Analysis of the Toroidal Magnetic Field in the Heliotron C Device

By

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An analysis of the magnetic field produced by a current flowing in the Heliotron C coils is carried out. Under the assumption that radius of coils is much smaller than that of torus, the calculation resorts to the perturbation method in which all quantities are expanded in κa, where κ is the curvature of torus and a the radius of coil.

The Heliotron C magnetic field is obtained analytically as the function of coil radius, the current ratio of adjacent coils, the distance of adjacent colis and the curvatre of torus.

The accuracy of this analyzed magnetic field is examined according to whether the position of neutral line calculated by the obtained formula agrees with that of experiment or not.

1. Introduction

The Heliotron C magnetic field was designed in order to be used in the Heliotron C device.[1)2)3)] This device was constructed in Kyoto University in colaboration with the Institute of Plasma Physics in Nagoya University and its aim was to investigate the spatial confinement of plasma.

When we investigate the plasma confinement in a nonuniform magnetic field such as the Heliotron C magnetic field or the bumpy field, it is necessary to get the exact formulas of the fields[4)]. But owing to a toroidal curvature effect, we can not easily obtain the analytical equations in general.

The main purpose in this paper is to get an analytical formula which describes the toroidal Heliotron C field and to evaluate the accuracy of the formula by comparing the analyzed magnetic field with that of the experiment.

The analytical formula is obtained by adopting the expansion techniques:[5)] all field quantities are written as power series in a small parameter κr which gives a measure to the magnitude of the toroidal curvature effect.

We analyze the toroidal Heliotron C magnetic field in Section 2 and compare

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the analytical results with that of the experiment in Section 3. The distribution of the field intensity is measured experimentally.

The obtained analytical formula to the first order of κ is found to be in good agreement with that of the experiment with respect to the location of the neutral line and the magnetic field intensity.

When considering the plasma confinement or particle confinement in the Heliotron C field, we can utuilize this first order analytical formula with good approximation. An extension to the calculation of the poloidal Heliotron magnetic field is carried out easily by using the same method.

2. Analysis of Heliotron C magnetic field

We take the (r, θ, z) toroidal coordinate such that z-axis lies along a circle of radius R and unit vector e_θ is perpendicular to e_r and e_z respectively. The toroidal perturbation introduces a small constant of curvature of z-axis, κ, so that $\kappa r \ll 1$. (Fig. 1)

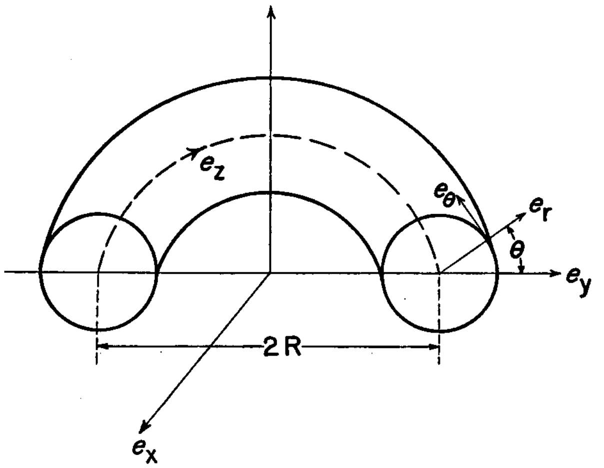


Fig. 1. Toroidal Coordinates.

The operator ∇ is defined as

$$\nabla = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + e_z \frac{1}{1+\kappa r \cos\theta}\frac{\partial}{\partial z} \tag{2-1}$$

wher $\kappa = 1/R$.

Then operating ∇ on the vector function F and scalar function f, respectively, we get next formulas with $\eta = 1 + \kappa r \cos\theta$;

$$\nabla\cdot \boldsymbol{F}=\frac{1}{r\eta}\left[\frac{\partial}{\partial r}(r\eta F_r)++\frac{\partial}{\partial \theta}(\eta F_\theta)+\frac{\partial}{\partial z}(rF_z)\right] \tag{2-2}$$

$$\nabla\times \boldsymbol{F}=\boldsymbol{e}_r\frac{1}{r\eta}\left[\frac{\partial}{\partial \theta}(\eta F_z)-\frac{\partial}{\partial z}(rF_\theta)\right]+\boldsymbol{e}_\theta\frac{1}{\eta}\left[\frac{\partial}{\partial z}F_r-\frac{\partial}{\partial r}\eta F_z\right]$$
$$+\boldsymbol{e}_z\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta)-\frac{\partial}{\partial \theta}F_r\right] \tag{2-3}$$

$$\nabla\cdot f=\boldsymbol{e}_r\frac{\partial}{\partial r}f+\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}f+\boldsymbol{e}_z\frac{1}{\eta}\frac{\partial}{\partial z}f \tag{2-4}$$

