Generalized Prandtl-Reuss Plastic Materials

III. Growth and Decay of Acceleration Waves and Propagating Boundary Surfaces

By

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The growth and decay of the acceleration waves in elastic materials and of the propagating acceleration boundary burfaces between elastic and plastic states are theoretically investigated. The deformation states of elastic materials before the waves or boundaries are assumed to be static and homogeneous. The principal transverse waves and boundaries can propagate with constant strength. The principal longitudinal waves and boundaries may grow boundless for one of dilatational or compressive initial disturbance and they may coalesce respectively into shock waves and shock boundaries. Numerical evaluations for iron are calculated.

1. Introduction

The variations in strength of acceleration waves in elastic materials and of propagating boundary surfaces between elastic and plastic states were analyzed by Thomas.[I.8)]** The elastic material is restricted to be isotropic with a linear stress-strain relation. The prandtl-Reuss material is restricted to be incompressible. For both materials he showed that the strength remains constant along propagating direction in plane wave and in plane sruface.

With respect to the effect of non-linearity to the variation in strength Chu[1)] treated shear wave in incompressible elastic material of stress free state and reported that the shear waves may coalesce into transverse shock waves. Varley and Dunwoody[2)] investigated also the possibility of formation of shock waves in hypoelastic materials of the hydrostatic state. They applied the technique described by Courant and Hilbert[3)] for bi-characteristic curve in linear equation to quasi-linear equation.

In this paper by the method given by Varley and Dunwoody I investigate the

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** I.8) denotes Reference 8) of the first paper[4)] of this series growth and decay of acceleration wave and boundary surface between elastic and plastic state of the generalized Prandtl-Reuss plastic materials proposed by Tokuoka.[4] The deformation states before the wave and boundary are assumed to be static and homogeneous and we restrict our attention to the principal waves and boundaries whose normals coincide with a principal axis of stress.

2. Fundamental Relations in Bi-Characteristics

As the technique used in this paper is the same given by Varley and Dunwoody,[2] only the essential results will be depicted here.

We adopt a new set of independent variable $(\boldsymbol{x}, \alpha)$ in place of $(\boldsymbol{x}, t)$, where $\alpha=\phi(\boldsymbol{x}, t)$ or $t=\Psi(\boldsymbol{x}, \alpha)$ and $\alpha=0$ represents the acceleration wave or the boundary surface. We have the important formula

$$V \operatorname{grad} \Psi = V \operatorname{grad} \bar{\Psi} - \Psi^{(1)} \boldsymbol{n}, \tag{2.1}$$

where

$$\Psi(\boldsymbol{x}, t) = \bar{\Psi}(\boldsymbol{x}, \alpha), \tag{2.2}$$

$$\Psi^{(1)} \equiv \frac{\partial \Psi}{\partial t} = \frac{\partial \phi}{\partial t} \frac{\partial \bar{\Psi}}{\partial \alpha} \tag{2.3}$$

and Ψ, V and $\boldsymbol{n}$ represent respectively any kind of scalar, vector or tensor quantity, the speed of propagation of surface $\alpha=0$ with respect to co-ordinates and the unit normal vector of the surface. Here we assume all of quantities, $\boldsymbol{S}$, $\boldsymbol{v}\equiv\dot{\boldsymbol{x}}$ ρ, $\boldsymbol{f}$, which were defined in the papers of Tokuoka,[4,5] are continuous across $\alpha=0$.

Applying (2.1) to (II.2.1)* and (II.2.2) yields that

$$\boldsymbol{S}^{(1)}\boldsymbol{n} + \frac{\rho}{\mu} U \boldsymbol{v}^{(1)} = V\left(\operatorname{div} \bar{\boldsymbol{S}} - \frac{\bar{\rho}}{\mu} \bar{\boldsymbol{v}} \cdot \operatorname{grad} \bar{\boldsymbol{v}}\right) \equiv \boldsymbol{A} \tag{2.4}$$

and

$$\rho \boldsymbol{n} \cdot \boldsymbol{v}^{(1)} - U \rho^{(1)} = V(\bar{\boldsymbol{v}} \cdot \operatorname{grad} \bar{\rho} + \bar{\rho} \operatorname{div} \bar{\boldsymbol{v}}) \equiv \boldsymbol{B}. \tag{2.5}$$

