The Theory of the Determination of Stress in an Anisotropic Elastic Medium Using an Instrumented Cylindrical Inclusion

By

Yoshiji NIWA* and Ken-ichi HIRASHIMA*

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The present paper describes the theoretical aspect of the determination of the stresses in rock unaffected by a borehole with an instrumented cylindrical inclusion. The stress fields of the elastic matrix (isotropic or anisotropic) containing a cylindrical inclusion are discussed and the formulae to be used in practice to determine the stresses in rock are presented. The results of calculation are shown by several numerical examples.

1. Introduction

The measurements of the state of initial or variational stresses in rock are of importance in obtaining design data for underground structures and in clarifying the problem of strata pressure that occurs in the earth's crust. Several authors[1~6] have been made attempts to analyze the stresses in rock by the use of boreholes or drifts. Since the rock is generally in a three-dimensional stress state and behaves frequently as an anisotropic elastic medium, the determination of the stresses in the rock is very complicated. Taking these conditions into consideration, the present paper describes a method of stress determination using a borehole with an instrumented inclusion such as in Rocha and Silverio's method.[3] This study has been carried out on the assumptions that:

(a) the rock is homogeneous and isotropic or anisotropic elastic body with arbitrary inclinations of the principal elastic axes,

(b) the stresses applied at infinity in the rock medium are three-dimensional, and

(c) the length of the inclusion is sufficiently larger than its diameter.

* Department of Civil Engineering

2. Method of Analysis

As in Fig. 1, we refer the body under consideration to a rectangular Cartesian coordinate system (x, y, z) where the z-axis is directed along the axis of cylindrical inclusion. In this case, the principal elastic axes of the matrix, assuming that the surrounding material is homogeneous and isotropic or anisotropic elastic body, incline in arbitrary directions against this coordinate system.

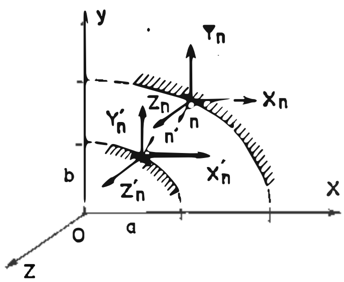


Fig. 1. Coordinates and boundaries of matrix and elliptical inclusion.

2.1 Basic Equations and Complex Analytic Functions

The stress-strain relations for an anisotropic elastic matrix are given by the generalized Hooke's law as follows.

$$\left.\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy},\\
\varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + \cdots\cdots + a_{26}\tau_{xy},\\
\varepsilon_z &= a_{13}\sigma_x + a_{23}\sigma_y + \cdots\cdots + a_{36}\tau_{xy},\\
&\cdots\cdots,\\
\gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{36}\sigma_z + a_{46}\tau_{yz} + a_{56}\tau_{xz} + a_{66}\tau_{xy}.
\end{aligned}\right\} \tag{2.1}$$

In which $a_{11}, a_{12}, \cdots\cdots, a_{66}$ are the elastic constants of the matrix under consideration. From the third equation of Eq. (2.1),

$$\sigma_z = \frac{\varepsilon_z}{a_{33}} - \frac{1}{a_{33}}(a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\tau_{yz} + a_{35}\tau_{xz} + a_{36}\tau_{xy}). \tag{2.2}$$

Substituting from Eq. (2.2) into the remaining equations of Eq. (2.1), we obtain:

$$\left.\begin{aligned}
\varepsilon_x &= \beta_{11}\sigma_x + \beta_{12}\sigma_y + \beta_{14}\tau_{yz} + \beta_{15}\tau_{xz} + \beta_{16}\tau_{xy} + \frac{a_{13}}{a_{33}}\varepsilon_z,\\
\varepsilon_y &= \beta_{12}\sigma_x + \beta_{22}\sigma_y + \cdots\cdots + \frac{a_{23}}{a_{33}}\varepsilon_z,\\
\gamma_{yz} &= \beta_{14}\sigma_x + \beta_{24}\sigma_y + \cdots\cdots + \frac{a_{34}}{a_{33}}\varepsilon_z,\\
&\cdots\cdots,\\
\gamma_{xy} &= \beta_{16}\sigma_x + \beta_{26}\sigma_y + \beta_{46}\tau_{yz} + \beta_{56}\tau_{xz} + \beta_{66}\tau_{xy} + \frac{a_{66}}{a_{33}}\varepsilon_z.
\end{aligned}\right\} \tag{2.3}$$