When these equations are expanded for small κr, the equations obtained are

$$\nabla\phi=\nabla^0\phi+\sum_{n=1}^{\infty}\phi_z(-\kappa r\cos\theta)^n\boldsymbol{e}_z \tag{2-5}$$

$$\nabla\cdot\boldsymbol{F}=\nabla^0\cdot\boldsymbol{F}-\sum_{n=1}^{\infty}\left[\frac{1}{n}\boldsymbol{F}\cdot\text{grad }(-\kappa r\cos\theta)^n-\frac{\partial}{\partial z}F_z(-\kappa r\cos\theta)^n\right] \tag{2-6}$$

and

$$\nabla\cdot\nabla\phi=\nabla^0\cdot\nabla^0\phi-\sum_{n=1}^{\infty}\left[\frac{1}{n}(\text{grad }\phi)\cdot\text{grad }(-\kappa r\cos\theta)^n\right.$$
$$\left.-\left(1+\frac{1}{\eta}\right)\phi_{zz}(-\kappa r\cos\theta)^n\right] \tag{2-7}$$

where $\phi_z\equiv\partial\phi/\partial z$ $\phi_{zz}\equiv\partial^2\phi/\partial z^2$ and

$$\nabla^0=\boldsymbol{e}_r\frac{\partial}{\partial r}+\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\boldsymbol{e}_z\frac{\partial}{\partial z} \tag{2-8}$$

Now the Heliotron C magnetic field is produced by the $2N$ filamentary current loops of radius a, which are equally spaced along the circumference of the torus.

N current loops carry the current I and another N current loops carry the current $-\lambda I$. The direction of current is reversed in adjacent coils.

For analytical convenience, a set of current loops is expanedd into Fourier series. If we assume the positive current loops locate at z so as to satisfy $\cos(mz)=1$ and the negative current loops locate at z so as to satisfy $\cos(mz)=-1$, then the equation for a set of current loops is obtained as

$$j(z)=(1+\kappa a\cos\theta)^{-1}(1/2\, a_0+a_1\cos(mz)+a_2\cos(mz)+\cdots) \tag{2-9}$$

2here

$$1/2\, a_0=Im\,(1-\lambda)/2\pi$$
$$a_n=Im\,(1-\lambda\cos n\pi)/\pi\,. \tag{2-10}$$

We rewrite $j(z)$ as

$$j(z)=\frac{Im}{2\pi}(1+\kappa a\cos\theta)^{-1}[(1-\lambda)+2(1+\lambda)\cos mz+\cdots] \tag{2-11}$$

The κ perturbation on magnetic scalar potential ϕ is

$$\phi = \phi^0 + \phi^\kappa + \phi^{\kappa\kappa} + \cdots \tag{2–12}$$

The ϕ^0 which satisfies the axisymmetric Heliotron field is given by the following equation:

$$\phi^0 = Cz + \sum_{n=1}^{\infty} [A_n I_0(nmr) + B_n K_0(nmr)] \sin(nmz) \tag{2–13}$$

where I_0 and K_0 are modified Bessel functions.
In the case of $n=1$, ϕ^0 is

$$\phi^0 = Cz + [AI_0(mr) + BK_0(mr)] \sin(mz) \tag{2–14}$$

where C, A and B are arbitrary constants.

