Generalized Prandtl-Reuss Plastic Materials

II. Characteristic Surfaces and Acceleration Wave Propagation

By

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The necessary condition for existence of characteristic surface is obtained for hypo-elastic materials and it is shown to coincide with the acceleration wave propagation condition. The wave speeds in plastic state are less than or equal to those in elastic state for the same stress state. The principal waves are analysed generally and classified into longitudinal or transverse waves for isotropic plastic materials. The principal transverse wave speeds have the same magnitudes for plastic and elastic state and they show identical acoustical birefringent effect for both states.

1. Introduction

Thomas[I.8]** investigated deeply and broadly the characteristic surfaces and acceleration waves in plastic materials. The constitutive equations adopted by him are linear for elastic deformation with the von Mises or Tresca yield condition. Furthermore he assumed the material to be incompressible. The last assumption excludes the existence of dilatational waves.

On the otherhand Tokuoka[1] proposed a generalized Prandtl-Reuss constitutive equation, which includes the non-linear elasticity and is consistent with the co-ordinate invariance. This is a special case of hypo-elasticity proposed by Truesdell.[I.1] This paper is concerned with the characteristic surfaces and acceleration waves in that generalized Prandtl-Reuss material.

In Sect. 2 the necessary condition for existence of characteristic surface is obtained. This coincides with the wave propagation condition obtained in Sec. 3 in general hypo-elastic material. Then characteristic surface, if it exists, must propagate with the same speed of acceleration wave. Two speeds of waves for elastic and plastic state in the same stress state are compared and the speed for plastic state is less than that for elastic state, if, and only if, the rates of work done by stress per niut volume are different on two sides of the wave. In Sed. 4 principal

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** 1.8) denotes Refernece 8) of the first paper[1] of this series waves are analyzed and they are longitudinal or transverse. The principal transverse waves have the same speeds for elastic and plastic states. The formula of acoustical birefringence in the first order approximation is obtained and this is identical to that in elastic materials, which was obtained by Tokuoka and Iwashimizu[2] and Tokuoka and Saito[3].

2. Characteristic Surfaces

As field equations we have the equation of continuity and first Cauchy's law of motion:

$$\dot{\rho}+\rho \operatorname{div} \dot{\boldsymbol{x}} = 0\,, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{S}+\frac{\rho}{\mu}\boldsymbol{f} = \frac{\rho}{\mu}\ddot{\boldsymbol{x}}\,, \tag{2.2}$$

where ρ, $\dot{\boldsymbol{x}}$, $\boldsymbol{f}$ and $\boldsymbol{S}$ are respectively mass density, vecloity of material particle, body force per unit mass and non-dimensionalized Cauchy stress, which is defined by the stress $\boldsymbol{T}$ divided by the shear elastic modulus μ.

A surface $\Sigma(t)$ in a material is called a characteristic surface when the basic equations (2.1), (2.2) and (I.3.16)* or (I.2.8) and the assigned data of ρ, $\dot{\boldsymbol{x}}$, $\ddot{\boldsymbol{x}}$, $\boldsymbol{f}$, $\boldsymbol{S}$ and $\overset{\circ}{\boldsymbol{S}}$ on $\Sigma(t)$ do not suffice for the determination of all first partial derivatives of ρ, $\dot{\boldsymbol{x}}$ and $\boldsymbol{S}$ over $\Sigma(t)$.

According to Thomas[I.8] we introduce a rectangular coordinate system y^1, y^2, y^3 which may move with uniform velocity relative to x system. The origin of the new system coincides with a point P on $\Sigma(t)$, y^1 and y^2 axes lie in the tangent plane to $\Sigma(t)$ at P, and we assume that the velocity of $\Sigma(t)$ relative to y system is zero at P at time t_0. The neighboring of $\Sigma(t)$ for P may be represented by

$$y^3 = f(y^1, y^2, t) \tag{2.3}$$

with

$$f_{,k} = \frac{\partial f}{\partial t} = 0 \qquad (k=1,\, 2). \tag{2.4}$$

The value of any quantity Ψ on $\Sigma(t)$ is expressed by

$$\Psi(y^1, y^2, y^3, t) = \overline{\Psi}(y^1, y^2, t)\,, \tag{2.5}$$

and we have

$$\frac{\partial \Psi}{\partial t} = \frac{\partial \overline{\Psi}}{\partial t}\,, \quad \Psi_{,k} = \overline{\Psi}_{,k} \qquad (k=1,\, 2) \tag{2.6}$$