In which,

$$\beta_{ij}=a_{ij}-\frac{a_{i3}a_{j3}}{a_{33}},\ (i,\ j=1,\ 2,\ 4,\ 5,\ 6). \tag{2.4}$$

The same will hold for the inclusion, but instead of a_{ij} and β_{ij} the elastic constants of the matrix are a'_{ij} and β'_{ij} respectively. In the case when the uniform stress components σ^0_x, σ^0_y, σ^0_z, τ^0_{yz}, τ^0_{xz} and τ^0_{xy} are applied at infinity in the matrix without the inclusion, the Airy's stress functions F^0 and ψ^0 can be given by

$$\left.\begin{aligned}&F^0=\frac{1}{2}(\sigma^0_y x^2-2\tau^0_{xy}xy+\sigma^0_x y^2),\\&\psi^0=\tau^0_{xz}y-\tau^0_{yz}x.\end{aligned}\right\} \tag{2.5}$$

From general results on elliptical or ellipsoidal inclusion in an infinite anisotropic medium, it is well known that the stress and strain fields in the inclusion are homogeneous.[7] This fact is used in the following analysis.

Assuming that the uniform stresses in the inclusion are represented as σ'_x, σ'_y, σ'_z, τ'_{yz}, τ'_{xz}, and τ'_{xy}, the stress functions F' and ψ' for the inclusion are also given by

$$\left.\begin{aligned}&F'=\frac{1}{2}(\sigma'_y x^2-2\tau'_{xy}xy+\sigma'_x y^2),\\&\psi'=\tau'_{xz}y-\tau'_{yz}x.\end{aligned}\right\} \tag{2.6}$$

Then the formulas for stresses and displacements in an infinite medium with an inclusion can be written in the following manner.[8,9]

$$\left.\begin{aligned}&\sigma_x=\sigma^0_x+2Re[\mu_1^2\phi_1'(z_1)+\mu_2^2\phi_2'(z_2)+\mu_3^2\lambda_3\phi_3'(z_3)],\\&\sigma_y=\sigma^0_y-2Re[\phi_1'(z_1)+\phi_2'(z_2)+\lambda_3\phi_3'(z_3)],\\&\tau_{yz}=\tau^0_{yz}-2Re[\lambda_1\phi_1'(z_1)+\lambda_2\phi_2'(z_2)+\phi_3'(z_3)],\\&\tau_{xz}=\tau^0_{xz}+2Re[\mu_1\lambda_1\phi_1'(z_1)+\mu_2\lambda_2\phi_2'(z_2)+\mu_3\phi_3'(z_3)],\\&\tau_{xy}=\tau^0_{xy}-2Re[\mu_1\phi_1'(z_1)+\mu_2\phi_2'(z_2)+\mu_3\lambda_3\phi_3'(z_3)],\end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned}&u=u^0+2Re\sum_{k=1}^{3}p_k\phi_k(z_k)-\omega^0 y+u_0,\\&v=v^0+2Re\sum q_k\phi_k(z_k)+\omega^0 x+v_0,\\&w=w^0+2Re\sum r_k\phi_k(z_k)+w_0.\end{aligned}\right\} \tag{2.8}$$

Where Re is the notation for the real part of the complex expressions, $\phi_k(z_k)$ are analytic functions with the argument of the complex variables $z_k=x+\mu_k y$, and μ_k, λ_k, p_k, q_k and r_k $(k=1,\ 2,\ 3)$, are complex constants related to the roots of the characteristic equation of the anisotropic elastic matrix and elastic constants β_{ij}. The real constants ω^0, u_0, v_0 and w_0 characterize rigid rotation and rigid displacements of the body which are not accompanied by deformations.

When the matrix and the inclusion are perfectly bonded, the boundary conditions on the contact surface can be written as (see Fig. 1),

$$\left.\begin{array}{lll} X_n=-X'_n, & Y_n=-Y'_n, & Z_n=-Z'_n, \\ u=u', & v=v', & w=w'. \end{array}\right\} \tag{2.9}$$

Here, X_n, Y_n, Z_n and u, v, w are the stress components and displacements along the directions of the coordinate axes on the contour of the matrix. The prime on the right sides of the above equation indicates the stresses and displacements for the inclusion. After some algebra, these boundary conditions reduce to the following equations.