We can apply (2.1) to the constitutive equation

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{H} : \boldsymbol{L}, \tag{2.6}$$

where $\boldsymbol{H}$ corresponds to $\boldsymbol{E}$ or $\boldsymbol{P}$, given in (I.2.9)** and (I.3.17), according as elastic and plastic region respectively. Then we have

* (II.2.1) denotes Equation (2.1) of the second paper[5] of this series

** (I.2.9) denotes Equation (2.9) of the first paper[4] of this series

$$US^{(1)}+\boldsymbol{H}:(\boldsymbol{v}^{(1)}\otimes\boldsymbol{n}) = V(\boldsymbol{H}:\operatorname{grad}\bar{\boldsymbol{v}})-\bar{\boldsymbol{v}}\cdot\operatorname{grad}\bar{\boldsymbol{S}}) \equiv \boldsymbol{C}, \tag{2.7}$$

where $\boldsymbol{A}$, B and $\boldsymbol{C}$ involve only derivatives for $\alpha=$constant, and

$$U \equiv V-\boldsymbol{v}\cdot\boldsymbol{n} \tag{2.8}$$

denotes the speed with respect to the material.

Eliminating $\boldsymbol{S}^{(1)}$ from (2.4) and (2.7) gives

$$\left(\boldsymbol{Q}(\boldsymbol{n})-\frac{\rho}{\mu}U^2\mathbf{1}\right)\boldsymbol{v}^{(1)} = -\boldsymbol{n}\cdot\boldsymbol{C}-U\boldsymbol{A}, \tag{2.9}$$

where

$$Q(\boldsymbol{n})^{km} \equiv H^{kpmq}n_pn_q \tag{2.10}$$

is the acoustical tensor for $\boldsymbol{n}$.

Differentiating (II.2.1) and (2.6) with respect to t and applying (2.1) to them, we have

$$\begin{aligned}\boldsymbol{S}^{(2)}\boldsymbol{n}+\frac{\rho}{\mu}U\boldsymbol{v}^{(2)} &= V(\operatorname{div}\bar{\boldsymbol{S}}^{(1)}-\frac{\bar{\rho}}{\mu}\bar{\boldsymbol{v}}\cdot\operatorname{grad}\bar{\boldsymbol{v}}^{(1)}\\ &-\left(\frac{\bar{\rho}}{\mu}\bar{\boldsymbol{v}}\right)^{(1)}\cdot\operatorname{grad}\bar{\boldsymbol{v}})+\left(\frac{\bar{\rho}}{\mu}\boldsymbol{n}\cdot\bar{\boldsymbol{v}}^{(1)}-\frac{\bar{\rho}^{(1)}}{\mu}U\right)\bar{\boldsymbol{v}}^{(1)} \equiv \boldsymbol{X}\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}US^{(2)}+\boldsymbol{H}:(\boldsymbol{v}^{(2)}\otimes\boldsymbol{n}) &= V(\boldsymbol{H}:(\operatorname{grad}\bar{\boldsymbol{v}}^{(1)})\\ &+\bar{\boldsymbol{H}}^{(1)}:(\operatorname{grad}\bar{\boldsymbol{v}})-\bar{\boldsymbol{v}}^{(1)}\cdot\operatorname{grad}\bar{\boldsymbol{S}}+\bar{\boldsymbol{v}}\cdot\operatorname{grad}\bar{\boldsymbol{S}}^{(1)})\\ &+(\bar{\boldsymbol{S}}^{(1)}\boldsymbol{n}\cdot\bar{\boldsymbol{v}}^{(1)}-\bar{\boldsymbol{H}}^{(1)}:(\bar{\boldsymbol{v}}^{(1)}\otimes\boldsymbol{n})) \equiv \boldsymbol{Y},\end{aligned} \tag{2.12}$$

where $\Psi^{(2)}\equiv\dfrac{\partial^2\Psi}{\partial t^2}$.

Eliminating $\boldsymbol{S}^{(2)}$ from (2.11) and (2.12) we obtain

$$\left(\boldsymbol{Q}(\boldsymbol{n})-\frac{\rho}{\mu}U^2\mathbf{1}\right)\boldsymbol{v}^{(2)} = \boldsymbol{Y}\boldsymbol{n}-U\boldsymbol{X}. \tag{2.13}$$

