Generalized Prandtl-Reuss Plastic Materials

I. Constitutive Equation for Finite Deformation

By

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A generalized constitutive equation of Prandtl-Reuss type plastic materials is derived. Von Mises yield condition and Lévy-St. Venant flow rule are assumed for isotropic and non-workhardening material and the rate type constitutive equation of elastic materials is modified to include the plastic deformation. The obtained equation is in frame-indifference and holds for finite deformation.

Introduction

The constitutive equation of the so-called Prandtl-Reuss type plastic materials has been adopted as a basic equation in the plastic deformation of metals. However this equation is applied within the infinitesimal elastic deformation and has no frame-indifference, i.e., it has a different form in changing of the reference frame. The frame-indifference is one of the most important axioms of modern continuum mechanics.

Truesdell[1] proposed the constitutive equation, which is called hypo-elasticity. The co-rotational stress rate is a function of stress and stretching and linear in the latter. Thomas[2] and Green[3,4] discussed relations between hypo-elasticity and the incremental theories of plasticity and they attempted to reduce the latter to a special form of the former. Other attempts to seek the consistent constitutive equations of plastic materials were published. Green and Naghdi[5] developed the rate type plastic theory in the spirit of modern nonlinear continuum mechanics, and Lee[6] and Freund[7] proposed the new constitutive equations involving finite elastic and plastic strain.

This paper is concerned with a generalization of the Prandtl-Reuss type equation revised by Thomas[8] who corrected the equation by replacing the material time derivative of stress with the co-rotational one. The material is restricted to be isotropic and non-workhardening but may deform finitely. Von Mises yield con-

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dition and flow rule, that is the proportionality of deviatoric stress and plastic stretching, are assumed and the rate-type constitutive equation of elasticity is supposed to hold for the assumed elastic part of stretching in the plastic deformation. The equation obtained is consistent with respect to co-ordinate invariance and includes the finite deformation.

2. Rate Type Constitutive Equation of Isotropic Elastic Material

When the deformation state remains within the yield state, which will be depicted in the next section, the material is assumed to be isotropic and perfectly elastic and has the forms:

$$\tilde{\boldsymbol{S}} = \boldsymbol{f}(\boldsymbol{C}), \quad \boldsymbol{S} = \boldsymbol{h}(\boldsymbol{B}), \tag{2.1}$$

where $\tilde{\boldsymbol{S}}$ and $\boldsymbol{S}$ are non-dimensional stresses defined by

$$\tilde{\boldsymbol{S}} \equiv \tilde{\boldsymbol{T}}/\mu, \quad \boldsymbol{S} \equiv \boldsymbol{T}/\mu,$$

where $\tilde{\boldsymbol{T}}$ and $\boldsymbol{T}$ are respectively second Piola-Kirchhoff and Cauchy stress tensor and μ is the shear elastic modulus. They are connected with each other

$$\boldsymbol{S} = J^{-1}\boldsymbol{F}\tilde{\boldsymbol{S}}\boldsymbol{F}^T, \tag{2.2}$$

where $\boldsymbol{C} \equiv \boldsymbol{F}^T\boldsymbol{F}$ and $\boldsymbol{B} \equiv \boldsymbol{F}\boldsymbol{F}^T$ are respectively the right and left Cauchy-Green tensor, $\boldsymbol{F}$ is the deformation gradient and $J \equiv \det \boldsymbol{B} = \det \boldsymbol{C}$ is the Jacobian.

Co-rotational time differentiation of (2.2) yields

$$\mathring{\boldsymbol{S}} = \boldsymbol{D}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{D} - \boldsymbol{S}\,\mathrm{tr}\,(\boldsymbol{D}) + \mathsf{C}(\boldsymbol{B}) : \boldsymbol{D}, \tag{2.3}$$

where

$$\mathring{\boldsymbol{S}} \equiv \dot{\boldsymbol{S}} - \boldsymbol{W}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{W} \tag{2.4}$$

is the co-rotational time rate of $\boldsymbol{S}$,

$$\boldsymbol{D} \equiv \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \quad \boldsymbol{W} \equiv \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T) \tag{2.5}$$

are respectively stretching and spin tensor, $\boldsymbol{L} \equiv \mathrm{grad}\,\dot{\boldsymbol{x}}$ is the spatial gradient of the velocity and

$$\mathsf{C}^{kmpq} \equiv 2J^{-1}x^k{}_{,\alpha}\, x^m{}_{,\beta}\, x^p{}_{,\gamma}\, x^q{}_{,\delta} \frac{\partial f^{\alpha\beta}}{\partial c_{\gamma\delta}} \tag{2.6}$$

is the elasticity tensor, which can be expressed by $\boldsymbol{B}$ for the isotropic material, and we define $(\mathsf{C} : \boldsymbol{D})^{km} \equiv \mathsf{C}^{kmpq} D_{pq}$. See Truesdell and Noll (p. 142).[9]