$$\left.\begin{array}{r} 2Re[\phi_1(z_1)+\phi_2(z_2)+\lambda_3\phi_3(z_3)]=\dfrac{\partial}{\partial x}(F'-F^0)+C_1, \\ 2Re[\mu_1\phi_1(z_1)+\mu_2\phi_2(z_2)+\mu_3\lambda_3\phi_3(z_3)]=\dfrac{\partial}{\partial y}(F'-F^0)+C_2, \\ 2Re[\lambda_1\phi_1(z_1)+\lambda_2\phi_2(z_2)+\phi_3(z_3)]=\psi'-\psi^0+C_3, \end{array}\right\} \tag{2.10}$$

$$\left.\begin{array}{l} 2Re\sum_{k=1}^{3} p_k\phi_k(z_k)=u'-u^0-(\omega'-\omega^0)y+(u_0'-u_0), \\ 2Re\sum q_k\phi_k(z_k)=v'-v^0+(\omega'-\omega^0)x+(v_0'-v_0), \\ 2Re\sum r_k\phi_k(z_k)=w'-w^0+(w_0'-w_0). \end{array}\right\} \tag{2.11}$$

C_1, C_2 and C_3 are determined on the basis of some additional conditions, depending on the matrix shape and distribution of forces. It is not necessary to consider these for the present case.

Let us assume that the shape of the inclusion is elliptical with semi-axes a and b along the coordinates (x, y). The contour of this inclusion is represented by

$$x_0=a\cos\theta, \quad y_0=b\sin\theta, \tag{2.12}$$

in which θ is an angle parameter varying from 0 to 2π in a counter-clockwise direction on the contour. By replacing the x and y of Eqs. (2.5) and (2.6) by the x_0 and y_0 of Eq. (2.12) in turn, and substituting into Eq. (2.10), we obtain the following.

$$\left.\begin{array}{r} 2Re[\phi_1(z_1)+\phi_2(z_2)+\lambda_3\phi_3(z_3)]\equiv 2Re[\bar{a}_1e^{-i\theta}], \\ 2Re[\mu_1\phi_1(z_1)+\mu_2\phi_2(z_2)+\mu_3\lambda_3\phi_3(z_3)]\equiv 2Re[\bar{b}_1e^{-i\theta}], \\ 2Re[\lambda_1\phi_1(z_1)+\lambda_2\phi_2(z_2)+\phi_3(z_3)]\equiv 2Re[\bar{c}_1e^{-i\theta}]. \end{array}\right\} \tag{2.13}$$

where,

$$\left.\begin{array}{l} \bar{a}_1=\dfrac{1}{2}\left\{a(\sigma'_y-\sigma^0_y)-ib(\tau'_{xy}-\tau^0_{xy})\right\}, \\ \bar{b}_1=\dfrac{1}{2}\left\{a(\tau'_{xy}-\tau^0_{xy})-ib(\sigma'_x-\sigma^0_x)\right\}, \\ \bar{c}_1=-\dfrac{1}{2}\left\{a(\tau'_{yz}-\tau^0_{yz})-ib(\tau'_{xz}-\tau^0_{xz})\right\}. \end{array}\right\} \tag{2.14}$$

Thus, we can express the analytic functions $\phi_k(z_k)$ in the case of an elliptical opening as

$$\phi_k(z_k)=\Gamma_k ln\zeta_k+\Gamma_{k1}\zeta_k^{-1},\ (k=1,2,3), \tag{2.15}$$

where,

$$z_k=\frac{1}{2}\left\{(a-i\mu_k b)\zeta_k+(a+i\mu_k b)\zeta_k^{-1}\right\}. \tag{2.16}$$

In which ζ_k on the contour of the cross section of the opening takes a value equal to $e^{i\eta}$. Thus, on substituting from Eqs. (2.14) and (2.15) into Eq. (2.11) and solving the simultaneous equation, it may be reduced to

$$\left.\begin{aligned}\Delta\cdot\Gamma_{11}&=(\mu_2-\mu_3\lambda_2\lambda_3)\bar{a}_1+(\lambda_2\lambda_3-1)\bar{b}_1+\lambda_3(\mu_3-\mu_2)\bar{c}_1,\\ \Delta\cdot\Gamma_{21}&=(\mu_3\lambda_1\lambda_3-\mu_1)\bar{a}_1+(1-\lambda_1\lambda_3)\bar{b}_1+\lambda_3(\mu_1-\mu_3)\bar{c}_1,\\ \Delta\cdot\Gamma_{31}&=(\mu_1\lambda_2-\mu_2\lambda_1)\bar{a}_1+(\lambda_1-\lambda_2)\bar{b}_1+(\mu_2-\mu_1)\bar{c}_1.\end{aligned}\right\} \tag{2.17}$$

$$\Delta=\mu_2-\mu_1+\lambda_2\lambda_3(\mu_1-\mu_3)+\lambda_1\lambda_3(\mu_3-\mu_2). \tag{2.17'}$$

