Matric Operational Calculus and Its Applications (Part II)

By

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This paper describes the fundamental properties of the operational calculus based on the Mikusinsky's method for the matrix functions (sequences and series of operators, operational functions and their derivatives), and then, presents the analysis of the multi-conductor transmission systems for its applications.

1. Introduction

In analysing the physical systems with many variables, matrix algebra provides a systematic method for the manipulation and solution of system equations. We have reported the matric operational calculus based on the Mikusinsky's method and applied it to the study of the linear lamped time-invariant systems[1]. There, the system equations are given by the simultaneous ordinary differential equations and their operational solutions are expressed by the rational functions of the operator s. But in the linear distributed time-invariant systems, typical examples are multi-conductor transmission systems, their properties are expressed by the simultaneous partial differential equations, and their operational equations become the simultaneous ordinary differential equations involving the operator s. Therefore, new mathematical concepts, which are, sequences and series of operators, operational functions and their derivatives are needed.

In this paper we shall try to apply new operational calculus to the study of the linear distributed systems. First, following the Mikusinsky's method[2] we shall present the basic definitions and the fundamental properties. In our case what differs from the scalar functions is that they are subjected by the matrix algebra. Next, using the results, we shall consider the multi-conductor transmission systems. Here, gothic letters represent matrices and sets of them

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are assumed to compose the integer domain.

2. Extension of Functions and Operators

2.1 Discontinuous functions. We shall consider a function $\{F(t)\}$ defined in the interval $0 \leq t < \infty$ satisfying the following conditions.

i) It is sectionally continuous in every finite interval.

ii) The integral $\int_0^t |F(\tau)|d\tau$ has a finite value for every $t>0$.

By the definition of convolution, we have

$$lF=\{1\}\{F(t)\}=\left\{\int_0^t F(\tau)d\tau\right\}.$$

The integral on the right side always represents a continuous function, denoting it by $\{H(t)\}$, we have

$$F=l^{-1}\{H(t)\}=sH$$

therefore $\{F(t)\}$ can be regarded as an operator.

When discontinuous functions $F\equiv\{F(t)\}$ and $G\equiv\{G(t)\}$ satisfy the above conditions, the following definitions are given.

$$F=G \quad \text{if} \quad \left\{\int_0^t F(\tau)d\tau\right\}=\left\{\int_0^t G(\tau)d\tau\right\}$$

$$F\pm G=l^{-1}\left[\left\{\int_0^t F(\tau)d\tau\right\}\pm\left\{\int_0^t G(\tau)d\tau\right\}\right]=l^{-1}\left\{\int_0^t [F(\tau)\pm G(\tau)]d\tau\right\}$$

By the above definition we can prove the formulas

$$\{F(t)\}\pm\{G(t)\}=\{F(t)\pm G(t)\}$$

$$a\{F(t)\}=\{aF(t)\} \qquad a \text{ constant.}$$

For a discontinuous function satisfying the above conditions the following theorem is given.

Theorem 2.1: If a function $\{X(t)\}$ has jumps $\beta_1, \beta_2, \cdots, \beta_n$ at the points $t_1, t_2, \cdots, t_n$, is elsewhere continuous and has a derivative $\{X'(t)\}$ satisfying the above conditions, then

$$s\{X(t)\}=\{X'(t)\}+X(0)+\sum_{\nu=1}^{n}\beta_\nu h^{t_\nu} \tag{2.1}$$

where h^{t_ν} is a translation operator.

2.2 Non integer power of the operator $s-a$. For all positive values λ, power of the operator $s-a$ is defined as

$$(s-a)^{-\lambda}=\left\{\frac{t^{\lambda-1}}{\Gamma(\lambda)}e^{at}\right\} \qquad \lambda>0 \tag{2.2}$$

where $\Gamma(\lambda)$ is the Euler's gamma function.

By the definition of the convolution, for all positive values of λ and μ, we have

$$(s-\alpha)^{-\lambda}(s-\alpha)^{-\mu}=(s-\alpha)^{-\lambda-\mu} \tag{2.3}$$

For all real numbers λ, powers of an operator $s-\alpha$ is defined by (2.2) and

$$(s-\alpha)^0=1,\ (s-\alpha)^{\lambda}=\frac{1}{(s-\alpha)^{-\lambda}}\quad \lambda>0 \tag{2.4}$$

therefore $(s-\alpha)^{-\lambda}$ represents a continuous function in the interval $0\leq t<\infty$ if $\lambda\geq 1$, and a function discontinuous at the point $t=0$ if $0<\lambda<1$, but if $\lambda\leq 0$ it does not representa function.