In the front side of $\alpha=0$ we assume the elastic deformation state is static and homogeneous. Therefore all of the quantities appearing in $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ vanish and they are continuous across $\alpha=0$, if grad Ψ exists and is continuous across $\alpha=0$, then on the contiguous back side of $\alpha=0$ they must vanish and we have from (2.4), (2.5), (2.7) and (2.9)

$$\boldsymbol{S}^{(1)}\boldsymbol{n} = -\frac{\bar{\rho}}{\mu}U\boldsymbol{v}^{(1)}, \tag{2.14}$$

$$\bar{\rho}\boldsymbol{n}\cdot\boldsymbol{v}^{(1)}-U\rho^{(1)} = 0, \tag{2.15}$$

$$US^{(1)} = -\bar{\boldsymbol{H}}: (\boldsymbol{v}^{(1)} \otimes \boldsymbol{n})\,, \tag{2.16}$$

$$\left(\boldsymbol{Q}(\boldsymbol{n}) - \frac{\rho}{\mu} U^2 \boldsymbol{1}\right) \boldsymbol{v}^{(1)} = 0\,, \tag{2.17}$$

where all of the quantities are evaluated on the contiguous back side of $\alpha=0$.

If we demand $|\boldsymbol{v}^{(1)}| \neq 0$, the propagation condition

$$\det\left(\boldsymbol{Q}(\boldsymbol{n}) - \frac{\rho}{\mu} U^2 \boldsymbol{1}\right) = 0 \tag{2.18}$$

must hold and $\boldsymbol{v}^{(1)}$ and $\frac{\rho}{\mu} U^2$ must coincide respectively with a right proper vector and the corresponding proper number of $\boldsymbol{Q}(\boldsymbol{n})$. If $\boldsymbol{a}$ is a left proper vector and U_a is the corresponding speed, operating $\boldsymbol{a}$ from left to (2.13) yields

$$\boldsymbol{a} \cdot (\boldsymbol{Y}\boldsymbol{n} - U_a \boldsymbol{X}) = 0\,. \tag{2.19}$$

This is the required compatibility condition.

Using (2.14), (2.15) and (2.16) and assuming $\bar{\boldsymbol{v}}=0$, $V=U=U_a$ and grad $\Psi=0$ we have

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{n}): \bar{\boldsymbol{H}}: (\mathrm{grad}\, \bar{\boldsymbol{v}}^{(1)}) + (\boldsymbol{a}\otimes \mathrm{grad}): \bar{\boldsymbol{H}}: (\boldsymbol{v}^{(1)}\otimes\boldsymbol{n}) \\ - U_a^{-2} \boldsymbol{a}\cdot\boldsymbol{Q}(\boldsymbol{n})\, \boldsymbol{v}^{(1)} (\boldsymbol{n}\cdot\boldsymbol{v}^{(1)}) - U_a^{-1} \boldsymbol{a}\cdot \boldsymbol{Q}^{(1)}(\boldsymbol{n})\, \boldsymbol{v}^{(1)} = 0\,.\end{aligned} \tag{2.20}$$

If we take the right unit proper vector as $\boldsymbol{b}$, we have

$$\bar{\boldsymbol{v}}^{(1)} = \sigma \boldsymbol{b}\,, \tag{2.21}$$

where σ denotes the magnitude of strength of acceleration induced by pass of the wave or the boundary. Because the state in front of $\alpha=0$ is homogeneous, unit vectors $\boldsymbol{n}$ and $\boldsymbol{b}$ of a plane wave hold the constant directions. The first and second terms in (3.20) are proportional to the slope of σ. $\boldsymbol{Q}(\boldsymbol{n})$ is a function of $\boldsymbol{S}$ then $\boldsymbol{Q}^{(1)}(\boldsymbol{n})$ is proportional to σ by insertion of (2.16). The third and fourth terms in (2.20) are therefore proportional to σ^2.

Hereafter we restrict our attention to a principal wave, which is defined by the condition that its normal $\boldsymbol{n}$ coincides with a principal direction of stress $\boldsymbol{n}_1$. In this case left and right proper vectors coincide with each other and are parallel to principal directions.

In the longitudinal wave, putting $\boldsymbol{n}=\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{n}_1$, we can obtain

$$2\frac{\rho}{\mu} U_{11}{}^2 \frac{d\sigma}{dl} - \frac{\rho}{\mu}\sigma^2 - U_{11}{}^{-1} \boldsymbol{n}_1 \cdot \boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\, \boldsymbol{n}_1 \sigma = 0\,, \tag{2.22}$$

and in the transverse wave, putting $\boldsymbol{n}=\boldsymbol{n}_1$, and $\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{n}_2$, we can obtain

$$\left\{2\frac{\rho}{\mu}U_{21}^{2}+(S_1-S_2)\right\}\frac{d\sigma}{dl}-U_{12}^{-1}\boldsymbol{n}_2\cdot\boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\boldsymbol{n}_2\sigma=0\,, \tag{2.23}$$

where we used (I.3.19) and (2.10),

$$\frac{d\sigma}{dl}\equiv n_k\frac{\partial\sigma}{\partial x^k} \tag{3.24}$$

means the variation of σ along the propagating path, U_{km} denotes the speed of the kth principal wave with the mth polarization direction and S_k is the kth principal stress.