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* (I.3.16) denotes Equation (3.16) of the first paper[1] of this series

on account of (2.4). Hence all derivatives of ρ, $\dot{\boldsymbol{x}}$ and $\boldsymbol{S}$ are determined at (P, t_0) from the assigned data on $\Sigma(t)$ with the exception of $\rho,_3$, $\dot{\boldsymbol{x}}_3$, and $\boldsymbol{S},_3$.

Equations (2.1), (2.2) and (I.3.16) give the conditions

$$\left.\begin{aligned} &\mathring{x}^3 \rho,_3 + \rho \mathring{x}^3,_3 = \cdots\cdots \\ &S^{k3},_3 - \frac{\rho}{\mu} \mathring{x}_3 \mathring{x}^k,_3 = \cdots\cdots \\ &\mathring{x}^3 S^{km},_3 - p^{kmp3} \mathring{x}_{p},_3 = \cdots\cdots \end{aligned}\right\} \quad (k,\ m=1,\ 2,\ 3), \tag{2.7}$$

at (P, t_0), where the dots are used to denote terms not involving differentiation with respect to y^3. Hence there are ten independent equations in (2.7) for the determination of the ten quantities $\rho,_3$, $\mathring{x}^k,_3$ and $S^{km},_3$.

The characteristic condition is therefore given by the vanishing of the determinant of the coefficients of these quantities. After some manipulation we have

$$\mathring{x}^3 = 0 \tag{2.8}$$

or

$$\det\left(p^{k3m3} - \frac{\rho}{\mu}(\mathring{x}^3)^2 \delta^{km}\right) = 0\,. \tag{2.9}$$

When (2.8) holds, from (2.7) we can assign arbitrary values to $\rho,_3$ $\mathring{x}^k,_3$ $S^{km},_3$ $(k, m=1, 2)$. When $\mathring{x}^3 \neq O$, it means the characteristic surface propagates normally with respect to material particle, relation (2.9) must hold necessarily.

The characteristic surface condition (2.9) is represented relative to the above special y system. In order to express it in invariant formulation, we substitute

$$\mathring{x}^3 = U, \quad p^{k3m3} = p^{kpmq} n_p n_q\,, \tag{2.10}$$

into (2.9), where U is the local speed of propagation and $\boldsymbol{n}$ denotes the unit normal on $\Sigma(t)$. Then we have

$$\det\left(Q^{km}(\boldsymbol{n}) - \frac{\rho}{\mu} U^2 g^{km}\right) = 0\,, \tag{2.11}$$

where

$$Q^{km}(\boldsymbol{n}) \equiv p^{kpmq} n_p n_q \tag{2.12}$$

is called the acoustical tensor. This is the necessary condition for the existence of a characteristic surface.

3. Acceleration Waves

We consider a singular surface across which the deformation $\boldsymbol{x}$ and its first derivative are continuous, but at least one of the second derivatives suffers a jump discontinuity. The geometrical and kinematical conditions of compatibility are

$$[\boldsymbol{L}] = -U\boldsymbol{a}\otimes\boldsymbol{n}\,, \quad [\ddot{\boldsymbol{x}}] = U^2\boldsymbol{a}\,, \tag{3.1}$$

where the bracket denotes the jump occasioned, $\boldsymbol{a}$ is called the amplitude and means the strength of discontinuity, $\boldsymbol{n}$ is a unit normal to the surface, and U is the local speed of propagation. See Truesdell and Toupin.[4]

In addition we assume

$$[\boldsymbol{S}] = 0 \tag{3.2}$$

and then the compatibility conditions are

$$[\dot{\boldsymbol{S}}] = -U\boldsymbol{A}\,, \quad [\operatorname{div}\boldsymbol{S}] = \boldsymbol{A}\boldsymbol{n}\,, \tag{3.3}$$

where $\boldsymbol{A}$ is a tensor specifying the jumps in the derivatives of $\boldsymbol{S}$.

