Effects of Couple Stresses on Stress Distributions in a Disc Specimen Subjected to Diametral Compression

By

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The present paper is concerned with the effects of couple stresses on the stress distribution in a disc subjected to diametral compression. An analytic solution was obtained by the Fourier-Bessel expansion method. Results are as follows.

(1) The effect of couple stresses is remarkable. The larger the material parameter l for bending rigidity becomes, the more the magnitude of stresses on the diametral plane in the loading direction is reduced, especially in tension. In the case, for instance, where the ratio of the material parameter l to the radius of the disc a, *i.e.* l/a, is 0.2, the tensile and the compressive stresses at the center are respectively reduced to about 50 percent and 80 percent of those obtained by the classical theory of elasticity. The stresses on the diametral plane perpendicular to the loading direction develop more uniformly with an increase of the ratio l/a.

(2) The effect of the loading width on the stress distribution is limited near the loaded boundary. As the loading width increases, the turning point of the circumferential stress from tensile to compressive is shifted toward the center and the radial compressive stress becomes more uniform. The influence of the loading width is less dominant in the couple stress theory.

(3) As the Poisson's ratio increases, the magnitude of both compressive and tensile stresses generally increases. The effect of the Poisson's ratio, however, is not very predominant.

1. Introduction

Knowledge of the stress distribution in a test specimen is a basic prerequisite to estimate the strength of the materials concerned. The diametral compression test of a disc, *i.e.* the so-called Brazilian test, is simple and easy to carry out and is widely accepted as a conventional test for estimating the tensile strength of rock-like materials. Several papers[1)~5)] were published on the usefulness of the test. In some of them,[1),2),4)] stress distributions were also discussed based on the classical theory of elasticity. As is well known, the classical theory of elasticity assumes the homogeneity of the constituent materials to the infinitesimal element of volume, that is, mass density is continuous and remains constant of any volume element is continuously shrunk to zero. This continuum approximation is violated

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for many materials composed of several distinct constituent materials, for example, for polycrystalline mixtures such as rocks, and for composite materials.

The classical theory of elasticity is successfully applied to the analysis of stress and strain wherever the overall dimension of the concerned problem is large as compared to an average dimension of the intrinsic discontinuities in the material, *e.g.* the average grain size or inter-grain distance, *i.e.* the average internal length of the constituent materials. However the ratio of the overall dimension to the average internal length decreases toward unity, the classical theory of elasticity is expected to fail. In such a case, the more precise theory which takes the effect of the constituent materials into consideration must be sought for.

The couple stress theory[6),7)] or micromorphic theory[8),9)] may find application in a wide variety of situations from crystal lattices to rocks or composite materials. As was discussed by Eringen[8)] and Cowin[10)], the couple stress theory is an extreme case of the micropolar (micromorphic) theory, and another extreme is the classical theory.

The present paper discusses the effects of the average internal length on the stress distributions in a disc specimen subjected to the diametral compression based on both extreme theories, *i.e.* the classical and the couple stress theories, since the average intrinsic discontinuity may well be predicted by them. On the effects of couple stresses on the stress distributions in the so-called uniaxial compression test specimen the reader is referred to the previous paper[11)].

2. Description of Problem

A disc specimen with radius a and unit thickness is subjected to diametral compression as shown in Fig. 1. The specimen is assumed to obey the couple stress theory of elasticity and in the state of plane strain.

According to Mindlin[6),7)], the fundamental equations of the couple-stress theory in plane strain are obtained as follows.

a) Kinematical relations

In general curvilinear coordinates x^α ($\alpha=1, 2,$ *i.e.* $r=x^1$, $\theta=x^2$ in Fig. 1), strains $e_{\alpha\beta}$, rotation ω_3 and curvatures $\kappa_{3\alpha}$ are expressed by displacements u_α as follows.

