

Stress Distribution for Anisotropic Elastic Plate Containing Two or More Arbitrary Elliptical Holes

By

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The theoretical solution by successive approximation utilizing the complex variable method is obtained for the case of an infinite anisotropic elastic plate containing two or more arbitrary elliptical holes under applied external stress at infinity. For the particular cases of isotropic plate involving two or three circular holes, the numerical results obtained by the present method are quite in agreement with the results obtained by others.

1. Introduction

The stress distribution in isotropic elastic plate containing groups of circular holes has been studied by several investigators. Howland and Knight¹⁾, Ishida²⁾ obtained solutions for a class of problems involving equal circular holes, and Green³⁾ obtained a general solution for an arbitrary group of circular holes. The general solution is, however, very complicated and consequently only a few numerical results were evaluated for special cases.

The general stress solution for plane problems containing two nonconcentric circular boundaries was first obtained by Jeffery⁴⁾. In order to obtain the stress distribution in an infinite isotropic plate with two circular holes, this solution in bipolar coordinates was used by Watanabe⁵⁾, Utoguchi⁶⁾ and Ling⁷⁾. Davies and Hoddinott⁸⁾, Haddon⁹⁾, Salerno and Mahney¹⁰⁾ have recently obtained approximate solutions for some cases involving two unequal circular holes using the complex variable method.

On the other hand, Kubo¹¹⁾ studied the theoretical solution by successive approximation for two equal elliptical holes in orthotropic elastic plate under a particular condition such that the direction of the applied normal stress at infinity perpendicular to the axis through the two origins of the holes coincides with that of the principal elastic modulus of the plate. And Kawamoto¹²⁾ carried out the

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experimental investigation for an orthotropic plate with two equal circular holes utilizing the photoelastic coating technique.

The present paper describes the theoretical solution by successive approximation to be obtained by the point matching approach using the solution of anisotropic elastic plate with an elliptical hole, and shows some numerical examples for an orthotropic plate containing two or three circular holes. In some particular cases of isotropic plate with two or three equal circular holes, the results obtained by the present method are compared with the results obtained by others.

2. Stress Analysis for an Infinite Anisotropic Plate Containing an Elliptical Hole

2.1 Stress Formulus for a Plate

The equations of the generalized Hooke's law for an anisotropic body in plane elasticity will be written in the following form:

$$\left. \begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \tau_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}, \end{aligned} \right\} \quad (2.1)$$

where $a_{11}, a_{12}, \dots, a_{66}$ are the elastic compliances.

In the case where the plane elastic body is orthotropic and the directions of their principal axes coincide with the ones of the x - and y -axes, the elastic compliances a_{16}, a_{26} vanish and the remaining compliances can be represented by Young's moduli E_1, E_2 , Poisson's ratios ν_{12}, ν_{21} and shear modulus G_{12} , as follows:

$$\left. \begin{aligned} a_{11} &= 1/E_1, & a_{22} &= 1/E_2, \\ a_{12} &= -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}, & a_{66} &= 1/G_{12}. \end{aligned} \right\} \quad (2.2)$$

In the particular case where the material is isotropic, independent elastic moduli are E ($=E_1=E_2$), ν ($=\nu_{12}=\nu_{21}$), and shear modulus G can be represented by E and ν as $2G=E/(1+\nu)$.

In order to obtain the stresses in an anisotropic elastic plate, let us introduce the complex variables

$$z_k = x + \mu_k y, \quad (k = 1, 2), \quad (2.3)$$

μ_1 and μ_2 and the complex conjugates $\bar{\mu}_1$ and $\bar{\mu}_2$ are the roots of characteristic equation for the anisotropic body under consideration as follows.

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0. \quad (2.4)$$

The stress components in rectangular Cartesian coordinates (x, y) are expressed in terms of two complex analytic functions $\phi_1(z_1)$ and $\phi_2(z_2)$ ¹³⁾.

$$\left. \begin{aligned} \sigma_x &= 2\text{Re}[\mu_1^2\phi_1'(z_1) + \mu_2^2\phi_2'(z_2)], \\ \sigma_y &= 2\text{Re}[\phi_1'(z_1) + \phi_2'(z_2)], \\ \tau_{xy} &= -2\text{Re}[\mu_1\phi_1'(z_1) + \mu_2\phi_2'(z_2)], \end{aligned} \right\} \quad (2.5)$$

where Re is the notation for the real part in the bracket.

2.2 Conformal Mappings and Complex Analytic Functions

We refer the plane body under consideration to a rectangular Cartesian coordinate system (x, y) where the origin lies in an elliptical cross section of the hole. We consider an infinite anisotropic elastic body containing the hole, the contour of which is given by equation:

$$x_0 = a \cos \theta, \quad y_0 = b \sin \theta. \quad (2.6)$$

In which θ is a parameter varying from 0 to 2π in a counter-clockwise direction, and a and b represent the semi-axes of the ellipse as shown in Fig. 1.

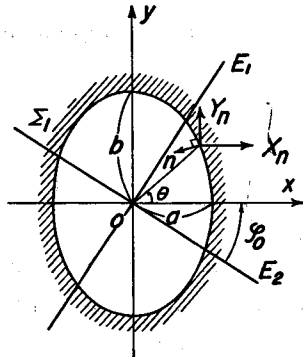


Fig. 1. Elliptical hole in an infinite anisotropic plate.

