence of the end constraint of the specimen by the loading platen including the partial slippage and the Poisson's ratio on the stress distributions of the specimen.

The present paper, based on the couple stress theory, presents the influence of internal structures of the specimen on the stress distributions of the specimen compressed between rigid rough platens, since the mechanical behaviour of the materials with internal structures may be expected to be explained to some extent by the couple stress theory of Mindlin's[2)].

2. Description of Problem

The specimen with $2a$ and $2b$ in width and in height, respectively, is compressed between parallel flat loading platens as shown in Fig. 1. The specimen is assumed to obey the couple stress theory of elasticity in the state of plane strain. The platen is assumed rigid and rough enough to completely prevent the specimen from slipping. The rectangular Cartesian coordinates are taken as a reference frame.

According to Mindlin[2)], the fundamental equations of the couple stress theory in plane strain are obtained as follows.

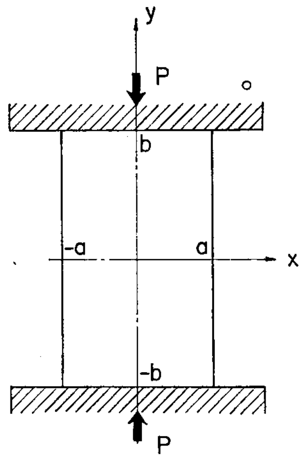


Fig. 1. Specimen and coordinates.

a) Kinematical relations

In the rectangular Cartesian coordinates x_α ($\alpha=1, 2$, i.e. $x=x_1$, $y=x_2$ in Fig. 1), strains $d_{\alpha\beta}$, rotation ω_3 and curvatures $\kappa_{3\alpha}$ are expressed by displacements u_α as follows.

$$d_{\alpha\beta}=u_{(\alpha,\beta)}=\frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}) \quad (\alpha,\ \beta=1,\ 2) \tag{2.1}$$

$$\omega_3=u_{[2,1]}=\frac{1}{2}(u_{2,1}-u_{1,2}) \tag{2.2}$$

$$\kappa_{3\alpha}=\omega_{3,\alpha}, \tag{2.3}$$

where $u_{\alpha,\beta}$ means the partial differentiation of u_α by x_β, and $u_{(\alpha,\beta)}$ and $u_{[\alpha,\beta]}$ mean the symmetrical and the antisymmetrical parts of $u_{\alpha,\beta}$, respectively.

b) Constitutive relations

The constitutive relations are

$$d_{\alpha\beta}=\frac{1}{2G}[\tau_{(\alpha\beta)}-\nu\delta_{\alpha\beta}\tau_{\gamma\gamma}] \tag{2.4}$$

$$\kappa_{3\alpha}=\frac{1}{4Gl^2}m_{\alpha 3}, \tag{2.5}$$

where $\tau_{\alpha\beta}$ and $m_{\alpha 3}$ mean the Cauchy stresses and the couple stresses, respectively, and G, ν and l mean the shear modulus, the Poisson's ratio and the material parameter for bending rigidity, respectively, and $\delta_{\alpha\beta}$ is the Kronecker's delta.

c) Equations of equilibrium

Disregarding the body force and body couple, the equilibrium equations are expressed as

$$\tau_{\beta\alpha,\beta}=0 \tag{2.6}$$

$$m_{\alpha 3,\alpha}+\varepsilon_{3\alpha\beta}\tau_{\alpha\beta}=0, \tag{2.7}$$

where $\varepsilon_{3\alpha\beta}$ means the permutation symbol.

d) Compatibility conditions

Compatibility conditions are

$$\varepsilon_{3\alpha\beta}\varepsilon_{3\delta\gamma}d_{\beta\gamma,\alpha\delta}=0 \tag{2.8}$$

$$\varepsilon_{3\alpha\beta}\kappa_{\alpha,\beta}=0, \tag{2.9}$$

which are expressed in terms of stresses as follows,

$$\varepsilon_{3\alpha\gamma}\varepsilon_{3\beta\delta}\tau_{(\alpha\beta),\gamma\delta}-\nu\nabla^2\tau_{\gamma\gamma}=0 \tag{2.10}$$

$$\varepsilon_{3\alpha\beta}m_{3\alpha,\beta}=0 \tag{2.11}$$

$$m_{\alpha 3}=\varepsilon_{3\beta\gamma}\tau_{(\alpha\gamma),\beta}+\nu\varepsilon_{3\alpha\beta}\tau_{\gamma\gamma,\beta}, \tag{2.12}$$

where $\nabla^2=\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}$. The two equations of Eqs. (2.10), (2.11) and (2.12) are independent.

e) Stress functions

Stresses are assumed to be expressed by such two potential functions ϕ and ψ as

$$\tau_{\alpha\beta}=\varepsilon_{3\alpha\gamma}\varepsilon_{3\beta\delta}\phi_{,\gamma\delta}+\varepsilon_{3\gamma\alpha}\psi_{,\beta\gamma} \tag{2.13}$$

$$m_{\alpha 3}=\psi_{,\alpha}. \tag{2.14}$$

The functions must satisfy the following differential equations in order to satisfy the equilibrium equations and compatibility conditions and *vice versa*,

$$\nabla^4\phi=0 \tag{2.15}$$

$$(l^2\nabla^2-1)\nabla^2\psi=0 \tag{2.16}$$

$$(l^2\nabla^2-1)\psi_{,\alpha}=2l^2(1-\nu)\varepsilon_{3\alpha\beta}\delta_{\beta\gamma}\nabla^2\phi_{,\gamma}. \tag{2.17}$$

When $l=0$, all the relations mentioned above reduce to those of the classical theory of elasticity.

The solution of the problem of the couple stress theory of elasticity in plane strain is obtained by solving the field equations (2.15) to (2.17) with appropriate boundary conditions.