$$e_{\alpha\beta} = u_{(\alpha}|_{\beta)} = \frac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha) \qquad (\alpha,\ \beta=1,\ 2) \tag{2.1}$$

$$\omega_3 = u_{[2}|_{1]} = \frac{1}{2}(u_2|_1 - u_1|_2) \tag{2.2}$$

$$\kappa_{3\alpha} = \omega_3|_\alpha\,, \tag{2.3}$$

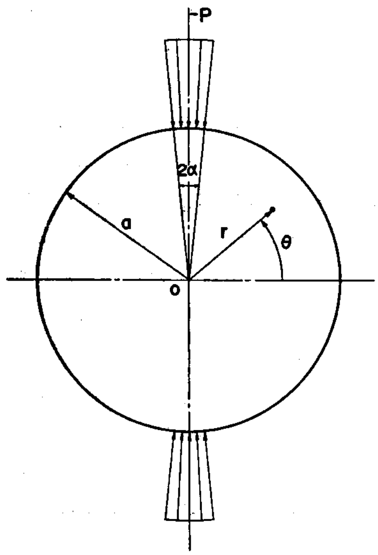

Fig. 1. Schematic diagram of the Brazilian test and the coordinate system.

where $u_\alpha|_\beta$ means the covariant derivative of u_α by x^β, and $u_{(\alpha}|_{\beta)}$ and $u_{[\alpha}|_{\beta]}$ mean the symmetrical and the antisymmetrical parts of $u_\alpha|_\beta$, respectively.

b) Constitutive relations

The constitutive relations are

$$e_{\alpha\beta} = \frac{1}{2G}[\tau_{(\alpha\beta)} - \nu g_{\alpha\beta}\tau^\gamma_\gamma] \tag{2.4}$$

$$\kappa_{3\alpha} = \frac{1}{4Gl^2}\mu_{\alpha 3}\,, \tag{2.5}$$

where $\tau_{\alpha\beta}$ and $\mu_{\alpha\beta}$ mean the Cauchy stress and the couple stress, respectively, and G, ν and l mean the shear modulus, the Poisson's ratio and the material parameter for bending rigidity, respectively, and $g_{\alpha\beta}$ is the metric tensor.

c) Equations of equilibrium

Disregarding the body force and the body couple, the equilibrium equations are expressed as

$$\tau^{\beta\alpha}|_\alpha = 0 \tag{2.6}$$

$$\mu^{\alpha 3}|_\alpha + \varepsilon^{3\alpha\beta}\tau_{\alpha\beta} = 0\,, \tag{2.7}$$

where $\varepsilon^{3\alpha\beta}$ means the permutation tensor.

d) Compatibility conditions

Compatibility conditions are

$$\varepsilon^{3\alpha\beta}\varepsilon^{3\delta\gamma}e_{\beta\gamma}|_{\alpha\delta} = 0 \tag{2.8}$$

$$\varepsilon^{3\alpha\beta}\kappa_{\alpha}|_{\beta} = 0\,, \tag{2.9}$$

which are expressed in terms of stresses as follows,

$$\varepsilon^{3\alpha\gamma}\varepsilon^{3\beta\delta}\tau_{(\alpha\beta)}|_{\gamma\delta} - \nu\nabla^2\tau^{\gamma}_{\gamma} = 0 \tag{2.10}$$

$$\varepsilon^{3\alpha\beta}\mu_{3\alpha}|_{\beta} = 0 \tag{2.11}$$

$$\mu_{\alpha 3} = \varepsilon^{3\beta\gamma}\tau_{(\alpha\gamma)}|_{\beta} + \nu\varepsilon_{3\alpha\beta}\tau^{\gamma}_{\gamma}|^{\beta}\,, \tag{2.12}$$

where $\nabla^2\tau = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{g}\,g^{\alpha\beta}\frac{\partial\tau}{\partial x^{\beta}}\right)$, $g = \det|g_{\alpha\beta}|$. The two of the three Eqs. (2.10), (2.11) and (2.12) are independent.

e) Stress functions

Stresses are assumed to be expressed by two such potential functions ϕ and ψ as

$$\tau^{\alpha\beta} = \varepsilon^{3\alpha\gamma}\varepsilon^{\beta\delta}\phi|_{\gamma\delta} + \varepsilon^{3\gamma\alpha}\psi|^{\beta}_{\gamma} \tag{2.13}$$

$$\mu_{3\alpha} = \psi|_{\alpha}\,. \tag{2.14}$$

The functions must satisfy the following differential equations in order to satisfy the equilibrium equations and the compatibility conditions and *vice versa*,

$$\nabla^4\phi = 0 \tag{2.15}$$

$$(1 - l^2\nabla^2)\nabla^2\psi = 0 \tag{2.16}$$

$$(1 - l^2\nabla^2)\psi|_{\alpha} = -2(1-\nu)l^2\varepsilon_{3\alpha\beta}g^{\beta\gamma}\nabla^2\phi|_{\gamma}\,. \tag{2.17}$$

When $l=0$, all the relations mentioned above reduce to those of the classical theory of elasticity.