The remaining task is to calculate the last terms in (2.22) and (2.23) for two cases, i.e., wave and boundary. If it is done we will have

$$\frac{d\sigma}{dl}+\gamma(\boldsymbol{S})\sigma^2=0\,, \tag{2.25}$$

where γ is in general a function of stress. We can obtain

$$\sigma=\frac{\sigma_0}{1+\sigma_0\gamma l}\,, \tag{2.26}$$

where σ_0 is the initial disturbance at $l=0$. When $\sigma_0\gamma>0$ σ decays monotonically, on the other hand when $\sigma_0\gamma<0$, σ will be unbounded at

$$l_{cr}=-\frac{1}{\sigma_0\gamma}\,. \tag{2.27}$$

At the critical point where the discontinuous accelerations grow unbounded, we assume as Chu[1] and Varley and Dunwoody[2] that the acceleration wave and boundary will be changed to shock wave and shock boundary.

3. Acceleration Waves in Elastic Materials

We will consider the plane principal acceleration wave in the elastic material governed by (II.3.4) in static and homogeneous state. From $(\text{I}.3.21)_1$, (2.10) and (2.18) we have

$$\frac{\rho}{\mu}U_{11}^{2}=\alpha_1+2+O(\kappa)\,,\quad \frac{\rho}{\mu}U_{21}^{2}=1+O(\kappa)\,, \tag{3.1}$$

where $O(\kappa)$ denotes the quantity of the first order of κ. We have in this connection

$$10^{-3}\lesssim\kappa\lesssim10^{-2} \tag{3.2}$$

for steel, iron, copper, aluminium etc.

3.1 Longitudinal Wave

After some manipulation with $(\mathrm{I}.3.21)_1$, $(\mathrm{I}.3.21)_2$, (2.10), (2.16) and (2.22) we have

$$\boldsymbol{n}_1\cdot\boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\,\boldsymbol{n}_1 = -U_{11}^{-1}(\alpha_1+2\alpha_3+2\alpha_5+2\alpha_6-1)\,\sigma \tag{3.3}$$

and

$$_E\gamma_0^l = -\frac{1}{2(\alpha_1+2)}\frac{\rho}{\mu}\,, \tag{3.4}$$

$$_E\gamma_1^l = \frac{\alpha_3+\alpha_5+\alpha_6-3/2}{(\alpha_1+2)^2}\frac{\rho}{\mu}\,, \tag{3.5}$$

where (3.4) corresponds to γ for the linear stress-strain relation and (3.5) for non-linear one. In this paper linearity means the linear relation between the stress rate and the stretching and does not mean that of the stress and infinitesimal strain. In the latter case Thomas[I.8)] proved that γ vanishes.

3.2 Transverse Wave

From $(\mathrm{I}.3.21)_2$, (2.10) and (2.23), we have

$$\boldsymbol{n}_2\cdot\boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\,\boldsymbol{n}_2 = 0 \tag{3.6}$$

$$_E\gamma_0^t = {}_E\gamma_1^t = 0 \tag{3.7}$$

and

$$\frac{d\sigma}{dl} = 0\,. \tag{3.8}$$

Thus the transverse wave holds its value along the propagating path.

4. Propagating Boundary Surfaces Between Elastic and Plastic State

We now consider the propagation of plastic region governed by (I.3.18) into elastic region governed by (I.2.7). We assume that the boundary surface is a plane and its normal coincides with a principal axis $\boldsymbol{n}_1$ and the elastic region is in static and homogeneous state. Then inserting $\boldsymbol{P}=\boldsymbol{E}-\boldsymbol{F}$ as $\boldsymbol{H}$ into (2.10) and (2.18) and referring $(\mathrm{I}.3.21)_1$ and $(\mathrm{I}.3.21)_2$, we have