Taking the jumps of (2.2) and (I.3.18) across the singular surface yields the equation

$$\boldsymbol{A}\boldsymbol{n} = \frac{\rho}{\mu}U^2\boldsymbol{a}, \quad -U\boldsymbol{A} = \mathsf{P}(\boldsymbol{S})(-U\boldsymbol{a}\otimes\boldsymbol{n})\,, \tag{3.4}$$

where we assume ρ, f, μ are continuous across the singular surface. From (3.4) we obtain the propagation condition

$$\left(\boldsymbol{Q}(\boldsymbol{n}) - \frac{\rho}{\mu}U^2\mathbf{1}\right)\boldsymbol{a} = 0\,, \tag{3.5}$$

where $\boldsymbol{Q}(\boldsymbol{n})$ is defined in (2.12) Then we have

$$\frac{\rho}{\mu}U^2 = \frac{\boldsymbol{a}\cdot\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{a}}{a^2}\,. \tag{3.6}$$

In Secs. 2 and 3 until now we have not considered the concrete structure of the response function P, therefore the concerned material includes broader class of materials, so-called hypo-elastic materials introduced by Truesdell.[I.1] The comparison of the characteristic surface condition (2.11) and the acceleration wave propagation condition (3.5) gives: *for any hypo-elastic material, if a characteristic surface exists, it is necessary that it propagates with the same speed as an acceleration wave front having the same normal of the characteristic surface.*

In an elastic state we have the propagation condition

$$\left({}_E\boldsymbol{Q}(\boldsymbol{n}) - \frac{\rho}{\mu}{}_EU^2\,\mathbf{1}\right)\boldsymbol{a} = 0\,, \tag{3.7}$$

where

$${}_EQ(\boldsymbol{n})^{km} = \mathsf{E}^{kpmq}n_pn_q\,. \tag{3.8}$$

If the stress state satisfies the yield condition (I.3.1) but the material remains in elastic state in the both sides of wave and if the proper vector is the same as that in plastic flow of the same stress state for the same normal $\boldsymbol{n}$, we have from (3.5), (3.7) and (I.3.17)

$$\begin{aligned}\frac{\rho}{\mu}({}_EU^2-U^2)a^2 &= \boldsymbol{a}({}_E\boldsymbol{Q}(\boldsymbol{n})-\boldsymbol{Q}(\boldsymbol{n}))\boldsymbol{a} \\ &= \frac{\mathsf{E}^{kpab}S^*_{ab}S^*_{cd}\mathsf{E}^{cd(mq)}a_k a_m n_p n_q}{\boldsymbol{S}^*:\mathsf{E}:\boldsymbol{S}^*} \\ &= [\boldsymbol{D}:\mathsf{E}:\boldsymbol{S}^*][\psi]\,, \end{aligned} \tag{3.9}$$

where we use the incompressibility of plastic flow and then we have the same numerical value of ρ in both cases and ψ is a material function, which relates the plastic stretching ${}_P\boldsymbol{D}$ and $\boldsymbol{S}^*$ as

$$ {}_P\boldsymbol{D} = \psi \boldsymbol{S}^* \tag{3.10}$$

and

$$\frac{-U\boldsymbol{S}^*:\mathsf{E}:(\boldsymbol{a}\otimes\boldsymbol{n})}{\boldsymbol{S}^*:\mathsf{E}:\boldsymbol{S}^*} = [\psi]\,, \quad -U(\boldsymbol{a}\otimes\boldsymbol{n}):\mathsf{E}:\boldsymbol{S}^* = [\boldsymbol{D}:\mathsf{E}:\boldsymbol{S}^*]\,. \tag{3.11}$$

From (3.10) and the yield condition (I.3.1) we have

$$[\psi] = \frac{1}{2\kappa^2}[\mathrm{tr}(\boldsymbol{S}_P\boldsymbol{D})]\,. \tag{3.12}$$

When the elastic materials in both sides of wave remains static, we have $\boldsymbol{D}={}_P\boldsymbol{D}=\psi\boldsymbol{S}^*$ and then

$$[\boldsymbol{D}:\mathsf{E}:\boldsymbol{S}^*] = [\psi]\boldsymbol{S}^*:\mathsf{E}:\boldsymbol{S}^*\,, \tag{3.13}$$