The solution of the problem of the couple stress theory of elasticity in plane strain is obtained by solving the field equations (2.15) to (2.17) with appropriate boundary conditions.

Let us consider a disc subjected to diametral compression. The compressive force is distributed uniformly and normally over the opposite sides of the surface of the disc, with a width $2a\alpha$ (α: rad) (Fig. 1).

The boundary conditions of the present problem are as follows, on $r=a$

$$\left.\begin{aligned} \sigma_r &= p(\theta) \\ \tau_{r\theta} &= 0 \\ \mu_r &= 0\,. \end{aligned}\right\} \tag{2.18}$$

In the above expressions, notations are redefined as σ_r, $\tau_{r\theta}$ and μ_r the components of the normal stress, the shear stress and the couple stress acting on the plane r=const., respectively.

3. Analytical Solutions

The solutions of the field equations (2.15) and (2.16) of the present problem are

$$\phi = A_0 r^2 + \sum_{n=2,4\cdots}^{\infty} (A_n r^{n+2} + B_n r^n) \cos n\theta \tag{3.1}$$

$$\psi = \sum_{n=2,4\cdots}^{\infty} \left\{ C_n r^n + D_n I_n\left(\frac{r}{l}\right)\right\} \sin n\theta\,, \tag{3.2}$$

where $I_n(r/l)$ is the modified Bessel function of the first kind and A_0, A_n, B_n, $\cdots$ are constants to be determined.

Substituting Eqs. (3.1) and (3.2) into Eq. (2.17), following relations are obtained

$$C_n = 8(n+1)(1-\nu)l^2 A_n\,. \tag{3.3}$$

Stresses of Eqs. (2.13) and (2.14) are expressed in the extended form as

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta} \\ &= 2A_0 + \sum_{n=2,4\cdots}^{\infty}\Big\{-(n-2)(n+1)r^n A_n - n(n-1)r^{n-2}B_n \\ &\quad - n(n-1)r^{n-2}C_n + \frac{n}{r^2}\Big[I_n\left(\frac{r}{l}\right) - \frac{r}{l}I_n'\left(\frac{r}{l}\right)\Big]D_n\Big\}\cos n\theta\end{aligned} \tag{3.4}$$

$$\begin{aligned}\sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta} \\ &= 2A_0 + \sum_{n=2,4\cdots}^{\infty}\Big\{(n+1)(n+2)r^n A_n + n(n-1)r^{n-2}B_n \\ &\quad + n(n-1)r^{n-2}C_n - \frac{n}{r^2}\Big[I_n\left(\frac{r}{l}\right) - \frac{r}{l}I_n'\left(\frac{r}{l}\right)\Big]D_n\Big\}\cos n\theta\end{aligned} \tag{3.5}$$

$$\begin{aligned}\tau_{r\theta} &= -\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \\ &= \sum_{n=2,4\cdots}^{\infty}\Big\{n(n+1)r^n A_n + n(n-1)r^{n-2}B_n \\ &\quad + n(n-1)r^{n-2}C_n + \frac{1}{r^2}\Big[n^2 I_n\left(\frac{r}{l}\right) - \frac{r}{l}I_n'\left(\frac{r}{l}\right)\Big]D_n\Big\}\sin n\theta\end{aligned} \tag{3.6}$$

$$\tau_{\theta r} = -\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{\partial^2\psi}{\partial r^2}$$

$$= \sum_{n=2,4\cdots}^{\infty} \Big\{ n(n+1)r^n A_n + n(n-1)r^{n-2} B_n$$

$$+ n(n-1)r^{n-2} C_n + \frac{1}{l^2} I_n''\left(\frac{r}{l}\right) D_n \Big\} \sin n\theta \tag{3.7}$$