Next in order to seek the κ order solution we use eqs. (2–7), (2–12), and (2–14), then the κ order Laplace's equation for ϕ is

$$(\nabla\cdot\nabla)^0\phi^\kappa = -\phi_r \cos\phi + 2\phi_{zz}\kappa r\cos\theta \tag{2–15}$$

$$= -Amk[I_1(mr) + 2mrI_0(mr)] \sin(mz)\cos\theta$$
$$+ Bmk[K_1(mr) - 2nmrK_0(mr)] \sin(mz)\cos\theta \tag{2–16}$$

where

$$(\nabla\cdot\nabla)^0\phi^\kappa \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi^\kappa\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^\kappa + \frac{\partial^2}{\partial z^2}\phi^\kappa\,. \tag{2–17}$$

The solution of ϕ^κ satisfying eq. (2–16) is composed of both homogeneous solution and inhomogeneous solution. The homogeneous solution may be written

$$\phi^\kappa{}_{\mathrm{homo}} = [A^\kappa I_1(mr) + B^\kappa K_1(mr)] \sin(mz)\cos\theta \tag{2–18}$$

since it satisfies the next differential equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\phi^\kappa\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta^2}\phi^\kappa + \frac{\partial}{\partial z^2}\phi^\kappa = 0\,. \tag{2–19}$$

Meanwhile inhomogeneous solution for eq. (2–16) is of the form:

$$\phi^\kappa{}_{\mathrm{inho}} = [f(mr) + g(mr)] \sin(mz)\cos\theta. \tag{2–20}$$

where $f(mr)$ and $g(mr)$ are functions of mr.

The differential equations obtained from eq. (2–16) for f and g are of the form:

$$f'' + \frac{1}{rm}f' - \left(1 + \frac{1}{(rm)^2}\right)f + \frac{A\kappa}{m}[3I_1(mr) + 2mrI_1'(mr)] = 0 \tag{2–21}$$

$$g''+\frac{1}{rm}g'-\left(1+\frac{1}{(rm)^2}\right)g+\frac{B\kappa}{m}[3K_1(mr)+2mrK_1'(mr)]=0 \qquad (2\text{–}22)$$

where the prime denotes differentiation by mr.

With the identity

$$I_\nu''+\frac{1}{hr}I_\nu'-\left[\left(\frac{\nu}{hr}\right)^2+1\right]I_\nu=0$$

and the similar relations for K_ν, K_ν'' and K_ν'', it can easily be verified that

$$f=-\frac{A\kappa}{2m}[mrI_1'(mr)+(mr)^2I_1(mr)] \qquad (2\text{–}23)$$

$$g=\frac{B\kappa}{2m}[mrK_1'(mr)+(mr)^2K_1(mr)] \qquad (2\text{–}24)$$

are the particular solutions of eqs. (2–21) and (2–22), respectively.

Then the solution of eq. (2–16) is

$$\begin{aligned}\phi^\kappa &= \phi^\kappa{}_{\text{homo}}+\phi^\kappa{}_{\text{inhomo}}\\ &= [A^\kappa I_1(mr)+B^\kappa K_1(mr)]\sin(mz)\cos\theta\\ &\quad -\frac{A\kappa}{2m}[mrI_1'+(mr)^2I_1]\sin(mz)\cos\theta\\ &\quad +\frac{B\kappa}{2m}[mrK_1'+(mr)^2K_1]\sin(mz)\cos\theta\end{aligned} \qquad (2\text{–}25)$$

For $a>r$, the magnetic potential ϕ^I is readily shown to be

$$\begin{aligned}\phi^I &= Cz+AI_0(mr)\sin(mz)+A^\kappa I_1(mr)\sin(mz)\cos\theta\\ &\quad -\frac{A\kappa}{2m}[mrI_1'+(mr)^2I_1]\sin(mz)\cos\theta\end{aligned} \qquad (2\text{–}26)$$

For $a<r$, the magnetic potential ϕ^{II} is

$$\begin{aligned}\phi^{II} &= BK_0(mr)\sin(mz)+B^\kappa K_1(mr)\sin(mz)\cos\theta\\ &\quad +\frac{B\kappa}{2m}[mrK_1'+(mr)^2K_1]\sin(mz)\cos\theta\end{aligned} \qquad (2\text{–}27)$$