The boundary conditions of the present problem are as follows,

$$\left.\begin{aligned} &u(x,\pm b)=0\\ &v(x,\pm b)=\mp v_0(=\text{const.})\\ &\kappa_y(x,\pm b)=0\\ &\sigma_x(\pm a,y)=0\\ &\tau_{xy}(\pm a,y)=0\\ &m_x(\pm a,y)=0.\end{aligned}\right\} \tag{2.18}$$

Instead of the conditions $v(x, \pm b) = \mp v_0$ and $\kappa_y(x, \pm b) = 0$, we may choose the conditions $v_{,x}(x, \pm b) = 0$ and $\int_{-a}^{a} \sigma_y dx = -P(=\text{const.})$, and $\kappa_y(x, \pm b) = 0$ or $\omega_3(x, \pm b) = 0$, although we did not use them in the present paper. In the above expressions, notations are redefined u, v, κ_y, σ_x, τ_{xy} and m_x for the components of displacement vectors in x and y directions, the component of the curvatures in y direction, the normal stress, the shear stress and the couple stress acting on the plane $x=$const., respectively.

3. Analytical Solutions

The general solutions of the field equations (2.15) and (2.16) are obtained in the rectangular Cartesian coordinates as follows,

$$\phi = \sum_{n=1}^{\infty} \left(A_n \begin{matrix} \sinh \alpha_n y \\ \cosh \alpha_n y \end{matrix} + B_n y \begin{matrix} \cosh \alpha_n y \\ \sinh \alpha_n y \end{matrix} \right) \begin{matrix} \sin \alpha_n x \\ \cos \alpha_n x \end{matrix}$$

$$+ \sum_{m=1}^{\infty} \left(A_m' \begin{matrix} \sinh \beta_m x \\ \cosh \beta_m x \end{matrix} + B_m' x \begin{matrix} \cosh \beta_m x \\ \sinh \beta_m x \end{matrix} \right) \begin{matrix} \sin \beta_m y \\ \cos \beta_m y \end{matrix}$$

$$\psi = \sum_{n=1}^{\infty} \left(C_n \begin{matrix} \sinh \alpha_n y \\ \cosh \alpha_n y \end{matrix} \right) \begin{matrix} \sin \alpha_n x \\ \cos \alpha_n x \end{matrix} + \sum_{m=1}^{\infty} \left(C_m' \begin{matrix} \sinh \beta_m x \\ \cosh \beta_m x \end{matrix} \right) \begin{matrix} \sin \beta_m y \\ \cos \beta_m y \end{matrix}$$

$$+ \sum_{n=1}^{\infty} \left(D_n \begin{matrix} \sinh \gamma_n y \\ \cosh \gamma_n y \end{matrix} \right) \begin{matrix} \sin \alpha_n x \\ \cos \alpha_n x \end{matrix} + \sum_{m=1}^{\infty} \left(D_m' \begin{matrix} \sinh \gamma_m x \\ \cosh \gamma_m x \end{matrix} \right) \begin{matrix} \sin \beta_m y \\ \cos \beta_m y \end{matrix},$$

where $\alpha_n = \frac{n\pi}{a}$, $\beta_m = \frac{m\pi}{b}$, $\gamma_n = \sqrt{\alpha_n^2 + \frac{1}{l^2}}$, $\gamma_m = \sqrt{\beta_m^2 + \frac{1}{l^2}}$, and A_n, B_n,$\cdots$ are constants to be determined.

In the present problem, the material is assumed centrosymmetric and the boundary conditions are symmetric with respect to x and y axes, so the stress functions are reduced to

$$\phi = \sum_{n=1,3,\cdots}^{\infty} (A_n \cosh \alpha_n y + B_n \alpha_n y \sinh \alpha_n y) \cos \alpha_n x$$

$$+ \sum_{m=1,3,\cdots}^{\infty} (A_m' \cosh \beta_m x + B_m' \beta_m x \sinh \beta_m x) \cdot \cos \beta_m y \tag{3.1}$$

$$\psi = \sum_{n=1,3,\cdots}^{\infty} C_n \sinh \alpha_n y \sin \alpha_n x + \sum_{m=1,3,\cdots}^{\infty} C_m' \sinh \beta_m x \sin \beta_m y$$

$$+ \sum_{n=1,3,\cdots}^{\infty} D_n \sinh \gamma_n y \sin \alpha_n x + \sum_{m=1,3,\cdots}^{\infty} D_m' \sinh \gamma_m x \sin \beta_m y. \tag{3.2}$$

Substituting Eqs. (3.1) and (3.2) into Eqs. (2.17), the following relations are obtained

$$\left.\begin{aligned} C_n &= -4(1-\nu) l^2 \alpha_n^2 B_n \\ C_m' &= 4(1-\nu) l^2 \beta_m^2 B_m'. \end{aligned}\right\} \tag{3.3}$$

Stresses of Eqs. (2.13) and (2.14) are expressed in the extended form as

$$\sigma_x=\phi,_{yy}-\psi,_{xy}$$
$$=\sum_{n=1,3,\cdots}^{\infty}\alpha_n^2\{(A_n+2B_n)\cosh\alpha_n y+B_n\alpha_n y\sinh\alpha_n y\}\cos\alpha_n x$$
$$-\sum_{m=1,3,\cdots}^{\infty}\beta_m^2\{A_m'\cosh\beta_m x+B_m'\beta_m x\sinh\beta_m x\}\cos\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}C_n\alpha_n^2\cosh\alpha_n y\cos\alpha_n x-\sum_{m=1,3,\cdots}^{\infty}C_m'\beta_m^2\cosh\beta_m x\cos\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}D_n\alpha_n\gamma_n\cosh\gamma_n y\cos\alpha_n x-\sum_{m=1,3,\cdots}^{\infty}D_m'\beta_m\gamma_m\cosh\gamma_m x\cos\beta_m y \tag{3.4}$$