$$\mu_r = \frac{\partial \psi}{\partial r}$$

$$= \sum_{n=2,4\cdots}^{\infty} \Big\{ n r^{n-1} C_n + \frac{1}{l} I_n'\left(\frac{r}{l}\right) D_n \Big\} \sin n\theta \tag{3.8}$$

$$\mu_\theta = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$= \sum_{n=2,4\cdots}^{\infty} \Big\{ n r^{n-1} C_n + \frac{n}{r} I_n\left(\frac{r}{l}\right) D_n \Big\} \cos n\theta \tag{3.9}$$

where the prime means the differentiation by r/l.

Strains are also easily obtained by substituting Eqs. (3.4) to (3.9) into Eq. (2.4), but are not expressed here.

The constants A_0, A_n, B_n, $\cdots$ will be determined from the three boundary conditions (2.8) and the compatibility relation (2.7), *i.e.* Eq. (3.3) in the final form. As the first of the boundary conditions (2.18) is expressed in Fourier cosine series as

$$\sigma_r = \frac{a_0}{2} + \sum_{n=2,4\cdots}^{\infty} a_n \cos n\theta , \tag{3.10}$$

where

$$a_0 = \frac{4p\alpha}{\pi}$$

$$a_n = (-1)^{\frac{n}{2}} \frac{4p}{n\pi} \sin n\alpha ,$$

the substitution of the Eqs. (3.4), (3.6) and (3.8) into the boundary conditions (2.8) with Eq. (3.10) leads to the following system of linear equations with respect to the unknown constants

$$A_0 = \frac{p\alpha}{\pi} \tag{3.11}$$

$$-(n-2)(n+1)a^n A_n - n(n-1)a^{n-2} B_n - n(n-1)a^{n-2} C_n$$

$$+ \frac{n}{a^2} \Big\{ I_n\left(\frac{a}{l}\right) - \frac{a}{l} I_n'\left(\frac{a}{l}\right) \Big\} D_n = (-1)^{\frac{n}{2}} \frac{4p}{n\pi} \sin n\alpha \tag{3.12}$$

$$n(n+1)a^n A_n + n(n-1)a^{n-2} B_n + n(n-1)a^{n-2} C_n$$

$$+ \frac{1}{a^2} \Big\{ n^2 I_n\left(\frac{a}{l}\right) - \frac{a}{l} I_n'\left(\frac{a}{l}\right) \Big\} D_n = 0 \tag{3.13}$$

$$na^{n-1}C_n+\frac{1}{l}I'_n\left(\frac{a}{l}\right)D_n=0\,. \tag{3.14}$$

In calculations, we are forced to truncate the higher terms of the equations because of the limited capacity of the computor. In the present computation, the terms higher than $n=50$ were truncated. Even in this computation we must adapt some technique.

In order to obtain the values of the modified Bessel functions up to the 50th order with sufficient accuracy, the values of the 50th and the 51st orders were calculated at first by Miller's method, then those of the lower orders were successively calculated through the recurrence formula down to that of the zero order. Although the values of the modified Bessel functions decrease very rapidly with an increase of their order, even the values of the relatively higher orders may not be disregarded because they play a considerable role in the solution of the problem.

In the determination of the constants, it is necessary to modify the coefficients of the system of equations (3.12~3.14) so that the difference of the maximum and the minimum is within several powers of ten. The coefficients accompanied with the modified Bessel function were too small for direct calculation, so they were reduced to the order of unity by dividing by an appropriate constant, say $I_n(a/l)$. Similar technique was also reported in the previous paper[11].