A complex plane $z_0 (=x+iy)$ with the hole as given by equation (2.6) is conformally transformed onto the exterior of a unit circle $|\zeta|=1$ in the ζ -plane ($\zeta = \rho e^{i\theta}$). The mapping function is expressed as

$$z_0 = x+iy = \frac{1}{2} \{ (a+b)\zeta + (a-b)\zeta^{-1} \}. \quad (2.7)$$

Similarly, we consider the solution of the problem by mapping conformally the plane $z_k (=x+\mu_k y)$ onto the exterior of unit circles $|\zeta_k|=1$ in the ζ_k -plane ($\zeta_k = \rho_k e^{i\theta_k}$). Then the mapping functions may be taken as the form:

$$z_k = x + \mu_k y = \frac{1}{2} \{ (a - i\mu_k b) \zeta_k + (a + i\mu_k b) \zeta_k^{-1} \}. \quad (2.8)$$

Each of these functions on the contour of the cross section of a hole takes a value equal to $\zeta = \zeta_k \equiv \sigma (=e^{i\theta})$.

Let us use curvilinear coordinates (ξ, η) which ξ and η define the point by means of two orthogonal intersecting curves such as two families of conformal ellipses and hyperbolas. One of their conformal ellipses consists of the contour of an elliptical hole under consideration. And let us specify the stresses as σ_ξ , the normal component on a curve $\xi = \text{constant}$; σ_η , the normal component on a curve $\eta = \text{constant}$; $\tau_{\xi\eta}$, the shear component on both curves. The following equations can be obtained from the fundamental relations of stresses.

$$\left. \begin{aligned} \sigma_\xi + \sigma_\eta &= \sigma_x + \sigma_y, \\ \sigma_\xi - \sigma_\eta + 2i\tau_{\xi\eta} &= \frac{\bar{\zeta}}{\zeta} \cdot \frac{dz_0}{d\zeta} \cdot \frac{d\zeta}{dz_0} (\sigma_x - \sigma_y + 2i\tau_{xy}). \end{aligned} \right\} \quad (2.9)$$

Where σ_x , σ_y and τ_{xy} are the stress components in rectangular Cartesian coordinates (x, y) as was defined by equation (2.5). If X_n and Y_n are the components of the external stresses applied on the contour of a hole as shown in Fig. 1, two complex analytic functions $\phi_k(z_k)$ to appear in these stress components must satisfy the conditions:

$$\left. \begin{aligned} 2\text{Re}[\phi_1(z_1) + \phi_2(z_2)] &= \int_0^s Y_n ds + C_1, \\ 2\text{Re}[\mu_1\phi_1(z_1) + \mu_2\phi_2(z_2)] &= -\int_0^s X_n ds + C_2. \end{aligned} \right\} \quad (2.10)$$

Where C_1 and C_2 are constants which can be fixed arbitrary on the contour, since the simple-connected region is to be considered; without loss of generality with respect to our case we can set these constants equal to zero. Integrations on right side of above equations are performed with respect to the arc-length s from a certain initial point ($s=0$) to the variable point s .

Whereas the external stresses X_n and Y_n may be given as follows:

$$\left. \begin{aligned} X_n &= \sigma_x \cos(n, x_0) + \tau_{xy} \cos(n, y_0), \\ Y_n &= \tau_{xy} \cos(n, x_0) + \sigma_y \cos(n, y_0). \end{aligned} \right\} \quad (2.11)$$

In which n is a unit vector directed to the inward normal to the contour of a hole. Using a tangential unit vector s in a clockwise direction on the contour, we have

$$\cos(n, x_0) = -\frac{dy_0}{ds}, \quad \cos(n, y_0) = \frac{dx_0}{ds}. \quad (2.12)$$

Considering equations (2.5), (2.6), (2.10), (2.11) and (2.12), we can seek the expressions for the functions $\phi_k(z_k)$ in the following forms:

$$\phi_k(z_k) = \frac{1}{\mu_2 - \mu_1} \sum_{m=1}^{\infty} \Gamma_{km} \zeta_k^{-m}, \quad (2.13)$$

The coefficients Γ_{km} are given as

$$\Gamma_{1m} = \mu_2 \bar{a}_m - \bar{b}_m, \quad \Gamma_{2m} = \bar{b}_m - \mu_1 \bar{a}_m. \quad (2.14)$$

Here \bar{a}_m and \bar{b}_m are the complex constants to be determined by external stresses on the contour of a hole and the shape of the hole. When the stress components equal and opposite sign to σ_x^0 , σ_y^0 and τ_{xy}^0 applied at infinity act on the contour of the elliptical hole with semi-axes of a and b , these constants take the values:

$$\left. \begin{aligned} \bar{a}_1 &= \frac{1}{2} (\sigma_y^0 a - i \tau_{xy}^0 b), \\ \bar{b}_1 &= \frac{1}{2} (\tau_{xy}^0 a - i \sigma_x^0 b), \\ \bar{a}_m &= \bar{b}_m = 0, \quad \text{for } m \geq 2. \end{aligned} \right\} \quad (2.15)$$

Thus we can obtain the stress components σ_x , σ_y and τ_{xy} in the coordinates (x, y) from equation (2.5) by the derivatives of the functions $\phi_k(z_k)$. And further we can also obtain the stress components σ_ξ , σ_η and $\tau_{\xi\eta}$ in the curvilinear coordinates (ξ, η) from equation (2.9).

In our case when the surface of the hole is not loaded and the stresses σ_x^0 , σ_y^0 and τ_{xy}^0 are applied at infinity, the components of resultant stress σ_x , σ_y and τ_{xy} can be determined by the formulas:

$$\left. \begin{aligned} \sigma_x &= \sigma_x^0 + \sigma_x, \\ \sigma_y &= \sigma_y^0 + \sigma_y, \\ \tau_{xy} &= \tau_{xy}^0 + \tau_{xy}, \end{aligned} \right\} \quad (2.16)$$

where σ_x , σ_y and τ_{xy} of right sides in above expressions are the stresses determined by equation (2.5) for the case when the stresses $-\sigma_x^0$, $-\sigma_y^0$ and $-\tau_{xy}^0$ act on the contour of the hole.