$$\sigma_y=\phi,_{xx}+\psi,_{xy}$$
$$=-\sum_{n=1,3,\cdots}^{\infty}\alpha_n^2(A_n\cosh\alpha_n y+B_n\alpha_n y\sinh\alpha_n y)\cos\alpha_n x$$
$$+\sum_{m=1,3,\cdots}^{\infty}\beta_m^2\{(A_m'+2B_m')\cosh\beta_m x+B_m'\beta_m x\sinh\beta_m x\}\cos\beta_m y$$
$$+\sum_{n=1,3,\cdots}^{\infty}C_n\alpha_n^2\cosh\alpha_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}C_m'\beta_m^2\cosh\beta_m x\cos\beta_m y$$
$$+\sum_{n=1,3,\cdots}^{\infty}D_n\alpha_n\gamma_n\cosh\gamma_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}D_m'\beta_m\gamma_m\cosh\gamma_m x\cos\beta_m y \tag{3.5}$$

$$\tau_{xy}=-\phi,_{xy}-\psi,_{yy}$$
$$=\sum_{n=1,3,\cdots}^{\infty}\alpha_n^2\{(A_n+B_n)\sinh\alpha_n y+B_n\alpha_n y\cosh\alpha_n y\}\sin\alpha_n x$$
$$+\sum_{m=1,3,\cdots}^{\infty}\beta_m^2\{(A_m'+B_m')\sinh\beta_m x+B_m'\beta_m x\cosh\beta_m x\}\sin\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}C_n\alpha_n^2\sinh\alpha_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}C_m'\beta_m^2\sinh\beta_m x\sin\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}D_n\gamma_n^2\sinh\gamma_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}D_m'\beta_m^2\sinh\gamma_m x\sin\beta_m y \tag{3.6}$$

$$\tau_{yx}=-\phi,_{xy}+\psi,_{xx}$$
$$=\sum_{n=1,3,\cdots}^{\infty}\alpha_n^2\{(A_n+B_n)\sinh\alpha_n y+B_n\alpha_n y\cosh\alpha_n y\}\sin\alpha_n x$$
$$+\sum_{m=1,3,\cdots}^{\infty}\beta_m^2\{(A_m'+B_m')\sinh\beta_m x+B_m'\beta_m x\cosh\beta_m x\}\sin\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}C_n\alpha_n^2\sinh\alpha_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}C_m'\beta_m^2\sinh\beta_m x\sin\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}D_n\alpha_n^2\sinh\gamma_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}D_m'\gamma_m^2\sinh\gamma_m x\sin\beta_m y \tag{3.7}$$

$$m_x=\psi_{,x}$$
$$=\sum_{n=1,3,\cdots}^{\infty} C_n\alpha_n \sinh\alpha_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} C_m'\beta_m\cosh\beta_m x\sin\beta_m y$$
$$+\sum_{n=1,3,\cdots}^{\infty} D_n\alpha_n\sinh\gamma_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} D_m'\gamma_m\cosh\gamma_m x\sin\beta_m y \tag{3.8}$$

$$m_y=\psi_{,y}$$
$$=\sum_{n=1,3,\cdots}^{\infty} C_n\alpha_n\cosh\alpha_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} C_m'\beta_m\sinh\beta_m x\cos\beta_m y$$
$$+\sum_{n=1,3,\cdots}^{\infty} D_n\gamma_n\cosh\gamma_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} D_m'\beta_m\sinh\gamma_m x\cos\beta_m y. \tag{3.9}$$

The displacements, the rotation and the curvatures are obtained as follows

$$Eu=\sum_{n=1,3,\cdots}^{\infty}\alpha_n[\{(1+\nu)A_n+2(1-\nu^2)B_n\}\cosh\alpha_n y+(1+\nu)B_n\alpha_n y\sinh\alpha_n y]\sin\alpha_n x$$
$$+\sum_{m=1,3,\cdots}^{\infty}\beta_m[\{-(1+\nu)A_m'+(1-\nu-2\nu^2)B_m'\}\sinh\beta_m x$$
$$-(1+\nu)B_m'\beta_m x\cosh\beta_m x]\cos\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}(1+\nu)C_n\alpha_n\cosh\alpha_n y\sin\alpha_n x-\sum_{m=1,3,\cdots}^{\infty}(1+\nu)C_m'\beta_m\sinh\beta_m x\cos\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}(1+\nu)D_n\gamma_n\cosh\gamma_n y\sin\alpha_n x-\sum_{m=1,3,\cdots}^{\infty}(1+\nu)D_m'\beta_m\sinh\gamma_m x\cos\beta_m y \tag{3.10}$$

$$Ev=\sum_{n=1,3,\cdots}^{\infty}\alpha_n[\{-(1+\nu)A_n+(1-\nu-2\nu^2)B_n\}\sinh\alpha_n y$$
$$-(1+\nu)B_n\alpha_n y\cosh\alpha_n y]\cos\alpha_n x$$
$$+\sum_{m=1,3,\cdots}^{\infty}\beta_m[\{(1+\nu)A_m'+2(1-\gamma^2)B_m'\}\cosh\beta_m x+(1+\nu)B_m'\beta_m x\sinh\beta_m x]\sin\beta_m y$$
$$+\sum_{n=1,3,\cdots}^{\infty}(1+\nu)C_n\alpha_n\sinh\alpha_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}(1+\nu)C_m'\beta_m\cosh\beta_m x\sin\beta_m y$$
$$+\sum_{n=1,3,\cdots}^{\infty}(1+\nu)D_n\alpha_n\sinh\gamma_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty}(1+\nu)D_m'\gamma_m\cosh\gamma_m x\sin\beta_m y \tag{3.11}$$