From (2.2) the following equalities are given.

$$\frac{1}{\sqrt{s+\alpha}}=\left\{\frac{t^{-1/2}}{\Gamma(1/2)}e^{-ct}\right\}=\left\{\frac{1}{\sqrt{\pi t}}e^{-\alpha t}\right\} \tag{2.5}$$

$$\binom{-\beta}{\nu}(s+\alpha)^{-(\nu+\beta)}=\left\{\frac{t^{\beta-1}(-t)^{\nu}}{\Gamma(\beta)\nu!}e^{-\alpha t}\right\}\quad \beta>0,\ \nu \text{ natural} \tag{2.6}$$

3. Sequences and Series of Operators

3.1 Sequences of operators. A sequence of operator A_n is termed convergent if every element $[A_n]_{ij}$ of A_n divided by a suitably chosen non-zero operator q_{ij} becomes a sequence of continuous functions uniformly convergent in every finite interval, and in this case A_n is said to have a limit and is written

$$\lim_{n\to\infty}[A_n]_{ij}=q_{ij}\lim_{n\to\infty}\frac{[A_n]_{ij}}{q_{ij}}. \tag{3.1}$$

Every element of A_n has only one limit, therefore A_n has only one limit.

If sequences of operators A_n and B_n have limits A and B i. e., $\lim_{n\to\infty}A_n=A$ and $\lim_{n\to\infty}B_n=B$, we have the following equalities.

$$\lim_{n\to\infty}(A_n\pm B_n)=A\pm B \tag{3.2}$$

$$\lim_{n\to\infty}A_nB_n=AB \tag{3.3}$$

3.2 Series of operators. For an infinite series of operators $\sum_{n=0}^{\infty}A_n$, if the sequence of partial sums

$$S_n=A_0+A_1+\cdots\cdots+A_n$$

converges to A, it is said to have the sum A and is written

$$\sum_{n=0}^{\infty}A_n=A_0+A_1+\cdots\cdots=A. \tag{3.4}$$

In applications power series of the following form is important

$$\Phi(w)=\alpha_0+\alpha_1 w+\alpha_2 w^2+\cdots\cdots \tag{3.5}$$

where α_0, $\alpha_1,\cdots$ are numerical coefficients and w an operator.

Specially for $w=F$ ($F\equiv\{F(t)\}$: continuous function in the interval $0\leq t<\infty$))

the following theorem is given.

Theorem 3.1 : If the convergence radius of the series

$$\Phi(\lambda)=a_0+a_1\lambda+a_2\lambda^2+\cdots\cdots \quad \lambda \text{ complex} \tag{3.6}$$

is positive, then series

$$\Phi(F)=a_0+a_1F+a_2F^2+\cdots\cdots \tag{3.7}$$

is operationally convergent for every continuous function F, where multiplications in F^2, F^3,…… mean the convolution.

3.3 Powers with arbitrary real exponents. From the binominal theorem for non-integer β and Theorem 3.1, we can define for an operator of the form $1+F(F\equiv\{F(t)\}$ continuous function in the interval $0\leq t<\infty)$ power with any real exponents as

$$(1+F)^\beta=\sum_{\nu=0}^{\infty}\binom{\beta}{\nu}F^\nu \tag{3.8}$$

and by the properties of the binominal expansion, the following relations are given.

$$(1+F)^{\beta_1}(1+F)^{\beta_2}=(1+F)^{\beta_1+\beta_2} \tag{3.9}$$

$$(1+F)^{-\beta}=\frac{1}{(1+F)^\beta} \tag{3.10}$$

Having defined the powers of the operators A and B, and they are commutative, we define the power of their product AB by the formula

$$(AB)^\beta=A^\beta B^\beta. \tag{3.11}$$

Using the equalities (3.8) (3.11) we have the following relations.

$$\frac{1}{(s-a)^\beta}=\frac{1}{s^\beta}\,\frac{1}{(1-al)^\beta}=\sum_{\nu=0}^{\infty}\binom{-\beta}{\nu}(-a)^\nu l^{\nu+\beta}=\left\{\frac{t^{\beta-1}}{\Gamma(\beta)}\sum_{\nu=0}^{\infty}\frac{(at)^\nu}{\nu!}\right\}=\left\{\frac{t^{\beta-1}}{\Gamma(\beta)}e^{at}\right\} \tag{3.12}$$

$$\frac{1}{\sqrt{s^2+a^2}}=l(1+(al)^2)^{-1/2}=\left\{\sum_{\nu=0}^{\infty}(-1)^\nu\frac{a^{2\nu}t^{2\nu}}{2^{2\nu}(\nu!)^2}\right\}=\{J_0(at)\} \tag{3.13}$$

$$\frac{1}{\sqrt{s^2-a^2}}=\{J_0(iat)\} \tag{3.14}$$

$$(\sqrt{s^2+a^2}-s)^n=\left\{\frac{na^n}{t}J_n(at)\right\} \quad n=1,\ 2,\cdots \tag{3.15}$$

4. Operational Functions and Their Derivatives

4.1 Operational functions. Given a function of two variables $F(\lambda,\ t)$ defined for $t\geq0$ and for some values of λ, we shall write it as

$$F(\lambda)=\{F(\lambda,\ t)\}. \tag{4.1}$$

Formula (4.1) defines an operational function which assigns an operator that is a function of the variable t to the value of λ and is called parametric.