3. Stress Analysis for an Infinite Anisotropic Plate Containing Two Elliptical Holes

From the stress values around or near the hole in an infinite anisotropic plate obtained in the previous section, we can solve the problem of multi-connected region such as an infinite anisotropic plate with several elliptical holes utilizing the point

matching technique. For simplicity, we shall describe the method of solutions for the problem of double-connected region such that an infinite anisotropic plate containing two arbitrary elliptical holes are applied the stresses σ_x^0 , σ_y^0 and τ_{xy}^0 at infinity.

Let us take two rectangular Cartesian coordinate systems (x, y) and (x', y') whose origins are respectively the centers o and o' of two arbitrary non-intersecting elliptical holes Σ_1 and Σ_2 , and whose centers are horizontal distance s_0 and vertical distance h_0 apart (see Fig. 2). The relation of these coordinates can be given by the expressions:

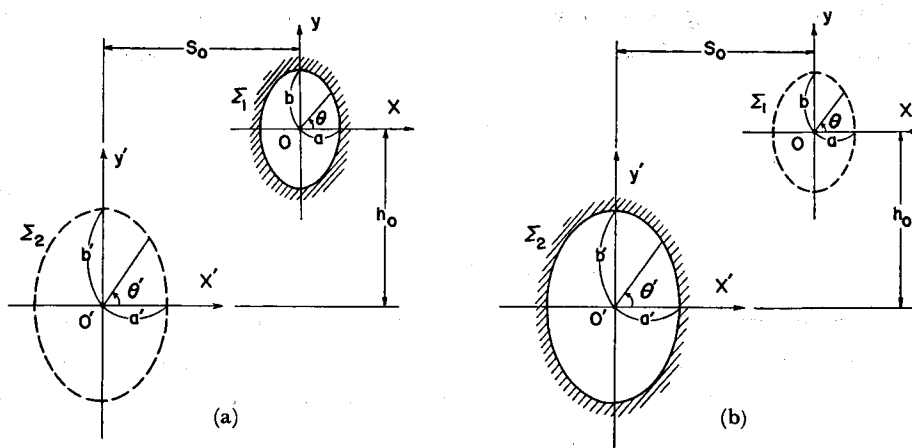


Fig. 2. Geometry of two arbitrary elliptical holes Σ_1 and Σ_2 .

$$x' = x + s_0, \quad y' = y + h_0. \quad (3.1)$$

Let the contour of an elliptical hole Σ_2 be given by

$$x'_0 = a' \cos \theta', \quad y'_0 = b' \sin \theta'. \quad (3.2)$$

In which θ' is a parameter varying from 0 to 2π in a counter-clockwise direction, and a' and b' represent the semi-axes of the ellipse Σ_2 .

In the same manner as was mentioned above, we consider the solution of the problem by mapping conformally the complex planes $z'_0 (=x' + iy')$ with the elliptical hole Σ_2 and $z'_k (=x'_k + iy'_k)$ onto the exteriors of unit circles $|\zeta'| = 1$ and $|\zeta'_k| = 1$, respectively. The relations between z'_0 and ζ' , and z'_k and ζ'_k are represented in equations (2.7) and (2.8) which replaced ζ and ζ_k by ζ' and ζ'_k , respectively. We also seek the expressions for the complex analytic functions $\phi_k^*(z'_k)$ correspondig to the infinite plate with an elliptical hole Σ_2 as follows:

$$\phi_k^*(z'_k) = \frac{1}{\mu_2 - \mu_1} \sum_{m=1}^{\infty} \Gamma'_{km} \zeta'_k{}^{-m}. \quad (3.3)$$

In which Γ'_{km} are complex constants to be represented by replacing \bar{a}_m and \bar{b}_m in equation (2.14) by \bar{a}'_m and \bar{b}'_m . The \bar{a}'_m and \bar{b}'_m are complex constants not yet determined.

For the complex variables z_k and z'_k , the relations may be deduced as

$$z'_k = z_k + (s_0 + \mu_k h_0), \quad (k = 1, 2). \quad (3.4)$$

And then the relations between ζ_k and ζ'_k must satisfy the following algebraic equations:

$$\begin{aligned} & \{(a - i\mu_k b) \zeta_k + (a + i\mu_k b) \zeta_k^{-1}\} \\ & - \{(a' - i\mu_k b') \zeta'_k + (a' + i\mu_k b') \zeta'_k{}^{-1}\} + 2(s_0 + \mu_k h_0) = 0. \end{aligned} \quad (3.5)$$

Thus, by substituting $e^{i\theta'}$ into ζ'_k and by solving these equations, we can obtain the positions ζ_k on the ζ_k -planes corresponding to the virtual contour of the elliptical hole Σ_2 (dotted boundary of Fig. 2 (a)). By use of these values, stress components σ_x , σ_y and τ_{xy} determined by equation (2.5) give the stress values on the contour of the hole Σ_2 in the coordinates (x, y) . On substituting from stresses equal and opposite sign to these stresses σ_x , σ_y and τ_{xy} into equation (2.11), considering the equation obtained from equation (2.12) by replacing x_0 and y_0 by x'_0 and y'_0 , and approximating with the summations by means of the point matching approach from which boundary stresses are given at a selected finite set of boundary points of the hole, the integrations on right sides of equation (2.11) may be represented by the following Fourier series.

$$\left. \begin{aligned} \int_0^s Y_n ds + C_1 &= a'_0 + \sum_{m=1}^{\infty} (a'_m e^{im\theta'} + \bar{a}'_m e^{-im\theta'}), \\ - \int_0^s X_n ds + C_2 &= b'_0 + \sum_{m=1}^{\infty} (b'_m e^{im\theta'} + \bar{b}'_m e^{-im\theta'}). \end{aligned} \right\} \quad (3.6)$$

On substituting from $\phi_k^*(z'_k)$ into $\phi_k(z_k)$ of left sides of equation (2.10), and by solving this equation (2.10), $\phi_k^*(z'_k)$ take the forms of equation (3.3), in which the constants \bar{a}'_m and \bar{b}'_m in the coefficients Γ'_{km} of equation (3.3) are now determined by the coefficients of right sides of equation (3.6).