$$2E\omega_3=-\sum_{n=1,3,\cdots}^{\infty}4(1-\nu^2)\alpha_n^2B_n\sinh\alpha_n y\sin\alpha_n x$$
$$+\sum_{m=1,3,\cdots}^{\infty}4(1-\nu^2)\beta_m^2B_m'\sinh\beta_m x\sin\beta_m y$$
$$-\sum_{n=1,3,\cdots}^{\infty}(1+\nu)(\alpha_n^2-\gamma_n^2)D_n\sinh\gamma_n y\sin\alpha_n x$$
$$-\sum_{m=1,3,\cdots}^{\infty}(1+\nu)(\beta_m^2-\gamma_m^2)D_m'\sinh\gamma_m x\sin\beta_m y \tag{3.12}$$

$$4Gl^2\kappa_x=$$

$$\sum_{n=1,3,\cdots}^{\infty} C_n\alpha_n \sinh\alpha_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} C_m'\beta_m\cosh\beta_m x\sin\beta_m y$$

$$+\sum_{n=1,3,\cdots}^{\infty} D_n\alpha_n\sinh\gamma_n y\cos\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} D_m'\gamma_m\cosh\gamma_m x\sin\beta_m y \qquad (3.13)$$

$$4Gl^2\kappa_y=$$

$$\sum_{n=1,3,\cdots}^{\infty} C_n\alpha_n\cosh\alpha_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} C_m'\beta_m\sinh\beta_m x\cos\beta_m y$$

$$+\sum_{n=1,3,\cdots}^{\infty} D_n\gamma_n\cosh\gamma_n y\sin\alpha_n x+\sum_{m=1,3,\cdots}^{\infty} D_m'\beta_m\sinh\gamma_m x\cos\beta_m y, \qquad (3.14)$$

where the terms for the rigid body displacements are eliminated. Strains are also easily obtained by substituting Eqs. (3.4) to (3.7) into Eq. (2.4), but are not expressed here.

The constants A_n, B_n,……will be determined from the six boundary conditions (2.18) and the two compatibility relations (2.17), i. e. Eq. (3.3) in the final form.

The substitution of the appropriate equations of Eqs. (3.4) to (3.14) into the boundary conditions (2.18) leads to the following system of linear equations with respect to the unknown constants.

$$\beta_m\cosh\beta_m a\,A_m'+\beta_m^2 a\sinh\beta_m a\,B_m'$$
$$+\beta_m\cosh\beta_m a\,C_m'+\gamma_m\cosh\gamma_m a\,D_m'=0 \qquad (3.15)$$

$$\beta_m\cosh\beta_m a\,C_m'+\gamma_m\cosh\gamma_m a\,D_m'=0 \qquad (3.16)$$

$$\alpha_n\cosh\alpha_n b\,A_n+\alpha_n\{2(1-\nu)\cosh\alpha_n b+\alpha_n b\sin h\alpha_n b\}B_n$$
$$-\alpha_n\cosh\alpha_n b\,C_n-\gamma_n\cosh\gamma_n b\,D_n=0 \qquad (3.17)$$

$$\alpha_n\cosh\alpha_n b\,C_n+\gamma_m\cosh\gamma_m b\,D_n=0 \qquad (3.18)$$

$$\sum_{n=1,3,\cdots}^{\infty}(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}\frac{\alpha_n^2}{\alpha_n^2+\beta_m^2}\frac{2\alpha_n}{b}\Big[\cosh\alpha_n b\,A_n+(\alpha_n b\sinh\alpha_n b$$
$$+\frac{2\beta_m}{\alpha_n^2+\beta_m^2}\cosh\alpha_n b)B_n\Big]$$
$$+\beta_m^2[\sinh\beta_m a\,A_m'+(\sinh\beta a+\beta_m a\cosh\beta_m a)B_m']$$
$$-\sum_{n=1,3,\cdots}^{\infty}(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}\frac{\alpha_n^2}{\alpha_n^2+\beta_m^2}\frac{2\alpha_n}{b}\cosh\alpha_n b\,C_n+\beta_m^2\sinh\beta_m a\,C_m'$$
$$-\sum_{n=1,3,\cdots}^{\infty}(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}\frac{\gamma_n^2}{\gamma_n^2+\beta^2_m}\frac{2\gamma_n}{b}\cosh\gamma_n b\,D_n+\beta_m^2\sinh\gamma_m a\,D_m'=0$$
$$(3.19)$$

$$-\alpha_n \sinh \alpha_n b\, A_n + \{\alpha_n(1-2\nu)\sinh \alpha_n b - \alpha_n b \cosh \alpha_n b\} B_n$$

$$+\sum_{m=1,3,\cdots}^{\infty}(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}\frac{\alpha_n}{\alpha_n^2+\beta_m^2}\frac{2\beta_m}{a}\Big[\cosh \beta_m a\, A_m'$$

$$+\Big\{2(1-\nu)\cosh \beta_m a + \beta_m a \sinh \beta_m a - \frac{2\beta_m^2}{\alpha_n^2+\beta_m^2}\cosh \alpha_n a\Big\} B_m'\Big]$$

$$+\alpha_n \sinh \alpha_n b\, C_n + \alpha_n \sinh \gamma_n b\, D_n$$

$$+\sum_{m=1,3,\cdots}^{\infty}(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}\frac{\alpha_n}{\alpha_n^2+\beta_m^2}\frac{2\beta_m}{a}\cosh \beta_m a\, C_m'$$

$$+\sum_{m=1,3,\cdots}^{\infty}(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}\frac{\alpha_n}{\alpha_n^2+\gamma_m^2}\frac{2\gamma_m}{a}\cosh \gamma_m a\, D_m'$$

$$=-\frac{4}{n\pi}(-1)^{\frac{n-1}{2}}\frac{E}{1+\nu}v_0. \tag{3.20}$$

In the derivation of the last two equations, we expanded the terms such as $\sinh \beta_m y$, $\cosh \alpha_n y$, ……, $x \sinh \beta_m x$, $y \cosh \alpha_n y$, …… into the Fourier sine or cosine series with a period $4a$ or $4b$, i.e. $\sin \alpha_n x$ or $\cos \beta_n y$, then satisfied the conditions $\tau_{xy}(a, y)=0$ and $v(x, b)=-v_0$ termwise in $\sin \alpha_n x$ and $\cos \beta_n y$, respectively.