4. Results and Discussions

The collected results of stresses on the diametral plane both in the directions of loading and of its perpendicular are shown in Fig. 2. This figure shows the remarkable influence of the material parameter l. When the ratio l/a is very small, the stress distribution is similar to the well-known one obtained by the classical theory of elasticity, *i.e.* on the diametral plane in the loading direction σ_θ is tensile and constant $P/\pi a$, where $P=2a\alpha p$, over a wide region about the center and σ_r is compressive, whose magnitude is $3P/\pi a$ at the center and gradually increases toward the boundary. As the ratio l/a increases, the magnitude of stresses on the diametral plane in the loading direction reduces, especially remarkably in tension, the turning point of σ_θ from tension to compression is shifted toward the center, and σ_θ compressive for small ratio of l/a turns into tensile near the loading boundary. When $l/a=0.2$, σ_θ and σ_r are respectively reduced to as much as about 0.5 and 0.8 of those obtained by the classical theory. The stress distribution on the diametral plane perpendicular to the loading direction becomes more uniform with an increase of the ratio l/a.

The effect of the loading width on the stress distribution on the diametral plane in the loading direction is shown in Figs. 3 and 4. The effect is limited

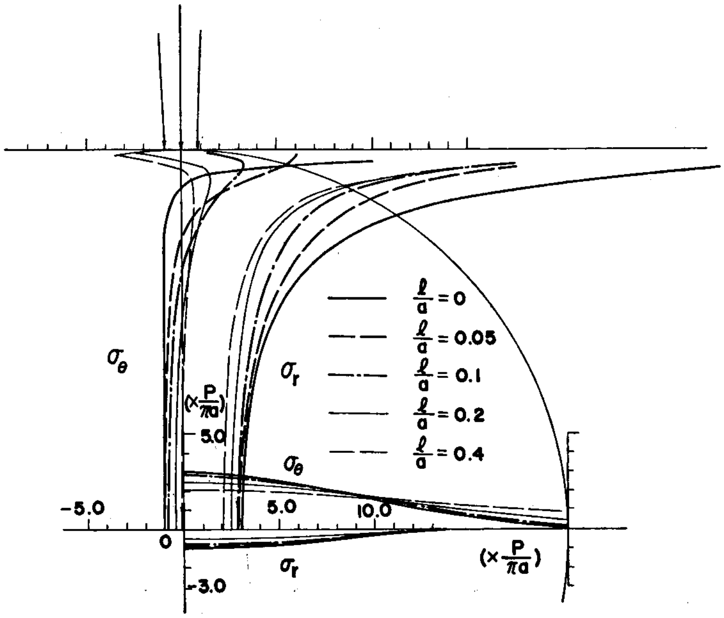

Fig. 2. Effects of the material parameter l on the circumferential and the radial stresses on the diametral planes in the loading direction and in its perpendicular.

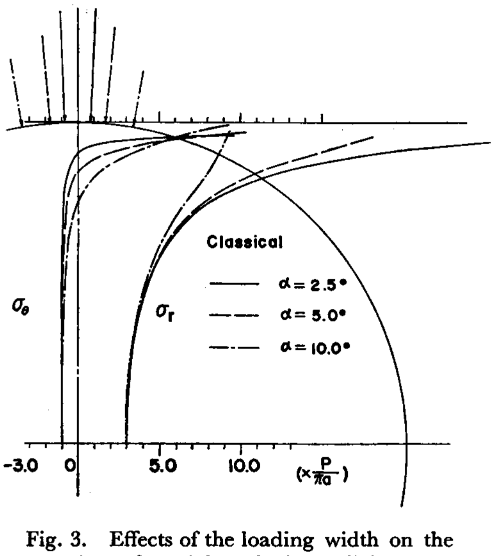

Fig. 3. Effects of the loading width on the circumferential and the radial stresses on the diametral plane in the loading direction (the classical theory).

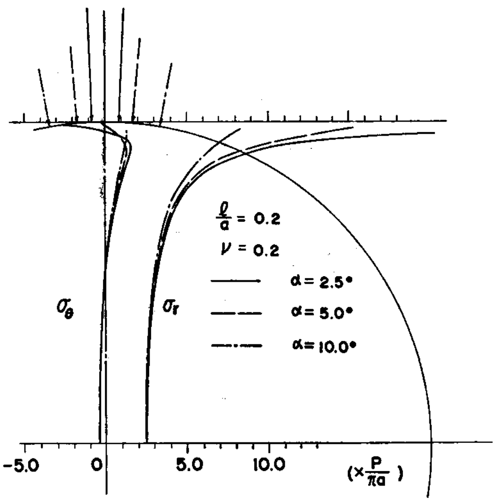

Fig. 4. Effects of the loading width on the circumferential and the radial stresses on the diametral plane in the loading direction (the couple stress theory, $l/a=0.2$ and $\nu=0.2$)

near the loaded boundary. As loading width increases, the turning point of σ_θ from tension to compression is shifted toward the center and the compressive stress becomes more uniform. The effect of the loading width is less dominant in the couple stress theory. It is noted here that the stress distribution obtained by the classical theory is valid for any value of the Poisson's ratio, although it is not for the couple stress theory.