We assume relation $(2.1)_2$ is invertible, then the elasticity tensor may be regarded as a function of stress and we have

$$\dot{S} = \mathsf{E}(S) : L \tag{2.7}$$

or

$$\mathring{S} = \mathsf{E}(S) : D \tag{2.8}$$

where

$$\mathsf{E}^{kmpq} \equiv \mathsf{C}^{kmpq} - S^{km} g^{pq} + S^{kq} g^{mp} + g^{kp} S^{mq} . \tag{2.9}$$

For the isotropic material we may assume that the stress has expansion

$$\begin{aligned}\tilde{S} &= \alpha_1 I_E \mathbf{1} + 2E + (\alpha_3 I_E^2 + \alpha_4 II_E)\mathbf{1} \\ &\quad + \alpha_5 I_E E + \alpha_6 E^2 + O(E^3) ,\end{aligned} \tag{2.10}$$

where $E = \frac{1}{2}(C - \mathbf{1})$ is the strain tensor, $\mathbf{1}$ is the unit tensor, I_E and II_E the invariants of E, $\alpha_1 = \lambda/\mu$, where λ and μ are Lamé constants, α_Γ ($\Gamma = 3, 4, 5, 6$) are the second-order elastic constants and $E \equiv \sqrt{\operatorname{tr}(EE^T)}$.

Substituting (2.10) into (2.6) and expressing the strain by stress, we have

$$\begin{aligned}\mathsf{C}^{kmpq}(S) &= \alpha_1 g^{km} g^{pq} + (g^{kp} g^{mq} + g^{kq} g^{mp}) \\ &\quad + \mathsf{C}_1^{kmpq}(S) + O(S^2) ,\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}\mathsf{C}_1^{kmpq}(S) &\equiv \{\beta_1 g^{km} g^{pq} + \beta_2 (g^{kp} g^{mq} + g^{kq} g^{mp})\} \operatorname{tr}(S) \\ &\quad + \frac{1}{2}(\alpha_5 S^{km} g^{pq} - \alpha_4 g^{km} S^{pq}) \\ &\quad + \frac{1}{4}\alpha_6 (S^{kp} g^{mq} + S^{kq} g^{mp} + g^{kp} S^{mq} + g^{kq} S^{mp}) ,\end{aligned} \tag{2.12}$$

and

$$\beta_1 \equiv \frac{\alpha_1(\alpha_4 - \alpha_5) + 4\alpha_3 + 2\alpha_4 - 2}{2(3\alpha_1 + 2)} , \quad \beta_2 \equiv \frac{\alpha_5 - \alpha_1 \alpha_6 - 2}{2(3\alpha_1 + 2)} \tag{2.13}$$

and where C is expanded with respect to stress and

$$S \equiv \sqrt{\operatorname{tr}(SS^T)} . \tag{2.14}$$

3. Prandtl-Reuss Type Constitutive Equation

We assume the von Mises yield condition and the isotropic flow rule for a perfectly plastic material. The former is given by

$$\mathrm{tr}\,(\boldsymbol{S}^{*2}) = 2\kappa^2\,, \tag{3.1}$$

where

$$\boldsymbol{S}^* \equiv \boldsymbol{S} - \frac{1}{3}\,\mathrm{tr}\,(\boldsymbol{S})\,\mathbf{1} \tag{3.2}$$

is the deviatoric stress tensor, and

$$\kappa \equiv \frac{k}{\mu} \tag{3.3}$$

is a non-dimensional material constant, where k equals the yield shear stress in pure shear test. The yield condition (3.1) shows the independence of yield on pressure. The flow rule is given by