An operational function $F(\lambda)$ will be termed continuous in an finite interval if every element $[F(\lambda)]_{ij}$ of $F(\lambda)$ can be represented in that interval as a product of a certain operator q_{ij} and a parametric function $G(\lambda)$ and written

$$[F(\lambda)]_{ij}=q_{ij}[G(\lambda)]_{ij} \quad q_{ij}\neq 0 \tag{4.2}$$

where $[G(\lambda)]_{ij}=\{[G(\lambda,\ t)]_{ij}\}$ is the function of two variables continuous in the domain $D(\lambda\epsilon I,\ 0\leq t<\infty)$.

4.2 Continuous derivative of an operational function. An operational function $F(\lambda)$ will be said to be continuously differentiable in a finite interval I if we can write for every element of it

$$[F(\lambda)]_{ij}=q_{ij}[G(\lambda)]_{ij} \quad q_{ij}\neq 0 \tag{4.3}$$

where q_{ij} is an operator and $[G(\lambda)]_{ij}$ is a parametric function $\{[G(\lambda,\ t)]_{ij}\}$ having a partial derivative $\left\{\frac{\partial}{\partial\lambda}[G(\lambda,\ t)]_{ij}\right\}$ continuous in the domain $D(\lambda\epsilon I,\ 0\leq t<\infty)$. In this case the function $F(\lambda)$ will be said to have in the interval I a continuous derivative $F'(\lambda)$ whose element is

$$[F'(\lambda)]_{ij}=q_{ij}\left\{\frac{\partial}{\partial\lambda}[G(\lambda,\ t)]_{ij}\right\} \quad q_{ij}\neq 0 \tag{4.4}$$

and from this definition $F'(\lambda)$ is determined uniquely.

Continuous derivative of high order of a function $F(\lambda)$ is defined as

$$[F^{(n)}(\lambda)]_{ij}=q_{ij}^{(n)}\left\{\frac{\partial^n}{\partial\lambda^n}[G(\lambda,\ t)]_{ij}\right\} \quad q_{ij}^{(n)}\neq 0 \tag{4.5}$$

where $[G(\lambda)]_{ij}=\{[G(\lambda,\ t)]_{ij}\}$ has n-th partial derivative $\left\{\frac{\partial^n}{\partial\lambda^n}[G(\lambda,\ t)]_{ij}\right\}$ in the domain D.

For the translation operator h^{λ} we can write

$$h^{\lambda}=s^2\{h_1(\lambda,\ t)\}=s^3\{h_2(\lambda,\ t)\} \tag{4.6}$$

where

$$h_1(\lambda,\ t)=\begin{Bmatrix}0 & : 0\leq t<\lambda \\ (t-\lambda)1 : & 0<\lambda<t\end{Bmatrix}, \qquad h_2(\lambda,\ t)=\begin{Bmatrix}0 & : 0\leq t<\lambda \\ \frac{1}{2}(t-\lambda)^2 1 : & 0<\lambda<t\end{Bmatrix}$$

therefore we have

$$(h^{\lambda})'=s^3\left\{\frac{\partial}{\partial\lambda}h_2(\lambda,\ t)\right\}=s^3\{-h_1(\lambda,\ t)\}=-sh^{\lambda}. \tag{4.7}$$

For continuous derivatives of the operational functions the following properties are given.

i) If the function $F(\lambda)$ is constant in a certain interval I, $F'(\lambda)=0$. Conversely if $F'(\lambda)=0$ in I then $F(\lambda)$ is constant.

ii) If the functions $F(\lambda)$ and $G(\lambda)$ have continuous derivatives $F'(\lambda)$ and $G'(\lambda)$ in I then

$$[F(\lambda)\pm G(\lambda)]'=F'(\lambda)\pm G'(\lambda)$$

$$[F(\lambda)G(\lambda)]'=F'(\lambda)G(\lambda)+F(\lambda)G'(\lambda)$$

iii) If the function $F(\lambda)$ has the continuous derivative $F(\lambda)$ in I and C is an arbitrary operator, then

$$[CF(\lambda)]'=CF'(\lambda)$$

A function $F(\lambda)$ is said to be differentiable at a point λ_0 if it can be represented in the neighbourhood of that point as the product

$$[F(\lambda)]_{ij}=q_{ij}\{[G(\lambda,\ t)]_{ij}\} \qquad q_{ij}\neq 0$$

where q_{ij} is an operator and $\{[G(\lambda,\ t)]_{ij}\}$ a parametric function such that the quotient

$$\frac{[G(\lambda,\ t)]_{ij}-[G(\lambda_0,\ t)]_{ij}}{\lambda-\lambda_0}$$

uniformly tends to the limit for $\lambda\to\lambda_0$ in every finite interval $0\leq t\leq t_0$. This is a more general definition of the operational derivative.