The components of magnetic flux density B in the two regions are given as

$$\begin{aligned}B_r^I &= AmI_1(mr)\sin(mz)+A^\kappa mI_1'(mr)\sin(mz)\cos\theta\\ &\quad -\frac{A\kappa}{2m}[mrI_1'+(mr)^2I_1]'m\sin(mz)\cos\theta\end{aligned} \qquad (2\text{–}28)$$

$$B_r^{II} = -BmK_1(mr)\sin(mz) + B^\kappa mK'(mr)\sin(mz)\cos\theta + \frac{B\kappa}{2m}[mrK_1' + (mr)^2K_1]'m\sin(mz)\cos\theta \tag{2–29}$$

$$rB_\theta^I = -A^\kappa I_1(mr)\sin(mz)\sin\theta + \frac{A\kappa}{2m}[mrI_1' + (mr)^2I_1]\sin(mz)\sin\theta \tag{2–30}$$

$$rB_\theta^{II} = -B^\kappa K_1(mr)\sin(mz)\sin\theta - \frac{B\kappa}{2m}[mrK_1' + (mr)^2K_1]\sin(mz)\sin\theta \tag{2–31}$$

$$(1+\kappa r\cos\theta)B_z^I = C + AmI_0(mr)\cos(mz) + A^\kappa mI_1(mr)\cos(mz)\cos\theta - \frac{A\kappa}{2m}[mrI_1' + (mr)^2I_1]m\cos(mz)\cos\theta \tag{2–32}$$

and

$$(1+\kappa r\cos\theta)B_z^{II} = BmK_0(mr)\cos(mz) + B^\kappa mK_1(mr)\cos(mz)\cos\theta + \frac{B\kappa}{2m}[mrK_1' + (mr)^2K_1]m\cos(mz)\cos\theta \tag{2–33}$$

Now, five arbitrary coefficients in the above equations are determined so as to satisfy the conditions:

$$\boldsymbol{n}\cdot[\boldsymbol{B}] = 0$$
$$\boldsymbol{n}\times[\boldsymbol{B}] = \mu_0\boldsymbol{j}\,.$$

at the current sources, $\boldsymbol{r} = \boldsymbol{a}$, where $[\boldsymbol{B}]$ is the jump at the current sources.

After the calculation, we get

$$C = \mu_0\frac{Im}{2\pi}(1-\lambda) \tag{2–34}$$

$$A = \mu_0\frac{Im}{2\pi}2a(1+\lambda)K_1(ma) \tag{2–35}$$

$$B = -\mu_0\frac{Im}{2\pi}2a(1+\lambda)I_1(ma) \tag{2–36}$$

$$A^\kappa = -ma\frac{\kappa}{2m}\mu_0\frac{Im}{2\pi}2a(1+\lambda)\hat{A} \tag{2–37}$$

$$B^\kappa = -ma\frac{\kappa}{2m}\mu_0\frac{Im}{2\pi}2a(1+\lambda)\hat{B} \tag{2–38}$$

where

$$\hat{A} = (-I_0K_1K_1 - I_1K_0K_1 - mrI_0K_0K_1 - mrK_1 - mrI_1K_0K_0)_{r=a}\,,$$
$$\hat{B} = \left[I_1' - \frac{I_1}{mr} - mrI_1''I_1K_1 + mrI_1I_1K_1'' - (mr)^2I_1I_0K_0 + (mr)^2I_1K_0I_1\right]_{r=a}$$

The above equations are just solutions that indicate the toroidal Heliotron C magnetic field to the first order of κ.

In the next section it will be shown that this set of equations (2–28) to (2–38) is quite in accordance with the experimental measurement of the toroidal magnetic field in the Heliotron C device.

3. Heliotron C magnetic field

A) Measurement of Heliotron C magnetic field

Magnetic field coils for Heliotron C device are designed so that the neutral lines may be laid inside a discharge tube far from a wall and the mirror ratio along field line may become considerably low. The coils used in the device are composed of positive coils and negative coils. Each of the former has eighty turns and each of the latter has forty turns with three auxiliary taps which have 45 turns, 50 turns and 55 turns respectively.

These coils are set with regular intervals to one another along the mean circumference of discharge tube. 64 coils are used in the normal operation, and carry the currents up to 2000 A, produced by the condenser discharge with energy 240 K joule.