All the constants are determined from Eqs. (3.15) to (3.20) and Eq. (3.3). The constant A_n, A_m', C_n, C_m', D_n and D_m' are easily expressed by B_n and B_m' through Eqs. (3.15) to (3.18) and Eq. (3.3), so finally a system of infinite linear equations with respect to B_n and B_m' is obtained. In the calculations, we are forced to truncate the higher terms of the equations because of the limited capacity of computor. In the present computation, we truncated the terms higher than $n=81$ and $m=81$. Even in this computation we must adapt a technique to divide the coefficients of the constants B_n and B_m' so as to make them almost the same order of unity, otherwise the system of the equations could not be solved.

4. Influence of Couple Stresses on Stress Distributions

In the numerical calculations, the material parameter l was chosen as $l/a=0$, 0.1, 0.2, and 0.4, and the Poisson's ratios $\nu=0.1$, 0.2, 0.3 and 0.4. The normalized stresses, i.e. the ratios of stresses σ_y, σ_x, τ_{xy} and τ_{yx} to the average axial stress σ_{y0}, are shown for some typical cases in Figs. 2 to 6.

On the influence of the couple stresses on the stress distributions the following may be concluded.

(1) As expected from the theory, stresses in the couple stress theory become closer to those in the classical theory as material parameter l approaches to zero.

(2) The larger the material parameter becomes, the more uniform stresses are expected to develop.

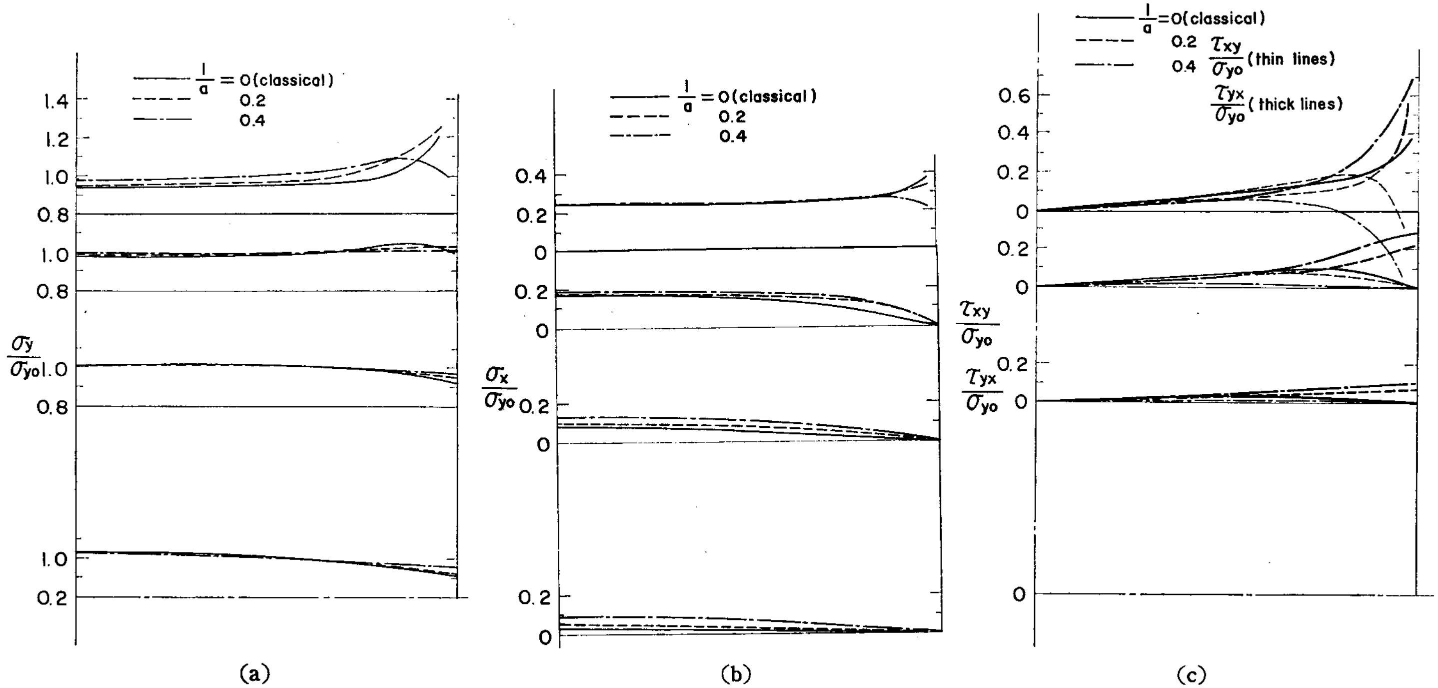


Fig. 2. Influence of couple stresses on normalized stresses with Poisson's ratio $\nu=0.2$ and height to width ratio $b/a=1.0$.