Inserting (3.12) and (3.13) into (3.9) we obtain

$$\frac{\rho}{\mu}({}_EU^2-U^2)a^2 = \frac{\boldsymbol{S}^*:\mathsf{E}:\boldsymbol{S}^*}{4\kappa^4 U^2}[\mathrm{tr}\,(\boldsymbol{S}\boldsymbol{D})]^2\,. \tag{3.14}$$

The quantity $\mathrm{tr}\,(\boldsymbol{S}\boldsymbol{D})$ denotes the rate of work done by stress per unit volume. Hence we can say in the case of $\boldsymbol{S}^*:\mathsf{E}:\boldsymbol{S}^*>0$ that *the speed of propagation of an acceleration wave in plastic flow is numerically less than that for a static elastic state, which has the same stress state and the same proper vector, if, and only if, the rates of work done by stress on the plastic flow per unit volume are different at adjacent points on the two sides of the wave.* In the zeroth order approximation in an isotropic material we have $\boldsymbol{S}^*:\mathsf{E}:\boldsymbol{S}^*=4\kappa^2$.

The obtained two conclusions in this section correspond to the generalizations proved by Thomas.[I.8)]

In the case of an isotropic material we can expand $\boldsymbol{Q}$ with respect to $\boldsymbol{S}$ by (I.3.17) and (2.12) and we have

$$
\begin{aligned}
{}_EQ_0(\boldsymbol{n})^{km} &= (\alpha_1+1)n^k n^m + g^{km}, \\
{}_PQ_0(\boldsymbol{n})^{km} &= -\frac{1}{\kappa^2} S^{*kp} S^{*mq} n_p n_q, \\
{}_EQ_1(\boldsymbol{n})^{km} &= (\beta_1+\beta_2)\,\mathrm{tr}(\boldsymbol{S}) n^k n^m + \left[\beta_2\,\mathrm{tr}\,(\boldsymbol{S}) + \left(\frac{1}{4}\alpha_6+1\right) S^{pq} n_p n_q\right] g^{km} \\
&\quad + \frac{1}{4}\alpha_6 S^{km} + \left(\frac{1}{2}\alpha_5 + \frac{1}{4}\alpha_6\right) S^{kp} n_p n^m + \left(-\frac{1}{2}\alpha_4 + \frac{1}{4}\alpha_6\right) n^k S^{mp} n_p, \qquad (3.15) \\
{}_PQ_1(\boldsymbol{n})^{km} &= -\frac{1}{2}(\alpha_5-2)\frac{\mathrm{tr}\,(\boldsymbol{S}^{*2})}{2\kappa^2} S^{*kp} n_p n^m + \frac{1}{2}\alpha_4 n^k S^{*mp} n_p \\
&\quad - \frac{\alpha_6+2}{2\kappa^2}\left(S^{*kp} n_p S^{*m}{}_r S^{*rq} n_q + S^{*kr} S^{*p}_r n_p S^{*mq} n_q\right) \\
&\quad + \left(\frac{\alpha_6+2}{4\kappa^4}\,\mathrm{tr}\,(\boldsymbol{S}^{*3}) - \frac{6\beta_2+\alpha_6+2}{6\kappa^2}\,\mathrm{tr}\,(\boldsymbol{S})\right) S^{*kp} S^{*mq} n_p n_q, \qquad (3.16)
\end{aligned}
$$

where ${}_E\boldsymbol{Q}_0$ and ${}_P\boldsymbol{Q}_0$ have the magnitude of zeroth order of κ and ${}_E\boldsymbol{Q}_1$ and ${}_P\boldsymbol{Q}_1$ that of first order of κ, α's are elastic constants and β_1 and β_2 are given in (I.2.13).

For usual metals, e.g. steel, iron, copper, brass and aluminum, the ratios of yield strength to shear modulus are between 10^{-3} and 10^{-2}. In this approximation we can adopt $\boldsymbol{Q}_0 \equiv {}_E\boldsymbol{Q}_0 + {}_P\boldsymbol{Q}_0$ as the acoustic tensor even in the yield stress state. Taking the principal axes of stress as coordinate system we have

$$
Q_0(\boldsymbol{n})^{km} = \left(\alpha_1 + 1 - \frac{S^*_k S^*_m}{\kappa^2}\right) n^k n^m + \delta^{km} \qquad \text{(no sum on } k \text{ and } m) \qquad (3.17)
$$

where S^*_Γ $(\Gamma=1, 2, 3)$ are the deviatoric principal stresses.