2.2 Determining Stresses in an Inclusion

Let us consider the displacements u^0, v^0 and w^0 in the matrix by the use of Eq. (2.1). Relations among strains, rotations, and displacements are generally written as follows:

$$\left.\begin{aligned}&\varepsilon_x^0=\frac{\partial u^0}{\partial x},\quad \gamma_{xy}^0=\frac{\partial u^0}{\partial y}+\frac{\partial v^0}{\partial x},\quad 2\omega_{xy}^0=\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y},\\ &\varepsilon_y^0=\frac{\partial v^0}{\partial y},\quad \gamma_{yz}^0=\frac{\partial v^0}{\partial z}+\frac{\partial w^0}{\partial y},\quad 2\omega_{yz}^0=\frac{\partial w^0}{\partial y}-\frac{\partial v^0}{\partial z},\\ &\varepsilon_z^0=\frac{\partial w^0}{\partial z},\quad \gamma_{xz}^0=\frac{\partial w^0}{\partial x}+\frac{\partial u^0}{\partial z},\quad 2\omega_{xz}^0=\frac{\partial u^0}{\partial z}-\frac{\partial w^0}{\partial x}.\end{aligned}\right\} \tag{2.18}$$

Since the external stresses in the matrix are applied at infinity, the body is considered to be in a state of plane strain, and we can assert that all components of stresses, rotations and displacements with one exceptional case related to the ε_z^0, will not depend on z. Thus,

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=0.$$

By integrating Eq. (2.18), we obtain displacements as follows.

$$\left.\begin{aligned}u^0&=\varepsilon_x^0 x+\frac{1}{2}\gamma_{xy}^0 y-\omega_{xy}^0 y+u_0,\\ v^0&=\varepsilon_y^0 y+\frac{1}{2}\gamma_{xy}^0 x+\omega_{xy}^0 x+v_0,\\ w^0&=\gamma_{xz}^0 x+\gamma_{yz}^0 y+\varepsilon_z^0 z+w_0.\end{aligned}\right\} \tag{2.18}$$

The displacements u', v' and w' for the inclusion are represented similarly as follows.

$$\left.\begin{aligned}u'&=\varepsilon_x' x+\frac{1}{2}\gamma_{xy}' y-\omega_{xy}' y+u_0',\\ v'&=\varepsilon_y' y+\frac{1}{2}\gamma_{xy}' x+\omega_{xy}' x+v_0',\\ w'&=\gamma_{xz}' x+\gamma_{yz}' y+\varepsilon_z' z+w_0'.\end{aligned}\right\} \tag{2.18'}$$

Therefore, substituting from expressions (2.18) and (2.18′) into the right side of Eq. (2.11), we obtain the following expressions

$$\left.\begin{aligned} u'-u^0-(\omega'-\omega^0)y+(u_0'-u_0)&=(\varepsilon_x'-\varepsilon_x^0)x+\frac{1}{2}(\gamma_{xy}'-\gamma_{xy}^0-2\omega)y,\\ v'-v^0+(\omega'-\omega^0)x+(v_0'-v_0)&=(\varepsilon_y'-\varepsilon_y^0)y+\frac{1}{2}(\gamma_{xy}-\gamma_{xy}^0+2\omega)x,\\ w'-w^0+(w_0'-w_0)&=(\gamma_{xz}'-\gamma_{xz}^0)x+(\gamma_{yz}'-\gamma_{yz}^0)y+(\varepsilon_z'-\varepsilon_z^0)z,\\ \omega&=(\omega'-\omega^0)+(\omega_{xy}'-\omega_{xy}). \end{aligned}\right\} \tag{2.19}$$

From the assumption that the length of the cylindrical (or elliptical) inclusion is sufficiently larger than its diameter, it is reasonable that the axial strain along the z-axis, ε_z', in the inclusion can be assumed to be equal to that in the matrix, ε_z^0; it is impossible to establish a theoretical determination if this assumption is not made.