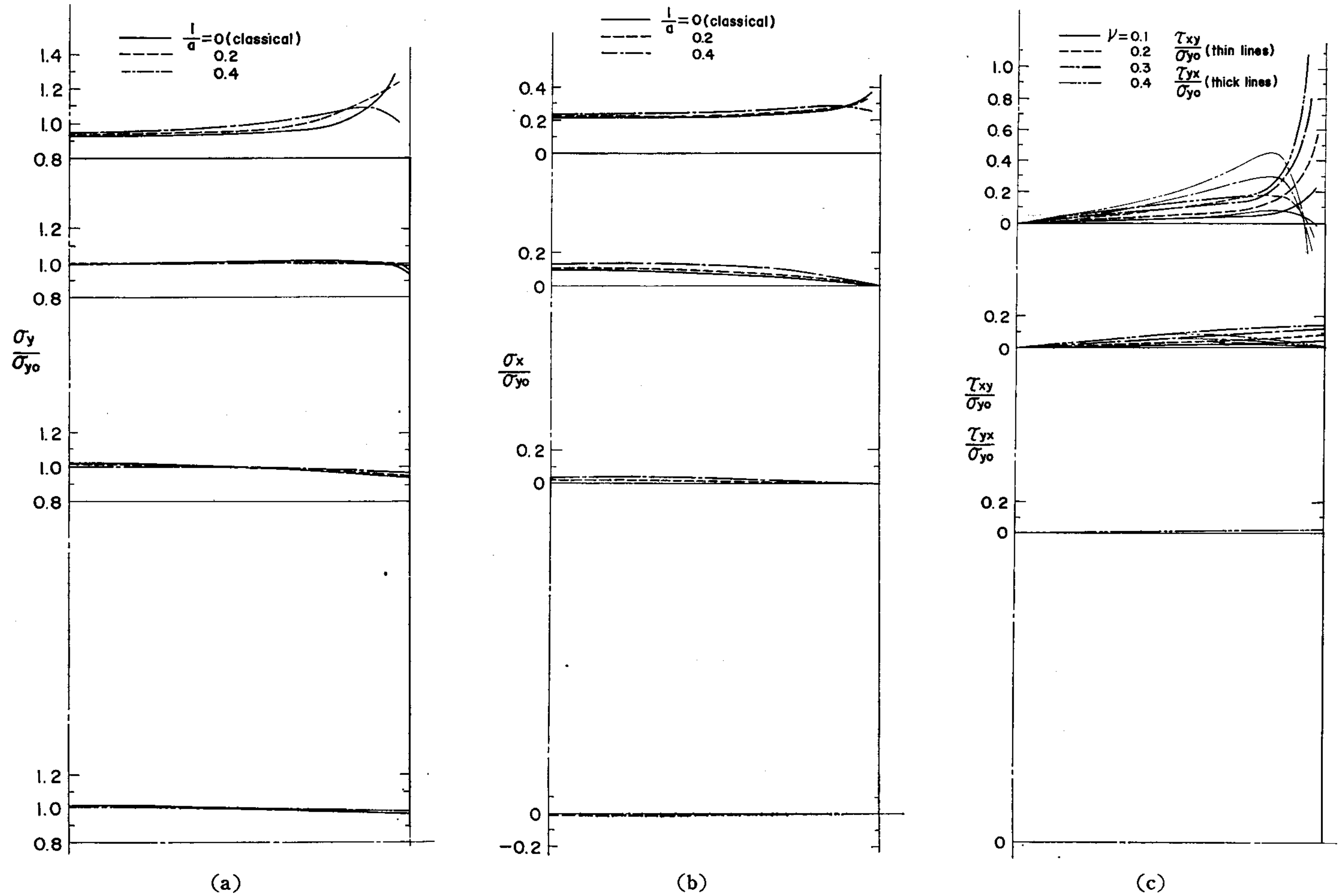
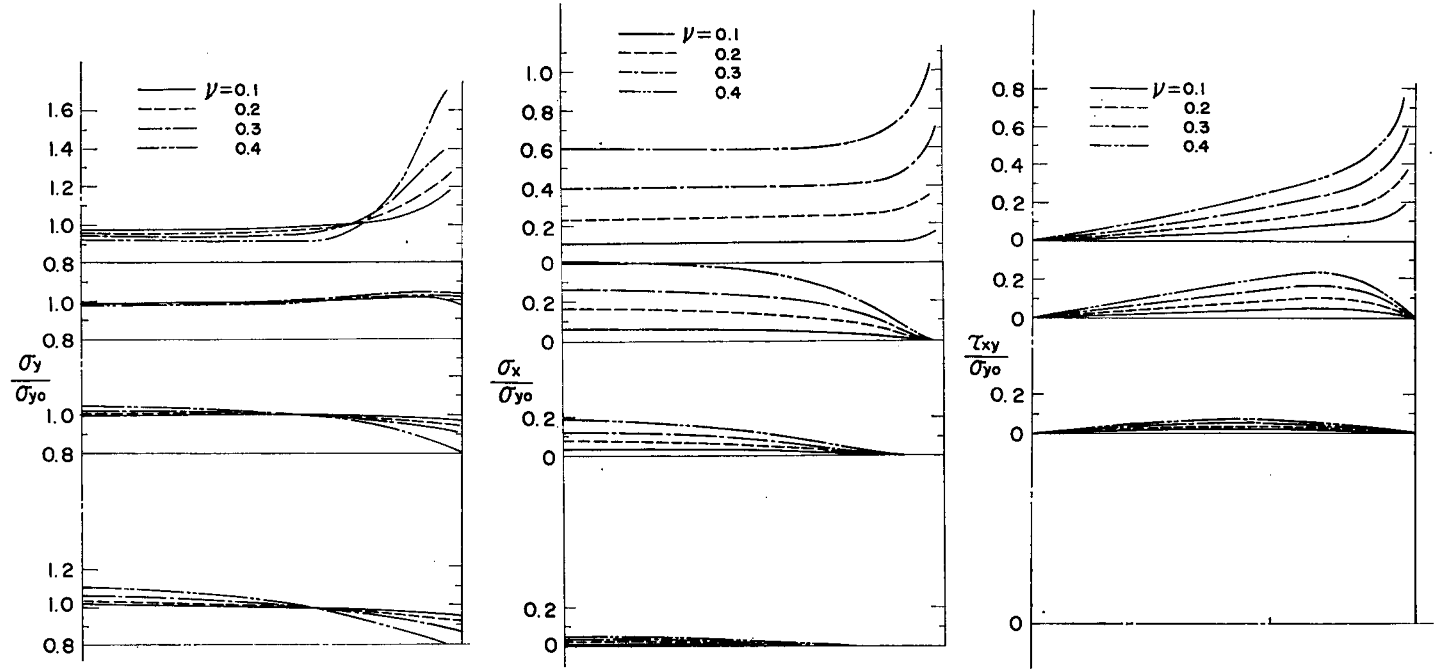


Fig. 3. Influence of couple stresses on normalized stresses with Poisson's ratio $\nu=0.2$ and height to width ratio $b/a=2.0$.