Then by superimposing the result of the previous section and the result obtained by the above process, the dotted ellipse Σ_2 of Fig. 2 (a) becomes the hole with free surface, while this disturbs the boundary condition of free surface around another hole Σ_1 . Therefore we must repeat the same process such as above for the case of Fig. 2(b).

By these repetitions up to the state that the normal and tangential shear stresses on the contour of the two holes Σ_1 and Σ_2 become excessively small compared with the applied stresses, and by superposition of their results, the stress distribution around two arbitrary elliptical holes may be obtained.

4. Numerical Examples

In order to check the accuracy of the results, the maximum and the minimum values of stresses around two equal circular holes under the applied stress at infinity were compared the results obtained by the present approach with the results obtained by others^{7),9)}. The results by the present method in which the angle of equal intervals $\Delta\theta$ sets equal to 2.5° (i.e. the number of selected boundary points is equal to 144), the number of terms in series (3.6) m equal to 25 and the number of repetitions i equal to 4, were in close agreement with the results of Ling⁷⁾, and Haddon⁹⁾ with an accuracy better than 0.5%.

A typical stress distribution around two unequal circular holes with dimensions $a'=2.5a$, $s_0=4.5a$ in isotropic plate under the normal stress $w_0=1.0$ applied at infinity with an inclined angle $\delta=45^\circ$ from the x -axis has been given by Haddon⁹⁾. The result by Haddon has been obtained by the calculation up to the state that the sum of the moduli of the normal and tangential shear stresses on the boundaries ($|\sigma_{\xi}| + |\tau_{\xi\eta}|$) is less than 10^{-4} compared with the applied stress

Table 1. Stress values of σ_η and $\sigma_{\eta'}$ around the two unequal circular holes Σ_1 and Σ_2 with the dimensions $a'=2.5a$, $s_0=4.5a$ in isotropic plate, when the normal stress $w_0=1.0$ applies at infinity with an inclined angle $\delta=45^\circ$ from the x -axis.

$\theta=\theta'$	σ_η on Σ_1		$\sigma_{\eta'}$ on Σ_2		$\theta=\theta'$	σ_η on Σ_1		$\sigma_{\eta'}$ on Σ_2	
	after Haddon ⁹⁾	$\Delta\theta=2.5^\circ$ $w=25$ $i=4$	after Haddon ⁹⁾	$\Delta\theta=2.5^\circ$ $w=25$ $i=4$		after Haddon ⁹⁾	$\Delta\theta=2.5^\circ$ $m=25$ $i=4$	after Haddon ⁹⁾	$\Delta\theta=2.5^\circ$ $m=25$ $i=4$
0°	1.432	1.423	1.273	1.183	180°	2.089	2.059	1.049	1.015
15°	0.159	0.155	-0.694	-0.645	195°	-0.067	-0.061	0.029	0.002
30°	-0.902	-0.903	-0.711	-0.717	210°	-1.563	-1.548	-0.728	-0.745
45°	-1.531	-1.531	-0.710	-0.731	225°	-1.932	-1.927	-1.030	-1.030
60°	-1.618	-1.619	-0.464	-0.461	240°	-1.335	-1.345	-0.808	-0.791
75°	-1.170	-1.172	0.207	0.222	255°	-0.137	-0.160	-0.138	-0.111
90°	-0.282	-0.286	1.159	1.184	270°	1.283	1.253	0.776	0.808
105°	0.900	0.895	2.128	2.151	285°	2.585	2.552	1.667	1.693
120°	2.209	2.200	2.840	2.852	300°	3.489	3.458	2.289	2.304
135°	3.408	3.386	3.092	3.089	315°	3.812	3.787	2.577	2.582
150°	4.101	4.057	2.811	2.792	330°	3.501	3.481	2.862	2.867
165°	3.736	3.682	2.065	2.035	345°	2.639	2.625	3.262	3.199

$w_0=1.0$. The comparison with the result obtained by the present method under the same conditions is shown in Table 1. The difference in both results is considerably small, and it may be concluded that the present method is sufficiently accurate for partial use.

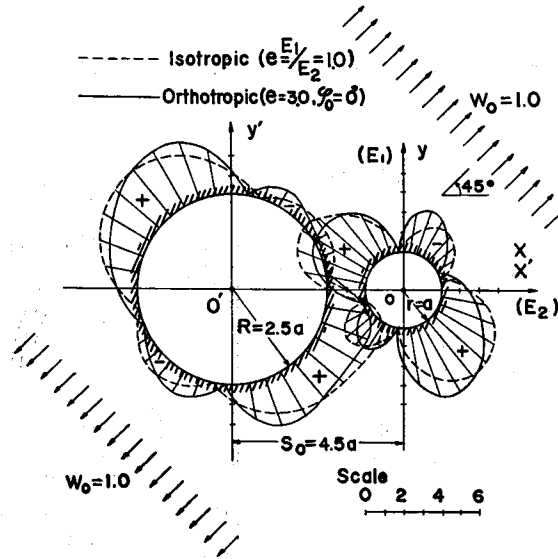


Fig. 3. Stress distributions tangential hoop stress σ_θ around the circular holes for the case of isotropic plate given by Table 1 and for the case of orthotropic plate with $e=E_1/E_2=3.0$, $\phi_0=0^\circ$.