The maximum magnetic field intensity under the negative coils becomes about 2000 G and the mirror ratio along the magnetic lines of force is about 3.5 on the axis. Fig. 2 shows the configuration of the Heliotron C device. The magnetic lines of force obtained in experiment are shown in Fig. 3. These lines

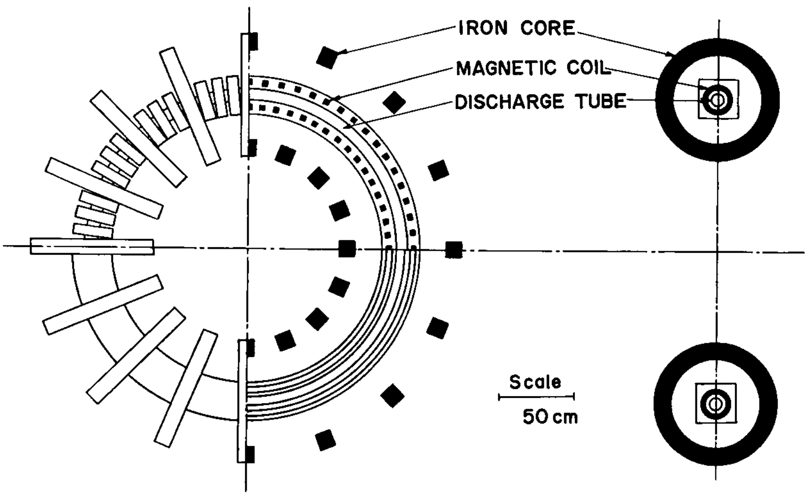
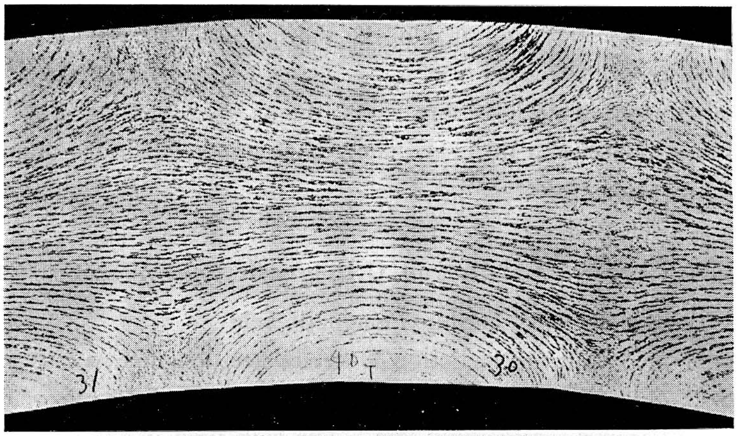
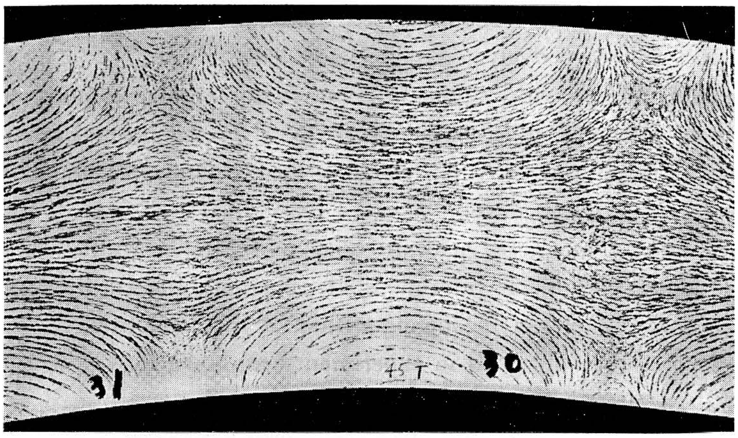


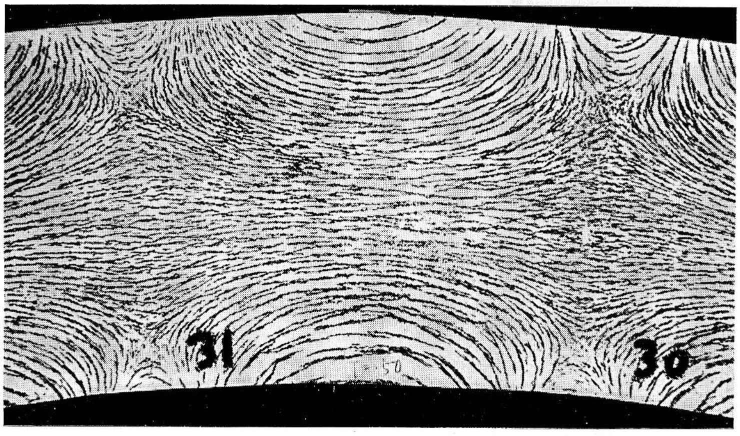
Fig. 2. The configuration of Heliotron C device.