$${}_P\boldsymbol{D} = \psi \boldsymbol{S}^*\,, \tag{3.4}$$

where ${}_P\boldsymbol{D}$ is the plastic part of total stretching $\boldsymbol{D}$ and ψ is a material scalar function of state and will be determined later. Relation (3.4) assures the incompressibility of ${}_P\boldsymbol{D}$ and is regarded as the definition of the plastic stretching and then

$$\left.\begin{aligned} {}_E\boldsymbol{L} &= \boldsymbol{L} - {}_P\boldsymbol{L} \\ \text{or} \qquad {}_E\boldsymbol{D} &= \boldsymbol{D} - {}_P\boldsymbol{D} \end{aligned}\right\} \tag{3.5}$$

may be regarded as a definition of the elastic part of velocity gradient, where ${}_P\boldsymbol{D} \equiv \frac{1}{2}({}_P\boldsymbol{L} + {}_P\boldsymbol{L}^T)$.

As Lee[6], if we assume an imaginary intermediate configuration ξ between the initial undeformed configuration $\boldsymbol{X}$ and the current one $\boldsymbol{x}$, we have

$$\boldsymbol{F} = \boldsymbol{F}_2 \boldsymbol{F}_1\,, \tag{3.6}$$

where

$$\boldsymbol{F} \equiv \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\,, \quad \boldsymbol{F}_1 \equiv \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{X}}\,, \quad \boldsymbol{F}_2 \equiv \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} \tag{3.7}$$

are deformation gradients. The total velocity gradient $\boldsymbol{L}$ is expressed by

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} = \boldsymbol{L}_2 + \boldsymbol{F}_2 \boldsymbol{L}_1 \boldsymbol{F}_2^{-1}\,, \tag{3.8}$$

where

$$\boldsymbol{L}_1 \equiv \dot{\boldsymbol{F}}_1 \boldsymbol{F}_1^{-1} \quad \text{and} \quad \boldsymbol{L}_2 \equiv \dot{\boldsymbol{F}}_2 \boldsymbol{F}_2^{-1} \tag{3.9}$$

are velocity gradients of two intermediate parts and the factors $\boldsymbol{F}_2$ and $\boldsymbol{F}_2^{-1}$ in the second term of $(3.8)_2$ indicate that $\boldsymbol{L}_1$ is transformed to the current configuration.

If we assume the first and the second intermediate parts as elastic and plastic ones respectively, we have

$$ {}_P\boldsymbol{L} = \boldsymbol{L}_2\,, \quad {}_E\boldsymbol{L} = \boldsymbol{F}_2\boldsymbol{L}_1\boldsymbol{F}_2^{-1} \tag{3.10} $$

and in inverse order we have

$$ {}_P\boldsymbol{L} = \boldsymbol{F}_2\boldsymbol{L}_1\boldsymbol{F}_1^{-1}\,, \quad {}_E\boldsymbol{L} = \boldsymbol{L}_2 \tag{3.11} $$

These separations of elastic and plastic parts are imaginary and depend on their order. Here we regard that (3.5) are a definition of ${}_E\boldsymbol{L}$ or ${}_E\boldsymbol{D}$.

It is reasonable in the microscopic point of view of crystal lattice as Lee[6] to assume that the elastic properties are almost invariant from pure elastic state to plastic flow. Thus we assume that the rate-type elastic constitutive equation (2.7) holds in the plastic flow when we substitute ${}_E\boldsymbol{L}=\boldsymbol{L}-{}_P\boldsymbol{L}$ in place of $\boldsymbol{L}$ in (2.7). Then we have

$$ \dot{\boldsymbol{S}} = \mathsf{E}(\boldsymbol{S}) : (\boldsymbol{L}-{}_P\boldsymbol{L}) \tag{3.12*} $$

and

$$ \overset{\circ}{\boldsymbol{S}} = \mathsf{E}(\boldsymbol{S}) : (\boldsymbol{D}-\psi\boldsymbol{S}^*)\,, \tag{3.13} $$