Let us consider the case of an orthotropic elastic plate such that the ratio of the principal elastic moduli $e (=E_1/E_2)$ and Poisson's ratio are equal to 3.0 and $\nu_{12}=0.150$, respectively, and shear modulus G_{12} is defined as

$$\frac{1}{G_{12}} = \frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} \quad (4.1)$$

When the stress $\sigma_y^0=1.0$, $\sigma_x^0=1.0$ or $\tau_{xy}^0=1.0$ at infinity is applied the orthotropic plate containing two equal circular holes located on the x -axis (i.e. $h_0=0$), the stress distributions of σ_θ around the right side hole are shown in Fig. 4, 6 or Fig. 7 with parameters of $s_0/2a$ and the angle ϕ_0 , in which s_0 is the horizontal distance between the two origins of the holes, and ϕ_0 the angle between the direction of the principal elastic modulus E_2 and the direction of the x -axis. Fig. 4 (a) and (b) are the cases under the applied normal stress $\sigma_y^0=1.0$ at infinity. The distributions of the upper parts of Fig. 4 (a) are the case of $\phi_0=90^\circ$, and the lower parts of this figure the case of $\phi_0=0^\circ$. These stress distributions are in

symmetry with respect to the x -axis. Fig. 4 (b) shows the stress distributions for the case of $\varphi_0=45^\circ$. It is known from this figure that the stress distributions of σ_y around the right side hole is not in symmetry with respect to the x -axis between the two equal circular holes with the spacing $s_0/2a=1.50$ for the case where the normal stress $\sigma_y^0=1.0$ applies at infinity. In this figure the dotted lines are the case of isotropic plate ($\nu=E_1/E_2=1.0$) under the same condition. It is interesting to note that the shear stress τ_{xy} on the x -axis for the case of $\varphi_0=45^\circ$ take the finite

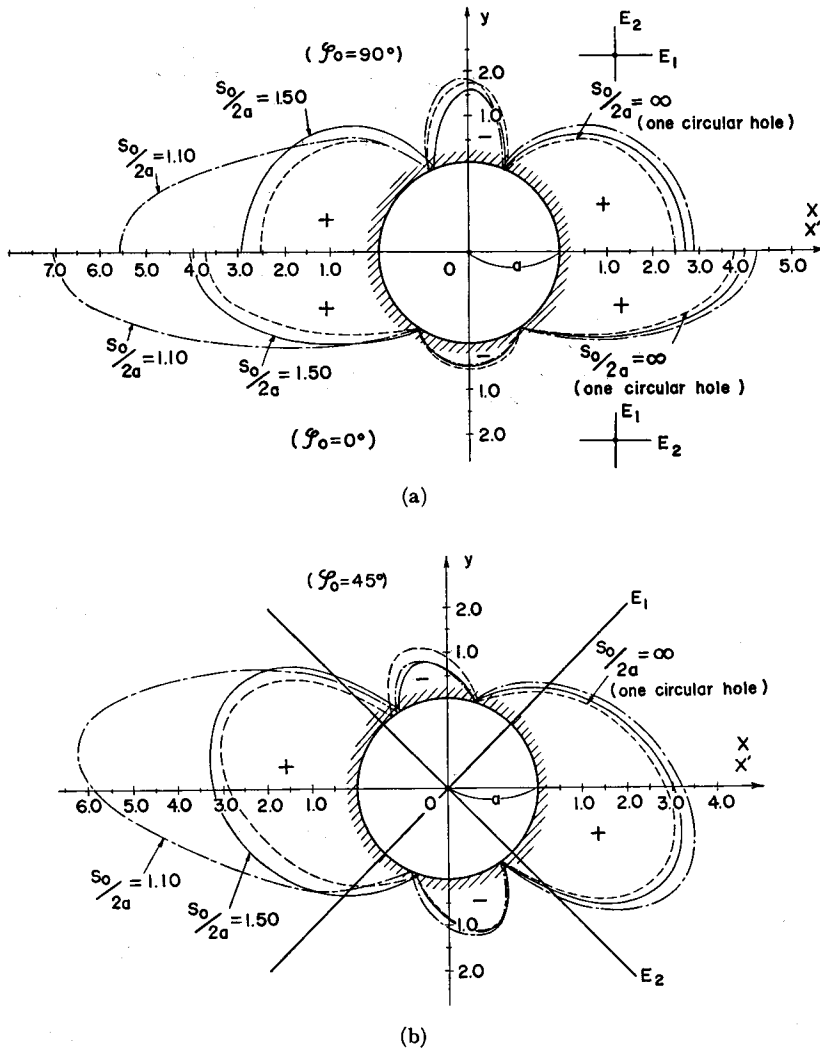


Fig. 4. Stress distributions of σ_y around the right side hole for the case of orthotropic plate ($\nu=3.0$) containing two equal circular holes located on the x -axes, when the normal stress $\sigma_y^0=1.0$ applies at infinity.

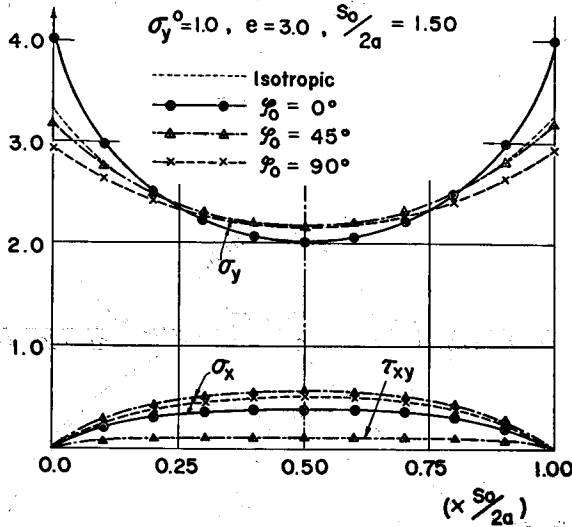


Fig. 5. Stress values of σ_x , σ_y and τ_{xy} on the x -axis between the two equal circular holes in orthotropic plate ($e=3.0$) with the spacing $s_0/2a=1.50$, when the normal stress $\sigma_y^0=1.0$ applies at infinity.

values as shown in this figure. The shear stress τ_{xy} usually does not generate on this line for the case such that isotropic body or orthotropic body coincided the principal axes of elastic moduli with the axes of the coordinates is treated.