4. Principal Waves and Acoustical Birefringence

When the normal of a wave is parallel to a principal axis of stress, it is called the principal wave.

In the isotropic elastic material the response function E of (I.2.9) is given explicitly by Truesdell and Noll (p. 142).[I.9] Let $\boldsymbol{n}_1$ be a unit vector parallel to a principal axis of stress, the acoustical axes of ${}_E\boldsymbol{Q}(\boldsymbol{n}_1)$ defined by (3.8) coincide with principal axes. On the other hand we have

$$
Q(\boldsymbol{n}_1)^{km} - {}_EQ(\boldsymbol{n}_1)^{km} = \frac{\mathsf{E}^{kpab} S^*_{ab} S^*_{cd} \mathsf{E}^{cd(mq)} n_{1p} n_{1q}}{\boldsymbol{S}^* : \mathsf{E} : \boldsymbol{S}^*}. \qquad (4.1)
$$

Therefore $\boldsymbol{Q}(\boldsymbol{n}_1) - {}_E\boldsymbol{Q}(\boldsymbol{n}_1)$ also has the same axes of ${}_E\boldsymbol{Q}(\boldsymbol{n}_1)$. Hence, *in an isotropic material, every principal wave is either longitudinal or transverse.*

From (4.1) it is easily verified that

$$\boldsymbol{n}_2 \cdot {}_P\boldsymbol{Q}(\boldsymbol{n}_1)\boldsymbol{n}_2 = \boldsymbol{n}_2 \cdot {}_E\boldsymbol{Q}(\boldsymbol{n}_1)\boldsymbol{n}_2 \tag{4.2}$$

in an isotropic material, where $\boldsymbol{n}_2$ is a unit vector perpendicular to $\boldsymbol{n}_1$. Hence we have, *in an isotropic material, transverse wave has the same speed with one in elastic state in the same stress state.* From this and the result obtained in Sec. 3 we can say that, *on the two adjacent sides of a principal transverse wave, the rates of work done by stress on the plastic flow per unit volume are the same.*

From (3.6) and (3.15) the propagation speeds of principal waves may be easily calculated and we have in the zeroth-order approximation

$$\begin{aligned} \frac{\rho}{\mu} U_{11}^{(0)} &= \alpha_1 + 2 - \frac{S_1^{*2}}{\kappa^2} = \frac{\lambda + 2\mu}{\mu} - \frac{T_1^{*2}}{k^2}, \\ \frac{\rho}{\mu} U_{1\nu}^{(0)} &= 1, \end{aligned} \tag{4.3}$$

where ν is any direction perpendicular to $\boldsymbol{n}_1$, $k = \mu\kappa$ and U_{km} means the kth principal wave having the mth polarization direction.

The first order approximations $U_{11}^{(2)}$ and $U_{12}^{(1)}$ $\cdots$ are also easily obtained and transverse waves are polarized into the principal axes of $\boldsymbol{S}$, which are the cause of the term $\boldsymbol{S}$ in ${}_E\boldsymbol{Q}_1$ and we have

$$\begin{aligned} U_{12}^{(1)} - U_{13}^{(1)} &= \sqrt{\frac{\mu}{\rho_0}} \frac{\alpha_6}{8} (S_2 - S_3) \\ &= \sqrt{\frac{\mu}{\rho_0}} \frac{\mu + \nu_3}{2\mu} \frac{T_2 - T_3}{\mu}, \end{aligned} \tag{4.4}$$

where ν_3 denotes a second-order elastic constant introduced by Toupin and Bernstein[5] and ρ_0 is the density in undeformed state.

Relation (4.4) indicates that the acoustical birefringence shows the same results for isotropic elastic material, that is, *two principal transverse waves polarize along the principal axes of stress and the difference of their propagation speeds is proportional to the difference of the principal stresses along the polarized directions.* These results coincide with those for elastic materials, which were obtained by Tokuoka and Iwashimizu[2] and by Tokuoka and Saito.[3]

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