In order to obtain ψ, we use the yield condition (3.1). Taking the time derivative produces

$$ \mathrm{tr}\,(\boldsymbol{S}^*\overset{\circ}{\boldsymbol{S}}) = 0\,, \tag{3.14} $$

and then we have

$$ \psi = \frac{\boldsymbol{S}^* : \mathsf{E} : \boldsymbol{D}}{\boldsymbol{S}^* : \mathsf{E} : \boldsymbol{S}^*}\,, \tag{3.15} $$

which gives the desired equation:

$$ \overset{\circ}{\boldsymbol{S}} = \mathsf{P}(\boldsymbol{S}) : \boldsymbol{D}, \tag{3.16} $$

where

$$ \mathsf{P}^{kmpq} = \mathsf{E}^{kmpq} - \frac{\mathsf{E}^{kmef} S^*_{ef} S^*_{gh} \mathsf{E}^{ghpq}}{S^*_{ab} \mathsf{E}^{abcd} S^*_{cd}}\,. \tag{3.17} $$

Relation (3.4) with (3.15) shows that flow rule is independent of time scale. Equation (3.16) is equivalent to

$$ \dot{\boldsymbol{S}} = \mathsf{P}(\boldsymbol{S}) : \boldsymbol{L}\,. \tag{3.18} $$

The constitutive equation (3.16) is consistent with co-ordinate invariance and a special case of hypo-elasticity. We have the following identities

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* The author wishes to thank Professor T.W. Ting for showing me his manuscript "Topics in the Mathematical Theory of Plasticity", which will appear in Handbuch der Physik, Bd. VI in 1971. In that article, he obtained also by an elastic continuation process a similar constitutive equation with (3.13) for infinitesimal elastic deformation.

$$E^{kmpq}=E^{mkpq}\,,\quad P^{kmpq}=P^{mkpq}\,,$$
$$E^{kmpq}-E^{kmqp}=S^{kq}g^{mp}+g^{kp}S^{mq}-S^{kp}g^{mq}-g^{kq}S^{mp}=P^{kmpq}-P^{kmqp}\,. \tag{3.19}$$

In the plastic state the components of deviatoric stress S^* have the order of magnitude κ by means of the yield condition (3.1). If we assume that the hydrostatic pressure has at largest the order of κ, we may assume the components of stress S are in order of κ. Substituting (2.9), (2.11), (2.12) and (3.15) into (3.17), expanding with respect to the components of S and S^* and arranging with the same order of κ, we have

$$P={}_EP_0+{}_PP_0+{}_EP_1+{}_PP_1+O(\kappa^2)\,, \tag{3.20}$$

where

$$
\begin{aligned}
{}_EP_0^{kmpq} &\equiv \alpha_1 g^{km}g^{pq}+(g^{kp}g^{mq}+g^{kq}g^{mp})\,,\\
{}_PP_0^{kmpq} &\equiv -\frac{1}{\kappa^2}S^{*km}S^{*pq}\,,\\
{}_EP_1^{kmpq} &\equiv [\beta_1 g^{km}\beta^{pq}+\beta_2(g^{kp}g^{mq}+g^{kq}g^{mp})]\ \mathrm{tr}\,(S)\\
&\quad+\frac{1}{2}[(\alpha_5-2)S^{km}g^{pq}-\alpha_4 g^{km}S^{pq}]\\
&\quad+\frac{1}{4}[\alpha_6 S^{kp}g^{mq}+(\alpha_6+4)S^{kq}g^{mp}+(\alpha_6+4)g^{kp}S^{mq}+\alpha_6 g^{kq}S^{mp}]\,,\\
{}_PP_1^{kmpq} &\equiv -\frac{1}{2}(\alpha_5-2)S^{*km}g^{pq}+\frac{1}{2}\alpha_4 g^{km}S^{*pq}\\
&\quad-\frac{\alpha_6+2}{2\kappa^2}(S^{*km}S^{*p}{}_rS^{*rq}+S^{*k}{}_rS^{*rm}S^{*pq})\\
&\quad+\left(-\frac{6\beta_2+\alpha_6+2}{6\kappa^2}\ \mathrm{tr}\,(S)+\frac{\alpha_6+2}{4\kappa^4}\ \mathrm{tr}\,(S^{*3})\right)S^{*km}S^{*pq}\,.
\end{aligned}
$$

In the region $\mathrm{tr}\,(S^{*2})<2\kappa^2$ and in the state of no plastic flow on the condition (3.1), the material is governed by the constitutive equation $(2.1)_2$ or (2.8). In the plastic flow region on the condition (3.1) the material is governed by the constitutive equation (3.16). Equation (3.16) satisfies (3.14) automatically.

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