By means of Eqs. (2.19) and (2.15) and by some calculations, the boundary conditions (2.11) are reduced to

$$\left.\begin{aligned} 2\sum_{k=1}^{3}p_k\Gamma_{k1}&=a\ (\varepsilon_x'-\varepsilon_x^0)\ +\frac{1}{2}ib(\gamma_{xy}'-\gamma_{xy}^0-2\omega),\\ 2\sum q_k\Gamma_{k1}&=\frac{1}{2}a(\gamma_{xy}'-\gamma_{xy}^0+2\omega)+ib(\varepsilon_y'-\varepsilon_y^0),\\ 2\sum r_k\Gamma_{k1}&=a(\gamma_{xz}'-\gamma_{xz}^0)+ib(\gamma_{yz}'-\gamma_{yz}^0). \end{aligned}\right\} \tag{2.20}$$

Substitution from Eqs. (2.14), (2.17) and (2.1) into Eq. (2.20) gives six simultaneous linear equations with unknown constants σ_x', σ_y', τ_{yz}', τ_{xz}', τ_{xy}' and ω. Thus by solving these equations, we can obtain the uniform stress components σ_x', σ_y', τ_{yz}', τ_{xz}' and τ_{xy}' in the inclusion and rotation ω. The stress component, σ_z', can be estimated as below, since the axial strain ε_z' is equal to ε_z^0.

$$\sigma_z'=\frac{\varepsilon_z^0}{a_{33}'}-\frac{1}{a_{33}'}(a_{13}'\sigma_x'+a_{23}'\sigma_y'+a_{34}'\tau_{yz}'+a_{35}'\tau_{xz}'+a_{36}'\tau_{xy}'), \tag{2.21}$$

where,

$$\varepsilon_z^0=a_{13}\sigma_x^0+a_{23}\sigma_y^0+a_{33}\sigma_z^0+a_{34}\tau_{yz}^0+a_{35}\tau_{xz}^0+a_{36}\tau_{xy}^0. \tag{2.21′}$$

Using the above method, we can also calculate the stress components σ_x^0, σ_y^0, σ_z^0, τ_{yz}^0, τ_{xz}^0 and τ_{xy}^0 applied at infinity in the matrix by means of the measured uniform stress components σ_x', σ_y', σ_z', τ_{yz}', τ_{xz}' and τ_{xy}'.

3. Numerical Examples

Let the stress components applied at infinity in the matrix be defined by σ_x^0, σ_y^0, σ_z^0, τ_{yz}^0, τ_{xz}^0 and τ_{xy}^0 as follows.

$$\left.\begin{aligned} \sigma_i^0&=F_i^1\sigma_x'+F_i^2\sigma_y'+F_i^3\sigma_z'+F_i^4\tau_{yz}'+F_i^5\tau_{xz}'+F_i^6\tau_{xy}',\\ &\qquad (i=x,\,y,\,z),\\ \tau_{ij}^0&=H_{ij}^1\sigma_x'+H_{ij}^2\sigma_y'+H_{ij}^3\sigma_z'+H_{ij}^4\tau_{yz}'+H_{ij}^5\tau_{xz}'+H_{ij}^6\tau_{xy}',\\ &\qquad (i,\,j=x,\,y,\,z,\ i\neq j). \end{aligned}\right\} \tag{3.1}$$

Here, F_i^k and H_{ij}^k ($k=1, 2, \cdots\cdots, 6$) are constants to be determined from the elastic constants and their directions with respect to the matrix and the inclusion. These coefficients can be easily calculated by the theoretical solution described above. The coefficients F_i^k and H_{ij}^k represent the influence coefficients of stress from which σ_i^0 and τ_{ji}^0 may be determined by measured stress components σ_i and τ_{ij} in the instrumented inclusion.

If these influence coefficients of stress are obtained, then we can find the stress components in the matrix from Eq. (3.1). In the first case, let us take an example such that the matrix and the inclusion are both isotropic bodies for which the elastic constants of the cylindrical inclusion are $E_0=3.0\times10^4$kg/cm^2 and $\nu_0=0.360$. Then, the influence coefficients F_i^4, F_i^5, H_{ij}^1, H_{ij}^2, H_{ij}^3, H_{yz}^5, H_{yz}^6, H_{xz}^4, H_{xz}^6, H_{xy}^4 and H_{xy}^5 are equal to zero in this case. The influence coefficients for this case are shown in Fig. 2 with varying parameters of