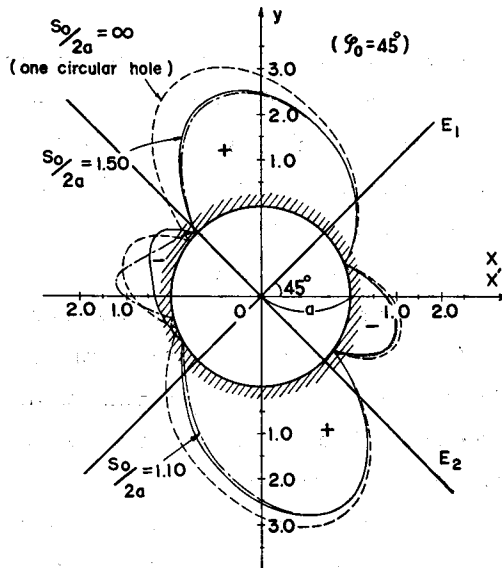


Fig. 6. Stress distributions of σ_η around the right side hole for the case of orthotropic plate ($e=3.0, \phi_0=45^\circ$) containing two equal circular holes located on the x -axis, when the normal stress $\sigma_x^0=1.0$ applies at infinity.

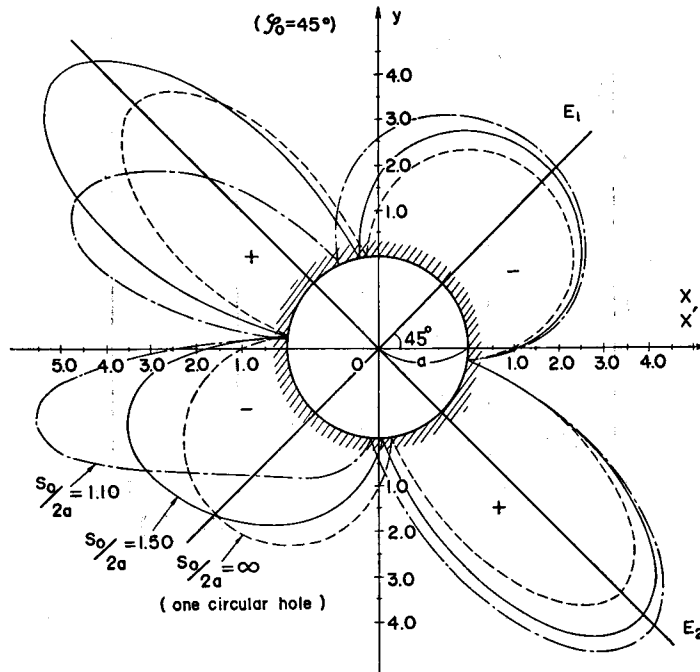


Fig. 7. Stress distributions of σ_{η} around the right side hole for the case of orthotropic plate ($e=3.0$, $\varphi_0=45^\circ$) containing two equal circular holes located on the x -axis, when the shear stress $\tau_{xy}^0=1.0$ applies at infinity.

Fig. 6 and Fig. 7 show respectively the stress distributions of σ_{η} around the right side hole for the case of orthotropic plate ($e=3.0$, $\varphi_0=45^\circ$) containing two equal circular holes under the normal stress $\sigma_x^0=1.0$ and the shear stress $\tau_{xy}^0=1.0$ applied at infinity. For the case of the applied normal stress $\sigma_x^0=1.0$ as in Fig. 5, the number of terms in series (3.6) $m=40$ and the number of repetitions $i=7$ were required for an accuracy of 0.5% on the holes.

As an illustration of two unequal circular holes in anisotropic plate, the stress distributions of σ_{η} and σ'_{η} around the holes in orthotropic plate ($e=3.0$, $\varphi_0=0^\circ$) under the same condition such as Table 1 given by Haddon, are shown by the solid lines in Fig. 3. And the dotted lines in the same figure show the stress distributions for the case of isotropic plate given by Table 1.

Fig. 8 shows the stress distributions of σ_{η} around the upper side hole for the case of orthotropic plate ($e=3.0$, $\varphi'_0=45^\circ$) containing two equal elliptical holes ($b/a=b'/a'=1.5$, $a'=a$) with the spacing $s_0/2a=h_0/2a=1.50$ under the applied normal stress $\sigma_y^0=1.0$ at infinity.

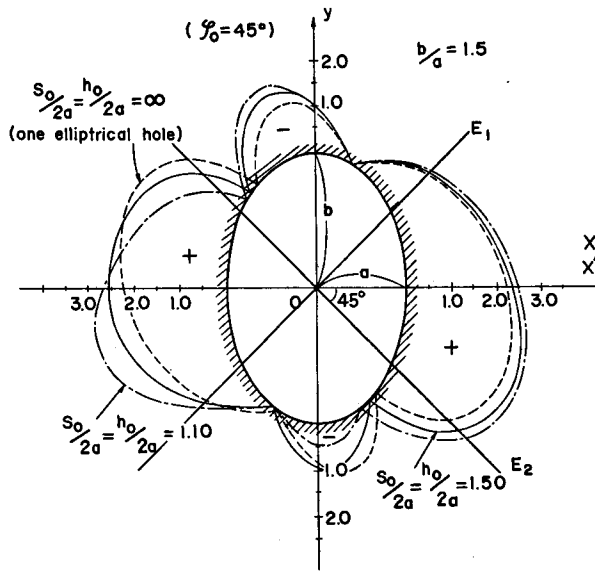


Fig. 8. Stress distributions of σ_η around the upper side hole for the case of orthotropic plate ($\epsilon=3.0$, $\varphi_0=45^\circ$) containing two equal elliptical holes with the spacing $s_0/2a=h_0/2a=1.50$, when the normal stress $\sigma_y^0=1.0$ applies at infinity.