(a) Toroidal Heliotron magnetic lines of force with $\lambda = 40/80$.



(b) Toroidal Heliotron magnetic lines of force with $\lambda = 45/80$.



(c) Toroidal Heliotron magnetic lines of force with $\lambda = 50/80$.

Fig. 3.

fo force are written with the iron powder method by changing the parameter of current ratio λ. The values of λ are 40/80, 45/80 and 50/80.

In the case of $\lambda=50/80$, the magnetic flux intensity distributions are shown in Fig. 4. An effect of the toriiodal curvature appears largely on the horizontal plane.

5) Comparison of the analyticzal results with that of experiment.

In order to check the accuracy of the set of eqs (2–28) to (2–33), we examine the difference between the analytical results and those of experiment. Of interest to us is the location of neutral line where the magnetic flux intensity becomes zero.

It is supposed that due to the toroidal curvature the neutral line is forced to move into the direction of the center of curvature to some extent.

Now, according to the analytical eq. (2–32), the z-component of magnetic flux intensity for $r<a$ is rewritten by introducing B_0, ε and α, as

$$B_z = B_0[1+\alpha I_0(mr)\cos(mz)][1-\kappa r\cos\theta] + \frac{B_0\kappa\varepsilon}{2m}\alpha I_1(mr)\cos\theta\cos(mz) \tag{3–1}$$

where

$$B_0 = \mu_0 Im(1-\lambda)/2\pi\ ,$$
$$\alpha = 2am(1+\lambda)K_1(ma)/(1-\lambda)\ ,$$
$$\varepsilon = ma\,\hat{A}/K_1((ma)\ .$$

If κ tends to zero, eq. (3–1) results in an axisymmetric case.

B-1) Position of neutral line in the vertical plane.

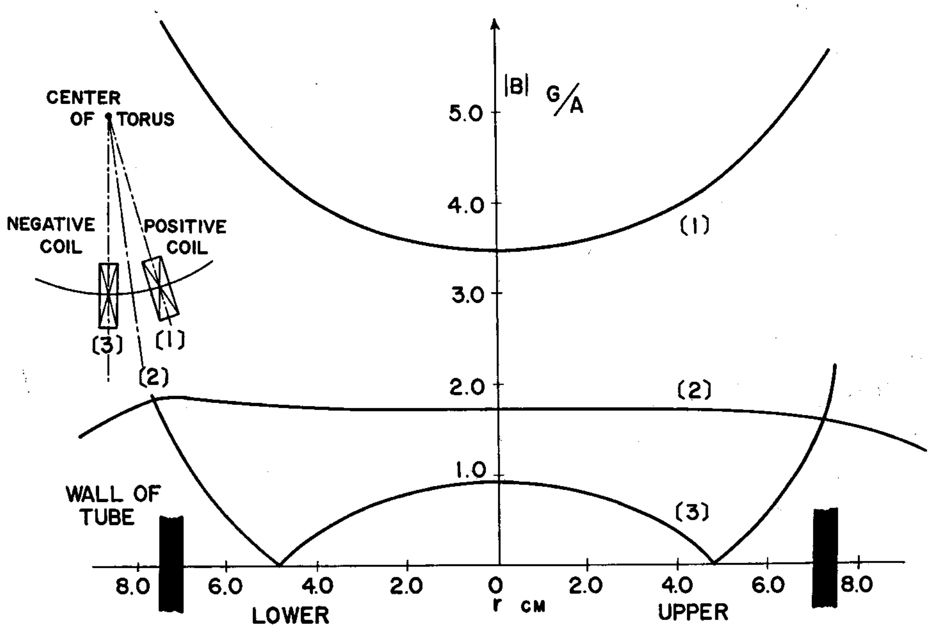
The magnetic flux intensity under the negative coils is given by putting $\cos(mz)=-1$ and in the case of $\cos\theta=0$, we get the value B_z in the vertical plane which includes the geometrical axis,