4.3 Exponential functions. For differential equations of the operational functions the following theorem is given.

Theorem 4.1: For given operators w, k and a real number λ_0 there exists at most one operational function $x(\lambda)$ satisfying for any real λ the equation

$$x'(\lambda)=wx(\lambda) \tag{4.8}$$

and the condition

$$x(\lambda_0)=k. \tag{4.9}$$

If w is constant the operational function $x(\lambda)=e^{\lambda w}$ satisfies (4.8) and the condition

$$x(0)=1 \tag{4.10}$$

and by this theorem this is the only function with these properties.

If w is an arbitrary operator equation (4.8) and condition (4.10) defines the generalised exponential function

$$x(\lambda)=e^{\lambda w}. \tag{4.11}$$

From (4.7) translation operator h^{λ} satisfies (4.8) and $h^0=1$, then by this theorem we have

$$h^{\lambda}=e^{-\lambda s}. \tag{4.12}$$

Furthermore the following theorem is given for the differential equations.

Theorem 4.2: For given operators w, k_0, k_1 and a real number λ_0, there exists at most one operational function $x(\lambda)$ satisfying for all real λ the equation

$$x''(\lambda)=wx(\lambda) \tag{4.13}$$

and the conditions

$$x(\lambda_0)=k_0,\ x'(\lambda_0)=k_1. \tag{4.14}$$

4.4 Derivatives of power series. For derivative of power series the following theorem is given.

Theorem 4.3: If a numerical series

$$\Phi(\lambda)=a_0+a_1\lambda+a_2\lambda^2+\cdots\cdots \qquad \lambda \text{ complex} \tag{4.15}$$

has a positive radius of convergence, then the function

$$\Phi(\lambda F)=a_0+a_1\lambda F+a_2\lambda^2F^2+\cdots\cdots \tag{4.16}$$

has a derivative in the form of a power series

$$\Phi'(\lambda F)=1a_1F+2a_2\lambda F^2+\cdots\cdots \tag{4.17}$$

where $F\equiv\{F(t)\}$ (continuous function in the interval $0\leq t<\infty$).

In particular if

$$\Phi(\lambda F)=1+\frac{\lambda F}{1!}+\frac{\lambda^2F^2}{2!}+\cdots\cdots$$

then by this theorem

$$\Phi'(\lambda F)=F+\frac{\lambda F^2}{1!}+\frac{\lambda^2F^3}{2!}+\cdots=F\Phi(\lambda F)$$

and $\Phi(0)=1$, therefore we have

$$e^{\lambda F}=1+\frac{\lambda F}{1!}+\frac{\lambda^2F^2}{2!}+\cdots\cdots. \tag{4.18}$$

Consequently the following relations are given if α and β are commutative and $\lambda\geq0$.

$$\frac{1}{s}e^{-\lambda\alpha^2/s}=\{J_0(2\sqrt{\lambda t}\,\alpha)\} \tag{4.19}$$

$$\exp\lambda\alpha(s-\sqrt{s^2+\beta^2})=1-\left\{\frac{\lambda\alpha}{\sqrt{t^21+2\lambda\alpha t}}\beta J_1(\beta\sqrt{t^21+2\lambda\alpha t})\right\} \tag{4.20}$$

$$\exp\lambda\alpha(s-\sqrt{s^2-\beta^2})=1-\left\{\frac{\lambda\alpha}{\sqrt{t^21+2\lambda\alpha t}}i\beta J_1(i\beta\sqrt{t^21+2\lambda\alpha t})\right\} \tag{4.21}$$

$$\exp(-\lambda\alpha\sqrt{s^2+\beta^2})=e^{-\lambda\alpha s}\exp\lambda\alpha(s-\sqrt{s^2-\beta^2}) \tag{4.22}$$

$$\exp(-\lambda\alpha\sqrt{s^2-\beta^2})=e^{-\lambda\alpha s}\exp\lambda\alpha(s-\sqrt{s^2-\beta^2}) \tag{4.23}$$

5. Multi-Conductor Transmission Systems

5.1 Telegraphic equations. Here we shall consider multi-conductor transmission systems stretched along the axis λ. Denote by $\{v(\lambda, t)\}$ and $\{i(\lambda, t)\}$ the voltage and the current at a point of coordinate λ and instant t. The following equations hold

$$\left.\begin{aligned} -\{v_\lambda(\lambda, t)\} &= L\{i_t(\lambda, t)\} + R\{i(\lambda, t)\} \\ -\{i_\lambda(\lambda, t)\} &= C\{v_t(\lambda, t)\} + G\{v(\lambda, t)\} \end{aligned}\right\} \tag{5.1}$$

where R denotes resistance, G leak-conductance, L inductance and C capacitance per unit length.