In the same manner as the case of two holes, we can solve the problem with three or more elliptical holes. We illustrate in the following the stress values obtained for several problems of an infinite isotropic and orthotropic plate involving three elliptical holes. Green³⁾, and Hulbert⁴⁾ solved the problem of isotropic plate under the applied stress $\sigma_y^0=1.0$ at infinity for which the three circular holes in a line on the x -axis had equal radii and the spacing was tanken to be such that $s_0/2a=3.3333$. In order to check with their results, we solved a problem under the same conditions. The results for these three cases are shown in Table 2, and the difference in them is minor; especially the results due to Green and the present method are in almost complete agreement.

Similarly, the results obtained by Hulbert and the present method for the case with the spacing $s_0/2a=1.50$ are given in Table 3 (a) and (b). Table 3 (b) gives the values of σ_x and σ_y at a few points on the x -axis between the center and outside holes. These are also in close agreement. This distribution of σ_η , σ'_η and σ''_η around the three holes is plotted graphically in Fig. 9 by the dotted lines with showing only the range of right halves with respect to the y -, y' - and y'' -axes. In this figure the case of orthotropic plate ($\epsilon=3.0$, $\varphi_0=0^\circ$) under the same conditions is illustrated by solid lines.

Table 2. Stress values of σ_η and σ_η' around the right side and center holes for the case of isotropic plate containing three equal circular holes with the spacings $s_0/2a=s_1/2a=3.333$ in a line on the x -axis, when the normal stress $\sigma_y^0=1.0$ applies at infinity.

$\theta=\theta'$	σ_η around the right side hole			σ_η' around the center hole		
	after Green ³⁾	after Hulbert ¹⁴⁾	$\Delta\theta=2.5^\circ$ $m=20$ $i=4$	after Green ³⁾	after Hulbert ¹⁴⁾	$\Delta\theta=2.5^\circ$ $m=20$ $i=4$
0°	2.997	2.996	2.995	3.009	3.009	3.009
10°	2.884	2.885	2.883	2.896	2.898	2.896
20°	2.559	2.560	2.558	2.570	2.572	2.570
30°	2.054	2.147	2.054	2.065	2.019	2.065
40°	1.428	1.451	1.428	1.439	1.462	1.439
50°	0.752	0.738	0.752	0.766	0.751	0.765
60°	0.108	0.108	0.109	0.126	0.126	0.126
70°	-0.426	-0.431	0.426	-0.399	-0.404	-0.399
80°	-0.782	-0.787	-0.781	-0.743	-0.748	-0.743
90°	-0.915	-0.914	-0.914	-0.863	-0.863	-0.863
100°	-0.807	-0.812	-0.807			
110°	-0.473	-0.479	-0.473			
120°	0.047	0.471	0.047			
130°	0.690	0.676	0.690			
140°	1.377	1.402	1.377			
150°	2.024	2.052	2.025			
160°	2.553	2.556	2.553			
170°	2.899	2.900	2.898			
180°	3.019	3.019	3.019			

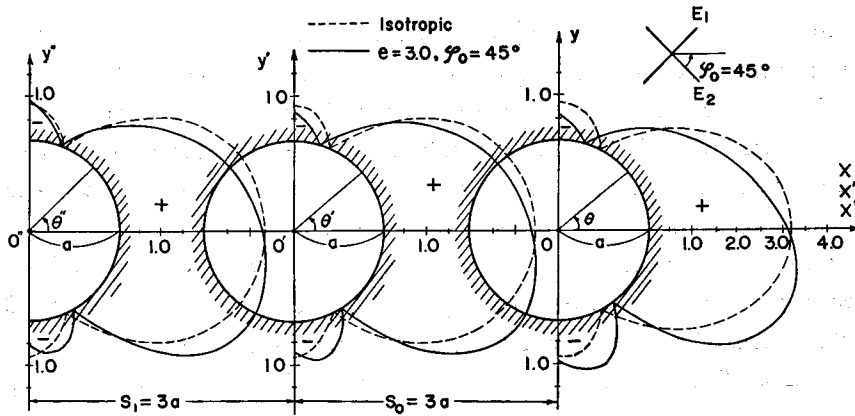


Fig. 9. Stress distributions of σ_η , σ_η' and σ_η'' around the holes for the case for isotropic plate and for the case of orthotropic plate ($e=3.0$, $\varphi_0=45^\circ$) containing three equal circular holes with the spacings $s_0/2a=s_1/2a=1.50$, when the normal stress $\sigma_y^0=1.0$ applies at infinity.

Table 3. Stress values of (a) σ_η and σ_η' around the right side and center holes, and (b) σ_x and σ_y on the x -axis between the center and outside holes, for the case of isotropic plate containing three equal circular holes with the spacings $s_0/2a=s_1/2a=1.50$ in a line on the x -axis, when the normal stress $\sigma_y^\circ=1.0$ applies at infinity.