$$\frac{\rho}{\mu}U_{11}^2 = \alpha_1+2-K^2+O(\kappa)\,,\quad \frac{\rho}{\mu}U_{21}^2 = 1+O(\kappa)\,, \tag{4.1}$$

where

$$K \equiv \frac{S_1^*}{\kappa} \tag{4.2}$$

4.1 The Zeroth Order Approximation

We can easily calculate the last terms of the left sides in (2.22) and (2.23) and we have

$$\boldsymbol{n}_1\cdot\boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\,\boldsymbol{n}_1 = 2\frac{K}{\kappa}\Big(2\alpha_1+\frac{8}{3}-K^2\Big)\sigma\,, \tag{4.3}$$

$$_P\gamma_0{}^l = -\frac{\frac{K}{\kappa}\Big(2\alpha_1+\frac{8}{3}-K^2\Big)+\frac{1}{2}(\alpha_1+2-K^2)}{(\alpha_1+2-K^2)^2}\,\frac{\rho}{\mu} \tag{4.4}$$

and

$$\boldsymbol{n}_2\cdot\boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\,\boldsymbol{n}_2 = 0\,, \tag{4.5}$$

$$_P\gamma_0{}^t = 0\,. \tag{4.6}$$

The transverse boundary disturbance remains constant along its propagation.

4.2 The First Order Approximation

Here we assume the expansion formulae (I.3.21) holds in the yield state. After some manipulations we have

$$\begin{aligned}\frac{U_{11}}{\sigma}\boldsymbol{n}_1\cdot\boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\,\boldsymbol{n}_1 &= 2\frac{K}{\kappa}\Big(2\alpha_1+\frac{8}{3}-K^2\Big)\\ &-\Big\{\beta_1+2\beta_2-\Big(\beta_2+\frac{1}{6}\alpha_6+\frac{1}{3}\Big)K^2\Big\}(3\alpha_1+2-K^2)\\ &-\Big(-\frac{1}{2}\alpha_4+\frac{1}{2}\alpha_5+\alpha_6+1\Big)(\alpha_1+2-K^2)\\ &-\frac{2}{3}\Big\{\frac{1}{2}\alpha_4-\frac{1}{2}\alpha_5+1-3(\alpha_6+2)K^2+\frac{1}{2}(\alpha_6+2)K\,\mathrm{tr}\Big(\Big(\frac{S^*}{\kappa}\Big)^3\Big)\\ &-\Big(2\beta_2+\frac{1}{3}\alpha_6+\frac{2}{3}\Big)K\,\mathrm{tr}\Big(\frac{S}{\kappa}\Big)\Big\}(2-K^2)-\frac{3}{4}(\alpha_6+2)K^2(2-K^2)\Big(K^2-\frac{2}{3}\Big)\\ &\equiv R\,,\end{aligned} \tag{4.7}$$

where we use

$$\boldsymbol{S}^{(1)} = -U_{11}{}^{-1}\{\alpha_1\boldsymbol{1}+(2-K^2)\,\boldsymbol{n}\otimes\boldsymbol{n}\}\,\sigma\,, \tag{4.8}$$

which is reduced from (2.16).

Substituting $(4.1)_1$ and (4.8) into (2.22) we have

$$_P\gamma_1{}^l = -\frac{R+(\alpha_1+2-K^2)}{2(\alpha_1+2-K^2)^2}\frac{\rho}{\mu}\,. \tag{4.9}$$

In the transverse disturbance case we can easily verify from (I.3.21) that

$$\boldsymbol{n}_2\cdot\boldsymbol{Q}^{(1)}(\boldsymbol{n}_1)\,\boldsymbol{n}_2 = 0\,, \tag{4.10}$$

and then

$$_P\gamma_1{}^t = 0\,. \tag{4.11}$$

5. Discussions

5.1 The Higher Order Approximation

When we take into account the second and higher order of $\boldsymbol{S}$ in the expansion of $\boldsymbol{E}$ and $\boldsymbol{P}$, the value of γ deviates at most one percent from (3.2). Therefore we can neglect them, because, as we will see in Subsec. 5.5, the experimental error latitude is larger than the corrections.

5.2 Transverse Disturbances

For all kinds of principal transverse waves and boundary surfaces, the coefficients γ vanish from (3.7), (4.6) and (4.11). Then the principal transverse disturbances can propagate with constant strength. But we note that Chu[1] verified that the transverse wave may grow boundless if there is shear stress.