$$B_z = B_0[1-\alpha I_0(mr)] \tag{3–2}$$

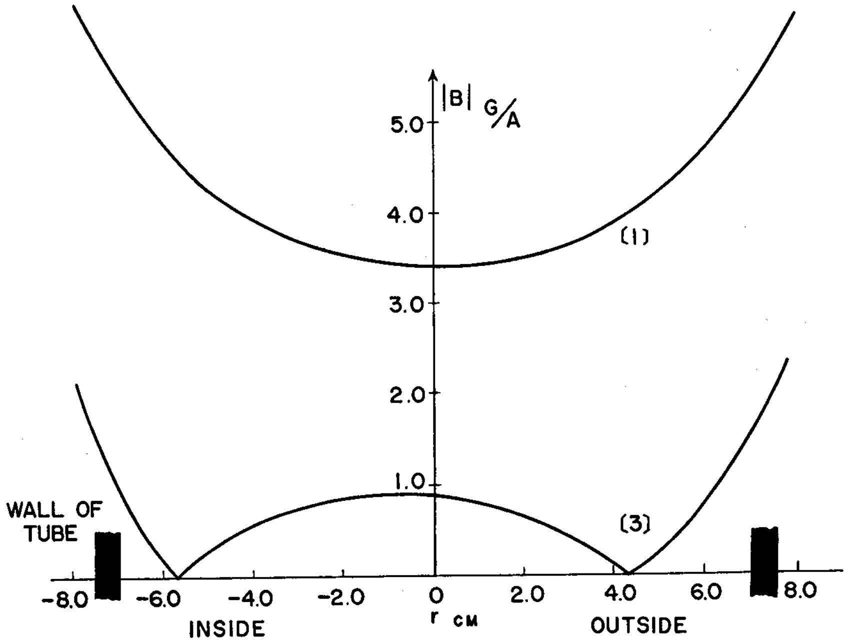
In the above equation, the effect of torus curvature does not appear. Using the numerical values in Table 1, the position of the neutral line can be claculated and results in $d=5.0$ cm, while the experimental reuslt is 4.9 cm according to Fig. 4.

Table 1. Numerical values for Heliotoron magnetid field.

$a=11.75$ cm	$L=10$ cm	$\lambda=0.625$	$ma=3.69$	$K_1(ma)=0.0178$	$D_1=maK_1(ma)$
$=0.0658$	$\alpha=2D_1/(1+\lambda)/(1-\lambda)=0.57$				



(a) Magnetic field intensity of Heliotron C with $\lambda=50/80$ in the vertical plane.



(b) Magnetic field intensity of Heliotron C with $\lambda=50/80$ in the horizontal plane.

Fig. 4.

B-2) Position of neutral line in the horizontal plane.

There are two cases in the position of neutral line; one is nearer to the center of tours, i.e., $\cos\theta=-1$ and the other is farther away from the center of torus, i.e., $\cos\theta=1$.

In the case of $\cos\theta=1$, the magnetic flux intensity is

$$B_z/B_0=[1-\alpha I_0(mr)](1-\kappa r)-\frac{\kappa\varepsilon}{2m}\alpha I_1(mr) + \kappa\alpha[mrI_1'+(mr)^2 I_1]/2m \tag{3-3}$$

and after the little calculation, $B_z=0$ occurs at $mr=1.38$. Then, the position of neutral line is $d=4.4$ cm, meanwhile the experimental result is 4.4 cm according to Fig. 4.

In the case of $\cos\theta=-1$, the magnetic flux intensity can be rewritten as

$$B_z/B_0=[1-\alpha I_0(mr)](1+\kappa r)+\frac{\kappa\varepsilon}{2m}\alpha I_1(mr) - \kappa\alpha[mrI_1'+(mr)^2 I_1]/2m \tag{3-4}$$

and after the calculation, $B_z=0$ occurs at $mr=1.81$. Then the position of neutral line is $d=5.76$ cm, meanwhile the experimental result is 5.8 cm according to Fig. 4.

Then as for the location of the neutral line, there is good agrerment between the experimental result and the analytical result.

4. Conclusion

An analysis of the magnetic field produced by a current flowing in the toroidal Heliotron coils is studied. Under the assumption that a radius of coils is much smaller than that of torus, the solution is expanded in powers of κa. The lowest and first order solutions of κ are obtianed so that magnetic scalar potential ϕ satisfies the boundary conditions on the circular current loops.

The Heliotron C magnetic field is given analytically as the function of coil radius, the current ratio of adjacent coils, the distance of adjacent coils and the curvature of torus.

The location of neutral lines calculated by the analetical formula shows good agreement with that of the experiment.

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