$\theta=\theta'$	σ_η around the right side hole		σ_η' around the center hole		Position on the x -axis ($\times s_0/2a$)	σ_y on the x -axis		σ_x on the x -axis	
	after Hulbert ⁽¹⁴⁾	$\Delta\theta=25^\circ$ $m=20$ $i=4$	after Hulbert ⁽¹⁴⁾	$\Delta\theta=2.5^\circ$ $m=20$ $i=4$		after Hulbert ⁽¹⁴⁾	$\Delta\theta=2.5^\circ$ $m=20$ $i=4$	after Hulbert ⁽¹⁴⁾	$\Delta\theta=2.5^\circ$ $m=20$ $i=40$
0°	3.355	3.323	3.455	3.468	0.000	3.455	3.468	0.000	0.000
10°	3.273	3.246	3.366	3.373	0.125	2.844	2.847	0.311	0.311
20°	3.007	2.996	3.082	3.081	0.250	2.496	2.496	0.465	0.463
30°	2.559	2.540	2.614	2.580	0.375	2.313	2.313	0.534	0.530
40°	1.901	1.893	1.938	1.901	0.500	2.252	2.251	0.553	0.546
50°	1.099	1.137	1.123	1.127	0.625	2.296	2.293	0.531	0.520
60°	0.368	0.380	0.385	0.375	0.750	2.459	2.448	0.459	0.445
70°	-0.623	-0.236	-0.243	-0.247	0.875	2.783	2.772	0.305	0.289
80°	-0.681	-0.658	-0.650	-0.653	1.000	3.355	3.323	0.000	-0.003
90°	-0.842	-0.829	-0.786	-0.794					
100°	-0.749	-0.735							
110°	-0.407	-0.396							
120°	0.137	0.104							
130°	0.788	0.803							
140°	1.539	1.512							
150°	2.210	2.168							
160°	2.729	2.722							
170°	3.084	3.072							
180°	3.205	3.199							

As a general example of three holes we solved the problem of isotropic plate and orthotropic plate ($e=3.0, \varphi_0=0^\circ$) under the applied normal stress $w_0=1.0$ at infinity with an inclined angle $\delta=45^\circ$ from the x -axis for which the three unequal holes with dimensions $b=a, a'=b'=2.5 a$ and $a''=2.0 a, b''/a''=1.5$ located with the spacings $s_0/2a=4.5, s_1/2a'=6.0$, and $h_0=h_1=0$. Table 4 gives the values of $\sigma_\eta, \sigma_\eta'$ and σ_η'' around the holes at an angle 15° equal intervals on the circumference of each hole.

Numerical calculations in the above examples were carried out by the aid of FACOM 230-60 computer in Kyoto University. Computing time for the case of two holes in orthotropic plate was about 80 seconds and for the case of three holes about 130 seconds.

Table 4. Stress values of σ_η , $\sigma_{\eta'}$ and $\sigma_{\eta''}$ around the three holes with the dimensions $a=b$, $a'=b'=2.5a$, $a''=2.0a$, $b''/a''=1.5$ and with the spacings $S_0/2a=4.5$, $S_1/2a=6.0$, when the normal stress $w_0=1.0$ applies at infinity with an inclined angle $\delta=45^\circ$ from the x -axis.

$\theta=\theta'=\theta''$	σ_η on Σ_1		$\sigma_{\eta'}$ on Σ_2		$\sigma_{\eta''}$ on Σ_3	
	Isotropic	$\epsilon=3.0$ $\varphi_0=0^\circ$	Isotropic	$\epsilon=3.0$ $\varphi_0=0^\circ$	Isotropic	$\epsilon=3.0$ $\varphi_0=0^\circ$
0°	1.491	1.654	1.313	1.512	0.986	1.024
15°	0.105	-0.785	-0.688	-1.687	-0.703	-1.510
30°	-1.058	-1.782	-0.744	-1.452	-1.309	-2.118
45°	-1.762	-1.705	-0.891	-0.924	-1.211	-1.457
60°	-1.879	-1.225	-0.837	-0.415	-0.668	-0.415
75°	-1.395	-0.617	-0.384	0.107	0.500	0.681
90°	-0.403	0.069	0.359	0.634	2.162	1.726
105°	0.940	0.888	1.211	1.182	3.406	2.629
120°	2.421	1.957	2.068	1.824	3.633	3.266
135°	3.746	3.403	2.938	2.278	3.144	3.477
150°	4.475	5.034	3.650	3.959	2.387	3.096
165°	4.038	5.461	3.434	4.445	1.560	2.075
180°	2.214	2.514	1.698	1.928	0.766	0.667
195°	-0.159	-1.516	-0.470	-1.451	0.025	-0.599
210°	-1.793	-2.795	-1.574	-2.196	-0.631	-1.275
225°	-2.163	-2.116	-1.539	-1.448	-1.117	-1.276
240°	-1.466	-0.952	-0.895	-0.524	-1.232	-0.779
255°	-0.123	0.167	0.037	0.274	-0.676	-0.017
270°	1.436	1.165	1.068	0.964	0.582	0.837
285°	2.839	2.081	1.988	1.588	1.888	1.680
300°	3.792	2.956	2.596	2.173	2.685	2.490
315°	4.112	3.767	2.869	2.775	3.122	3.315
330°	3.750	4.284	3.204	3.654	3.320	3.993
345°	2.803	3.908	3.634	4.626	2.741	3.526

5. Concluding Remarks

A method of successive approximations utilizing the complex analytic function has been used to obtain the stress distributions around two or more arbitrary elliptical holes in an infinite anisotropic plate, when the external stress applies at infinity. The results obtained by the present approach are compared with the results obtained by others in which isotropic plate with two or three circular holes were treated, and their results are in quite agreement. It is believed that the present method can be used to solve the problem of multi-connected region such as an infinite anisotropic plate with several elliptical holes.

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