When the initial values are

$$v(\lambda, 0)=0, \quad i(\lambda, 0)=0 \tag{5.2}$$

then equations (5.1) have the following operational form.

$$\left.\begin{aligned} v'(\lambda) &= -(Ls+R)i(\lambda) \\ i'(\lambda) &= -(Cs+G)v(\lambda) \end{aligned}\right\} \tag{5.3}$$

Differentiating (5.3) with respect to λ we have

$$\left.\begin{aligned} v''(\lambda) &= [LCs^2+(RC+LG)s+RG]\, v(\lambda) \\ i''(\lambda) &= [CLs^2+(CR+GL)s+GR]\, i(\lambda) \end{aligned}\right\}. \tag{5.4}$$

5.2 Hyperbora type equations. If R and G can be neglected (5.4) becomes

$$v''(\lambda)=a^2s^2v(\lambda) \quad a^2=LC. \tag{5.5}$$

Suppose that the boundary conditions are given as

$$\{v(0, t)\}=\{v_1(t)\}, \quad \{v(\lambda_0, t)\}=\{v_2(t)\} \tag{5.6}$$

or as operational form

$$v(0)=v_1, \quad v(\lambda_0)=v_2. \tag{5.7}$$

Considering the operational function $e^{\lambda w}$ satisfying (5.5), we have $w=a^2s^2$. Since a^2 is positive definite, $w=\pm as$, therefore we have

$$v(\lambda)=e^{\lambda as}C_1+e^{-\lambda as}C_2 \tag{5.8}$$

where C_1 and C_2 are arbitrary operators and from (5.7)

$$v(\lambda)=(1-e^{-2\lambda_0 as})^{-1}[(e^{-\lambda as}-e^{-(2\lambda_0-\lambda)as})v_1+(e^{-(\lambda_0-\lambda)as}-e^{-(\lambda_0+\lambda)as})v_2]. \tag{5.9}$$

The operator $(1-e^{-2\lambda_0 as})^{-1}$ is expanded to the following infinite series

$$(1-e^{-2\lambda_0 as})^{-1}=\sum_{k=0}^{\infty} e^{-2k\lambda_0 as} \tag{5.10}$$

therefore (5.9) becomes

$$v(\lambda)=\sum_{k=0}^{\infty}[(e^{-(2k\lambda_0+\lambda)as}-e^{-(\overline{2k+1}\lambda_0-\lambda)as})v_1+(e^{-(\overline{2k+1}\lambda_0-\lambda)as}-e^{-(\overline{2k+1}\lambda_0+\lambda)as})\,v_2]. \tag{5.11}$$

Denote by $\alpha_1, \alpha_2, \cdots, \alpha_n$ eigen values of a and when they are distinct we have

$$e^{\alpha s}=Q^{-1}[e^{\alpha_1 s}, e^{\alpha_2 s}, \quad , e^{\alpha_n s}]Q \tag{5.12}$$

where Q denotes a non-singular numerical matrix and $[e^{\alpha_1 s}, e^{\alpha_2 s}, \quad , e^{\alpha_n s}]$ diagonal matrix.

Suppose that $v_2=0$ then (5.11) can be solved to

$$v(\lambda)=\{v(\lambda, t)\}=\sum_{k=0}^{\infty} Q^{-1}\left[\begin{Bmatrix} 0 & : 0\leq t<\overline{2k\lambda_0+\lambda\alpha_1} \\ u_1(t-\overline{2k\lambda_0+\lambda\alpha_1}) & : \overline{2k\lambda_0+\lambda\alpha_1}<t \end{Bmatrix},\right.$$

$$\left.\cdots\cdots, \begin{Bmatrix} 0 & : 0\leq t<\overline{2k\lambda_0+\lambda\alpha_n} \\ u_n(t-\overline{2k\lambda_0+\lambda\alpha_n}) & : \overline{2k\lambda_0+\lambda\alpha_n}<t \end{Bmatrix}\right] \tag{5.13}$$

where $u=Qv_1$ and $u=[u_1 \ u_2\cdots u_n]^t$.