(a) (b) (c)

Fig. 4. Influence of Poisson's ratio on normalized stresses in the classical theory with height to width ratio $b/a=1.0$.

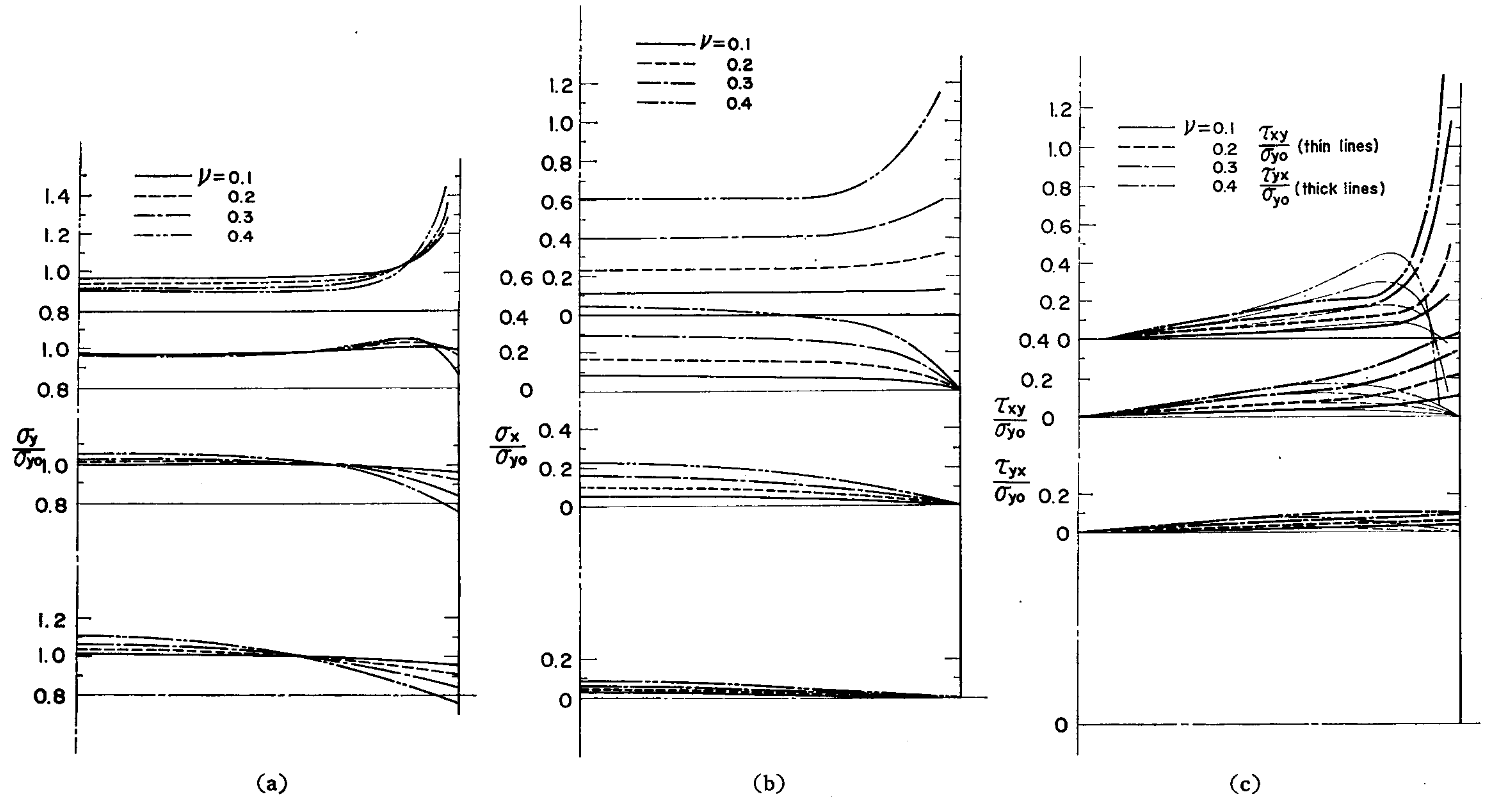


Fig. 5. Influence of Poisson's ratio on normalized stresses in the couple stress theory with $\nu=0.2$ and height to width ratio $b/a=1.0$.