5.3 Longitudinal Disturbances

For all kind of longitudinal waves and boundary surfaces, the coefficients γ have non-zero values from (3.4), (3.5), (4.4) and (4.9). Therefore they may grow boundless for appropriate sign of initial disturbance σ_0. From (2.15) and(2.21) $\sigma_0 \gtreqless 0$ correspond respectively to $\rho_0^{(1)} \gtreqless 0$ and then to compressive and dilatational disturbance. Therefore from (2.26) we can conclude that the longitudinal disturbance may produce infinite acceleration deviation and form shock wave or shock boundary on the contiguous rear surface of wave or boundary,

$$\begin{aligned} &\text{when } \gamma>0 \text{ for dilatational disturbance} \\ \text{and}\quad & \\ &\text{when } \gamma<0 \text{ for compressive disturbance.} \end{aligned} \tag{5.1}$$

For the acceleration wave in elastic region, γ is independent of the stress state, but for the acceleration boundary between elastic and plastic region, γ depends essentially on the stress state.

5.4 Numerical Evaluation

In order to obtain the numerical values of γ's, we must know the first and second order elastic constants and the yield stress. Fortunately the elastic constants of iron and copper are given by Seeger and Buck[6] in terms of μ, bulk modulus k, Murnaghan's second order elastic constants l, m, n. α's are connected with them by the formulae

$$\lambda = k-\frac{2}{3}\mu\,, \quad \alpha_1 = \frac{\lambda}{\mu}\,, \quad \alpha_3 = -\alpha_1+24\frac{l}{\mu}+8\frac{m}{\mu}\,,$$

$$\alpha_4 = 8\left(\frac{m}{\mu}+\frac{n}{\mu}\right), \quad \alpha_4 = 2(\alpha_1-1)-\alpha_4\,, \quad \alpha_6 = 4-8\frac{n}{\mu}\,, \tag{5.2}$$

where we must assume the restriction

$$\alpha_4-\alpha_5 = 2(\alpha_1-1)\,. \tag{5.3}$$

See Truesdell and Noll (p. 230).[I.9]

Then we can obtain Table 1 for iron, where we adopt the yield stress $\tau = 3.2 \times 10\ kg/mm^2$ in simple tension test and have $\kappa = \tau/\sqrt{3}\,\mu$ and where the experimental error latitudes are omitted.

Table 1.

κ	α_1	α_3	α_4	α_5	α_6
2.24×10^{-3}	1.4	-1.3×10^2	-2.2×10^2	2.2×10^2	1.5×10^2

For simplicity we consider a special stress state in the case of elastic-plastic boundary surface, that is, the principal stresses take values κ, 0, $-\kappa$. Then we have

$$\mathrm{tr}\,(\boldsymbol{S}) = \mathrm{tr}\,(\boldsymbol{S}^{*3}) = 0 \tag{5.4}$$

and

$$K = 1,\ 0,\ -1\,. \tag{5.5}$$

Substituting Table 1, (I.2.13), (5.4) and (5.5) into (3.4), (3.5), (4.4), (4.7) and (4.9), we have Table 2. The values in the lowest line of Table 2 mean the initial acceleration magnitudes to coalesce into shock after one centimeter propagation for respective cases, where they are calculated by (2.27) and g means the acceleration of gravitation of earth.

Table 2 shows that 1. in elastic wave the effect of non-linearity is predominant and the necessary initial disturbance to break out shock is reduced to one sixtieth in comparison with linear case; 2. in propagating elastic and plastid boundary

Table 2.

	Elastic Wave		Elastic-Plastic Boundary					
	Zeroth Order	First Order	Zeroth Order			First Order		
K			1	0	-1	1	0	-1
$\gamma \frac{\mu}{\sigma}$	-1.4×10^{-1}	8.4	-3.5×10^{2}	-1.4×10^{-1}	3.5×10^{2}	-3.8×10^{2}	1.3	3.2×10^{2}
$\frac{\sigma_0}{g}$	7.6×10^{8}	-1.3×10^{7}	3.1×10^{5}	7.6×10^{8}	-3.1×10^{5}	2.8×10^{5}	-8.1×10^{7}	-3.3×10^{5}

the effect of yielding is predominant and the necessary initial disturbance is reduced to one sixtieth in comparison with elastic non-linear case; 3. but, when the deviatoric stress along the propagation direction vanishes, a large disturbance is demanded to break out shock; 4. for the elastic and plastic boundary, when the material is tensioned, the necessary disturbance to break out the shock is compressive and when it is compressed, it is dilatational.

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