If infinitely long system is considered the operational solution is given as

$$v(\lambda)=e^{-\lambda\alpha s}v_0 \tag{5.14}$$

where $v_0=v(0)=\{v(0, t)\}$ and it can be solved to

$$v(\lambda)=\{v(\lambda, t)\}=\sum_{k=0}^{\infty} Q^{-1}\left[\begin{Bmatrix} 0 & : 0\leq t<\lambda\alpha_1 \\ u_1(t-\lambda\alpha_1) & : \lambda\alpha_1<t \end{Bmatrix},\right.$$

$$\left.\cdots\cdots, \begin{Bmatrix} 0 & : 0\leq t<\lambda\alpha_n \\ u_n(t-\lambda\alpha_n) & : \lambda\alpha_n<t \end{Bmatrix}\right] \tag{5.15}$$

where $u=Qv_0$ and $u=[u_1 \ u_2\cdots u_n]^t$.

5.3 Parabora type equations. If L and G can be neglected (5.4) becomes

$$v''(\lambda)=a^2 s v(\lambda) \quad a^2=RC. \tag{5.16}$$

The boundary conditions are given as operational form

$$\left.\begin{aligned} v(0)&=v_1=\{v_1(t)\}=\{v(0, t)\} \\ v(\lambda_0)&=v_2=\{v_2(t)\}=\{v(\lambda_0, t)\} \end{aligned}\right\}. \tag{5.17}$$

Since a^2 is positive definite, the operational solution is

$$v(\lambda)=e^{\lambda a\sqrt{s}}C_1+e^{-\lambda a\sqrt{s}}C_2. \tag{5.18}$$

Here let us consider the parametric function

$$F(\lambda)=\{F(\lambda, t)\}=\left\{\frac{\lambda a}{2\sqrt{\pi t^3}}\exp\left(-\frac{\lambda^2 a^2}{4t}\right)\right\} \quad 0<\lambda<\infty. \tag{5.19}$$

Then we have the equality

$$l^{1/2}F(\lambda)=\left\{\frac{1}{\sqrt{\pi t}}\right\}F(\lambda)=\left\{\frac{\lambda a}{2\pi}\int_0^t (t-\tau)^{-1/2}\tau^{-3/2}\exp\left(-\frac{a^2\lambda^2}{4\tau}\right)d\tau\right\}.$$

Substituting $\lambda^2/4\tau=\lambda^2/4t+\sigma^2$, we have

$$l^{1/2}F(\lambda)=\left\{\frac{2a}{\pi\sqrt{t}}\exp\left(-\frac{\lambda^2 a^2}{4t}\right)\int_0^{\infty} e^{-\sigma^2 a^2}d\sigma\right\}=\left\{\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\lambda^2 a^2}{4t}\right)\right\}. \tag{5.20}$$

Further we have

$$l^{3/2}F(\lambda)=\left\{\int_0^t \frac{1}{\sqrt{\pi\tau}}\exp\left(-\frac{\lambda^2 a^2}{4\tau}\right)d\tau\right\} \tag{5.21}$$

and

$$\frac{\partial}{\partial\lambda}\int_0^t \frac{1}{\sqrt{\pi\tau}}\exp\left(-\frac{\lambda^2 a^2}{4\tau}\right)d\tau=-\int_0^t \frac{\lambda a^2}{2\sqrt{\pi\tau^3}}\exp\left(-\frac{\lambda^2 a^2}{4\tau}\right)d\tau.$$

By the operational form this equality is written as $l^{3/2}F'(\lambda)=-alF(\lambda)$ and consequently

$$F'(\lambda)=-a\sqrt{s}\,F(\lambda). \tag{5.22}$$

From (5.21)

$$F(0)=s^{3/2}\left\{\int_0^t \frac{1}{\sqrt{\pi\tau}}d\tau\right\}=s^{3/2}ll^{1/2}=1$$

then we have

$$e^{-\lambda a\sqrt{s}}=F(\lambda)=\left\{\frac{\lambda a}{2\sqrt{\pi t^3}}\exp\left(-\frac{\lambda^2 a^2}{4t}\right)\right\}. \tag{5.23}$$

From (5.20) we have

$$\frac{1}{\sqrt{s}}e^{-\lambda a\sqrt{s}}=\left\{\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{\lambda^2 a^2}{4t}\right)\right\}. \tag{5.24}$$

Further we have

$$\frac{1}{s}e^{-\lambda a\sqrt{s}}=\left\{\int_0^t \frac{\lambda a}{2\sqrt{\pi\tau^3}}\exp\left(-\frac{\lambda^2 a^2}{4\tau}\right)d\tau\right\}$$

and substituting $\sigma=1/2\sqrt{\tau}$ we have

$$\frac{1}{s}e^{-\lambda a\sqrt{s}}=\left\{\frac{2}{\sqrt{\pi}}\int_{1/2\sqrt{t}}^{\infty}\lambda a\exp\left(-\lambda^2 a^2\sigma^2\right)d\sigma\right\}. \tag{5.25}$$