The effect of the Poisson's ratio on the stress distribution on the diametral

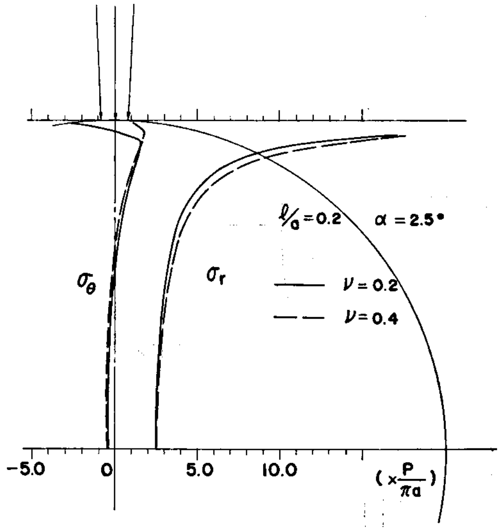

Fig. 5. Effects of the Poisson's ratio on the circumferential and the radial stresses on the diametral plane in the loading direction when $l/a=0.2$ and $\alpha=2.5°$.

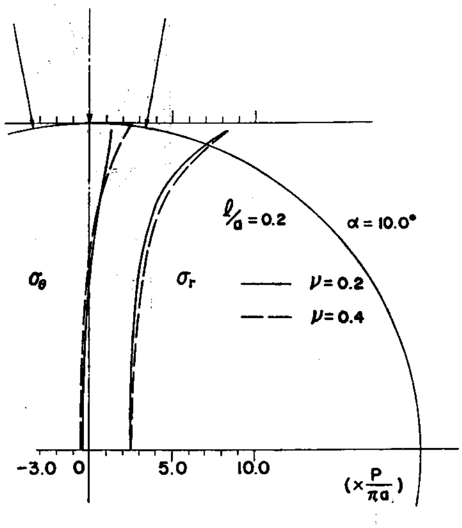

Fig. 6. Effects of the Poisson's ratio on the circumferential and the radial stresses on the diametral plane in the loading direction when $l/a=0.2$ and $\alpha=10.0°$.

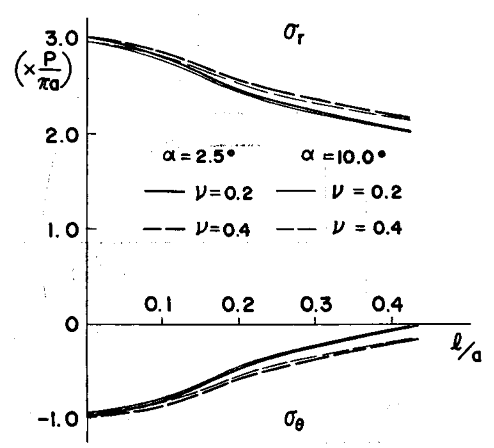

Fig. 7. Variations of the circumferential and the radial stresses at the center with l/a.

plane in the loading direction is observed in Figs. 5 and 6. As the Poisson's ratio increases, the magnitude of both compressive and tensile stresses, σ_r and σ_θ, generally increases. The effect of the Poisson's ratio, however, is not so predominant.

The variations of the tensile and the compressive stresses σ_θ and σ_r at the center of the specimen with the ratio l/a are shown in Fig. 7. The magnitude of both stresses decreases rapidly as the ratio l/a increases.

5. Concluding Remarks

The analytical results indicate that couple stresses have remarkable effects on the stress distributions in the Brazilian test specimen, in particular on the tensile stress in the central region of the specimen. It is advisable to take the effects of couple stresses into consideration in the interpretation of the experimental results of the Brazilian test.

Acknowledgements

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