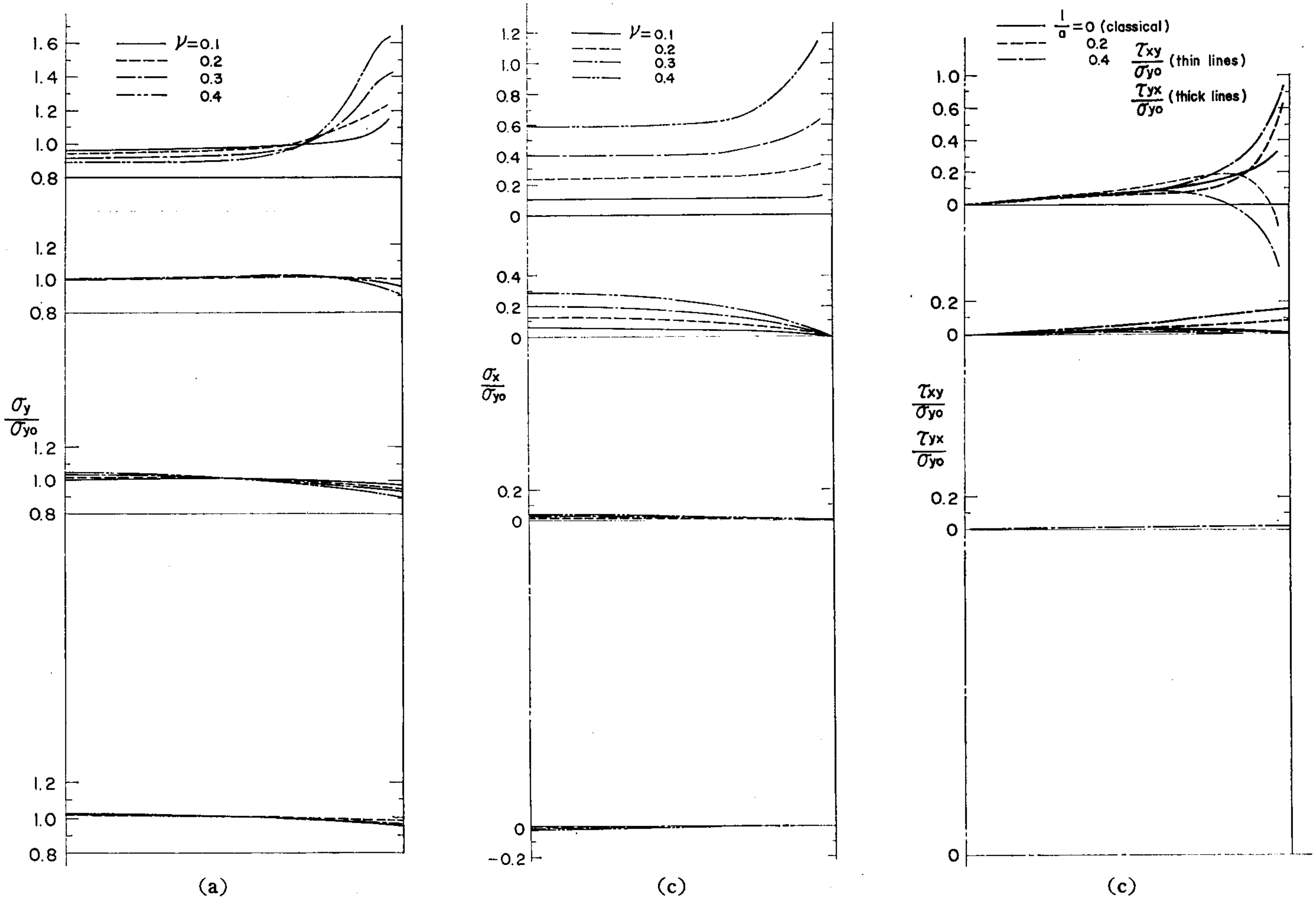


Fig. 6. Influence of Poisson's ratio on normalized stresses in the couple stress theory with $\nu=0.2$ and height to width ratio $b/a=2.0$.

(3) The influence of the couple stresses is limited near the boundaries of the specimen and rapidly fades out as it goes away from the boundaries.

(4) Stresses in the couple stress theory near the corner of the specimen behave differently from those in the classical theory.

(5) The magnitude of the shear stress acting on the perpendicular plane to the specimen axis is larger in general than that of the shear stress acting on the plane parallel to the specimen axis. The shear stress in the classical theory approximately falls between the above two.

On the influence of the Poisson's ratio on the stress distributions the followings may be concluded.

(1) The larger the Poisson's ratio is, the larger the magnitude of stresses becomes.

(2) The smaller the Poisson's ratio is, the more uniform stress distribution is expected.

(3) The magnitude of stress is affected not only by the Poisson's ratio, but also by the material parameter. For the fixed material parameter, the magnitude of stress becomes larger as the Poisson's ratio increases.

(4) Differing from the influence of the couple stresses, the Poisson's ratio has predominant influence on the stresses throughout the specimen.

It is of some interest from the viewpoint of the brittle fracture that there appears a tensile zone in σ_x as the height to width ratio increases.

The influence of the couple stresses and the Poisson's ratio on the apparent Young's modulus E', i. e. the average axial stress divided by the average axial strain, is observed in Fig. 7. As the Poisson's ratio increases the apparent Young's modulus decreases. The larger the material parameter is, the more

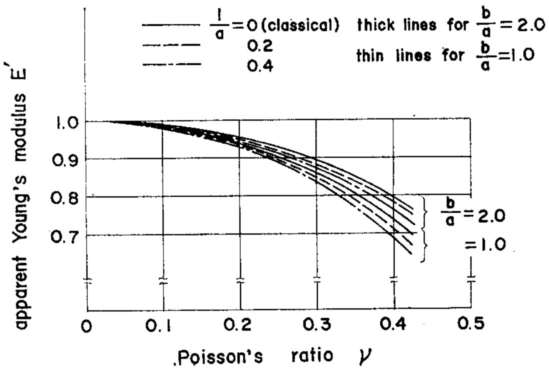


Fig. 7. Influence of couple stresses and Poisson's ratio on the apparent Young's modulus.

rapidly the apparent Young's modulus decreases. As the height to width radio increases, the influence of the Poisson's ratio and the couple stresses on the apparent Young's modulus, as expected, becomes less dominant.

5. Concluding Remarks

As theoretically predicted, the couple stresses have only second order effect on the overall stress distributions. However, the influence on the stresses near the boundaries, especially near the corner, is predominant and may not be disregarded. The results obtained in the present paper may help to interpret the experimental results of rock-like materials.

Reference

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