From (5.23) the operator $e^{-\lambda a\sqrt{s}}$ represents the continuous function in $0\leq t<\infty$, therefore the operator $(1-e^{-\lambda a\sqrt{s}})^{-1}$ can be expanded to the infinite series and (5.18) with the conditions (5.17) becomes

$$\begin{aligned} v(\lambda)=\sum_{k=0}^{\infty}\Big[&\left(e^{-(2k\lambda_0+\lambda)a\sqrt{s}}-e^{-(\overline{2k+1}\lambda_0-\lambda)a\sqrt{s}}\right)v_1\\ &+\left(e^{-(\overline{2k+1}\lambda_0-\lambda)a\sqrt{s}}-e^{-(\overline{2k+1}\lambda_0+\lambda)a\sqrt{s}}\right)v_2\Big] \end{aligned} \tag{5.26}$$

and can be solved to

$$\begin{aligned} v(\lambda)=\{v(\lambda,t)\}=\sum_{k=0}^{\infty}\Big\{&\int_0^t[F(2k\lambda_0+\lambda,\ t-\tau)-F(\overline{2k+1}\lambda_0-\lambda,\ t-\tau)]v_1(\tau)d\tau\\ &+\int_0^t[F(\overline{2k+1}\lambda_0-\lambda,\ t-\tau)-F(\overline{2k+1}\lambda_0+\lambda,\ t-\tau)]v_2(\tau)d\tau\Big\}. \end{aligned} \tag{5.27}$$

Considering infinitly long system we have

$$v(\lambda)=e^{-\lambda a\sqrt{s}}v_0=\{v(\lambda,t)\}=\left\{\int_0^t F(\lambda,\ t-\tau)v_0(\tau)d\tau\right\} \tag{5.28}$$

where $\{v_0(t)\}=\{v_1(0,\ t)\}$ and if $v_0(t)=V_0$ (constant) then we have

$$\{v(\lambda,\ t)\}=\left\{\frac{2}{\sqrt{\pi}}\int_{1/2\sqrt{t}}^{\infty}\lambda\alpha\exp(-\lambda^2\alpha^2\sigma^2)d\sigma V_0\right\}. \tag{5.29}$$

5.4 Operation T^α and its applications. Here we shall introduce an operation T^α defined as

$$T^\alpha F=T^\alpha\{F(t)\}=\{e^{\alpha t}F(t)\} \tag{5.30}$$

where α is an arbitrary numerical matrix and F an arbitrary sectionally continuous integrable function, and this operation has the following properties.

i) $T^\alpha(F+G)=T^\alpha F+T^\alpha G$

ii) $T^\alpha T^\beta F=T^{\alpha+\beta}F$ if α and β are commutative.

iii) $T^\alpha(FG)=(T^\alpha F)(T^\alpha G)$ if α and F are commutative.

iv) $T^\alpha R(s)=R(s-\alpha)$, $T^\alpha R(s)=R(s-\alpha)$ where $R(s)$ is a rational function of s and commutative with α.

v) $T^\alpha e^w=e^{T^\alpha w}$ where w is an operator and commutative with α.

Using this operation we shall consider the solution of (5.4) in some special cases.

When L can be neglected and an infinitly long system is considered we have

$$v(\lambda)=e^{-\lambda\alpha\sqrt{s+\beta}}v_0 \quad \alpha^2=RC,\ \beta=C^{-1}G \tag{5.31}$$

where $v_0=\{v_0(t)\}=\{v(0,\ t)\}$.

If α and β are commutative $e^{-\lambda\alpha\sqrt{s+\beta}}=T^{-\beta}e^{-\lambda\alpha\sqrt{s}}$, then

$$v(\lambda)=\{v(\lambda,\ t)\}=\left\{e^{-\beta t}\int_0^t F(\lambda,\ t-\tau)v_0(\tau)d\tau\right\} \tag{5.32}$$

where $F(\lambda,\ t)$ is defined by (5.23).

Further when G can be neglected and an infinitly long system is considered we have

$$v(\lambda)=e^{-\lambda\alpha}\sqrt{(s+\beta)^2-\beta^2}v_0 \quad \alpha^2=LC,\ \beta=\frac{1}{2}C^{-1}L^{-1}RC \tag{5.33}$$

and if α and β are commutative $e^{-\lambda\alpha\sqrt{(s+\beta)^2-\beta^2}}=T^{-\beta}e^{-\lambda\alpha\sqrt{s^2-\beta^2}}$, then we have

$$v(\lambda)=e^{-\lambda\alpha s}\left\{e^{-\beta t}\left[1-\frac{\lambda\alpha}{\sqrt{t^2 1+2\lambda\alpha t}}i\beta J_1(i\beta\sqrt{t^2 1+2\lambda\alpha t})\right]\right\}v_0. \tag{5.34}$$

6. Conclusion

As mentioned above, a method to apply the Mikusinsky's operational calculus to the matrix functions in order to study the physical systems with distributed constants has been presented, and using these results, multi-conductor transmission systems are analysed.

The operations used here are restricted by the matrix algebra, and then commutative matrices are considered, this method is very useful to get the numerical solutions of systems with many variables.

References

1) Ichikawa, S. and Kishima, A.: This Memoirs., Vol. XXXII, Part 2 (1970).
2) Mikusinsky, J.: "Operational Calculus" Press (1959).

Appendix

Proof of The Theorems

Theorem 2.1: Denote a continuous part of $\{X(t)\}$ by $\{Y(t)\}$, then

$$\{X(t)\}=\{Y(t)\}+\sum_{\nu=1}^{n}\beta_\nu\{H_{t_\nu}(t)\}=\{Y(t)\}+\frac{1}{s}\sum_{\nu=1}^{n}\beta_\nu h^{t_\nu}.$$

Multiplying by s and considering that $\{X'(t)\}=\{Y'(t)\}$ and $X(0)=Y(0)$, we have

$$s\{X(t)\}=s\{Y(t)\}+\sum_{\nu=1}^{n}\beta_\nu h^{t_\nu}=\{X'(t)\}+X(0)+\sum_{\nu=1}^{n}\beta_\nu h^{t_\nu}.$$

Theorem 3.1: We shall consider the following power series

$$\alpha_1\lambda G+\alpha_2\lambda^2G^2+\cdots\cdots \tag{A-1}$$

where $G\equiv\{G(t)\}$ continuous function in the interval $0\leq t<\infty$.

Denote by M the maximum absolute value of all elements of $G(t)$ in an arbitrarily fixed interval $0\leq t\leq t_0$, then

$$|G(t)|\leq MH \quad H=[H_{ij}] \quad H_{ij}=1$$

$$|G^2(t)|=\left|\int_0^t G(t-\tau)G(\tau)d\tau\right|\leq m\int_0^t M\cdot Md\tau\, H=mM^2\frac{t}{1!}H$$

where m is the order of G, and generally

$$|G^n(t)|\leq m^{n-1}M^n\frac{t^{n-1}}{(n-1)!}H\leq m^{n-1}M^n\frac{t_0^{n-1}}{(n-1)!}H \quad n=1,\ 2,\cdots.$$

Denote by ρ the convergence radius of the series (3.6), then the sequence

$$\left(\frac{2\lambda_0}{\rho}\right)^n m^{n-1}M^n\frac{t_0^{n-1}}{(n-1)!}$$

tends to 0, *ie.*, it is bounded by a certain number K, therefore

$$|\alpha_n\lambda^nG^n(t)|\leq|\alpha_n\lambda_0^n m^{n-1}M^n\frac{t_0^{n-1}}{(n-1)!}H|=|\left(\frac{2\lambda_0}{\rho}\right)^n m^{n-1}M^n\frac{t_0^{n-1}}{(n-1)!}$$

$$\left(\frac{\rho}{2}\right)^n\alpha_nH|\leq k|\alpha_n|H\left(\frac{\rho}{2}\right)^n\leq k'|\alpha_n|\left(\frac{\rho}{2}\right)^n$$

and ρ is the convergence radius of the series (3.6) then the series

$$k'|\alpha_1|\frac{\rho}{2}+k'|\alpha_2|\left(\frac{\rho}{2}\right)^2+\cdots$$

is convergent. Hence the convergence of the series (A-1) follows and from this fact follows the convergence of the series (3.7).

Theorem 4.1 and 4.2: The operator w is constant with respect to λ, then from the theorem of the ordinary differential equations uniqueness of the solution is assured.

Theorem 4.3: From theorem 3.1, the power series (4.15) represents the parametric function in every domain

$$|\lambda| \leq \lambda_0, \quad 0 \leq t \leq t_0 \tag{A-2}$$

and a numerical series

$$\boldsymbol{\Phi}'(\lambda) = 1 \cdot \boldsymbol{\alpha}_1 + 2 \cdot \boldsymbol{\alpha}_2 \lambda + \cdots$$

has the same radius of convergence as the series (4.15), therefore the power series (4.17) represents the parametric function in the domain (A-2), then the theorem is proved by differentiating term by term.