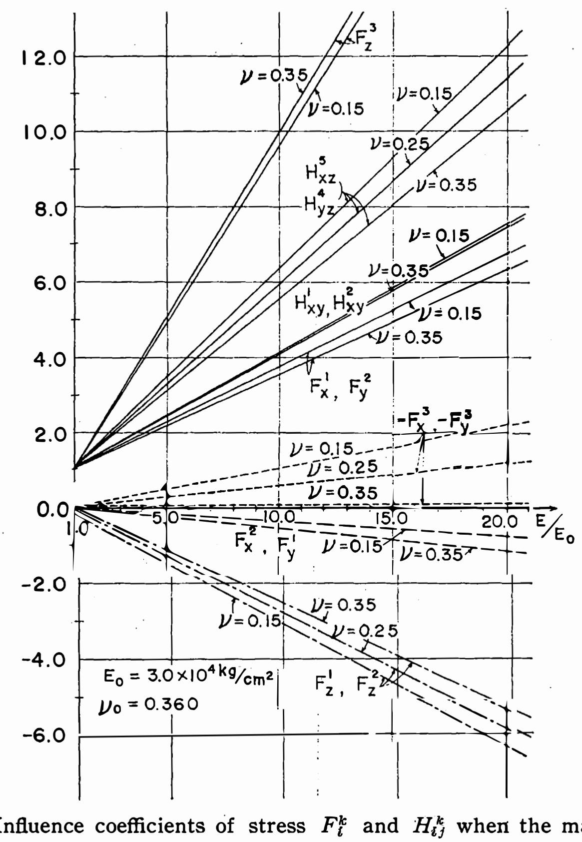
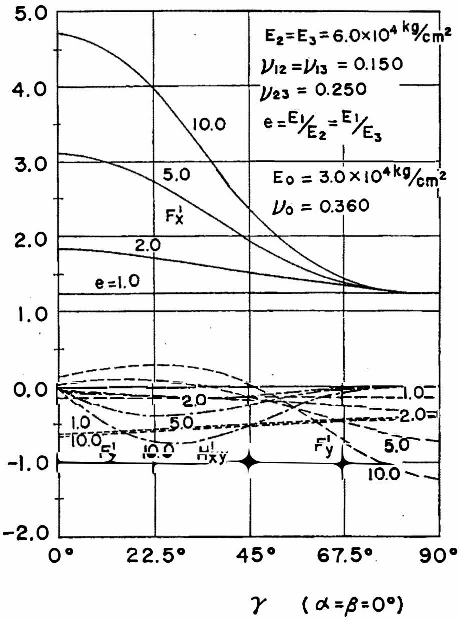
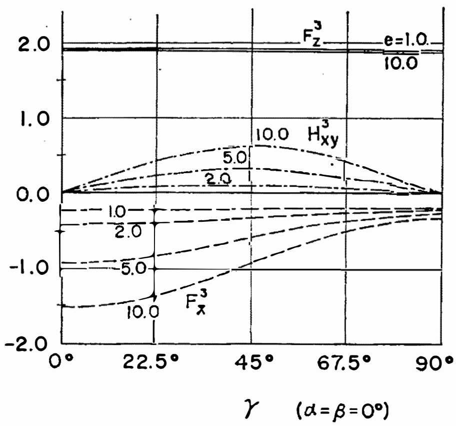


Fig. 2. Influence coefficients of stress F_i^k and H_{ij}^k when the matrix and cylindrical inclusion are both isotropic bodies (Young's modulus and Poisson's ratio of the inclusion are respectively, $E_0=3.0\times10^4$kg/cm^2 and $\nu_0=0.360$).

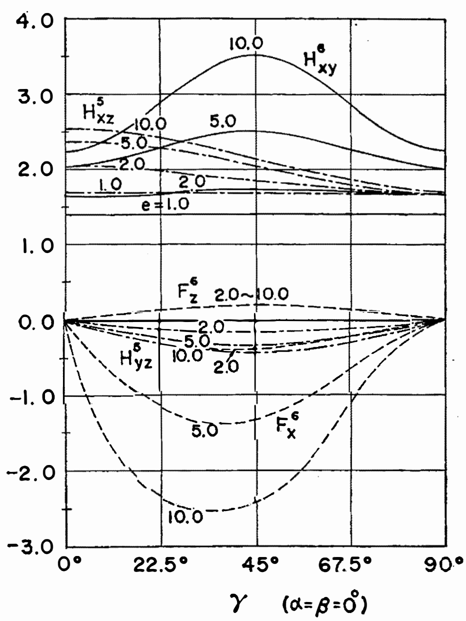
Young's modulus E and Poisson's ratio ν of the matrix as an isotropic elastic body.



(a)



(b)



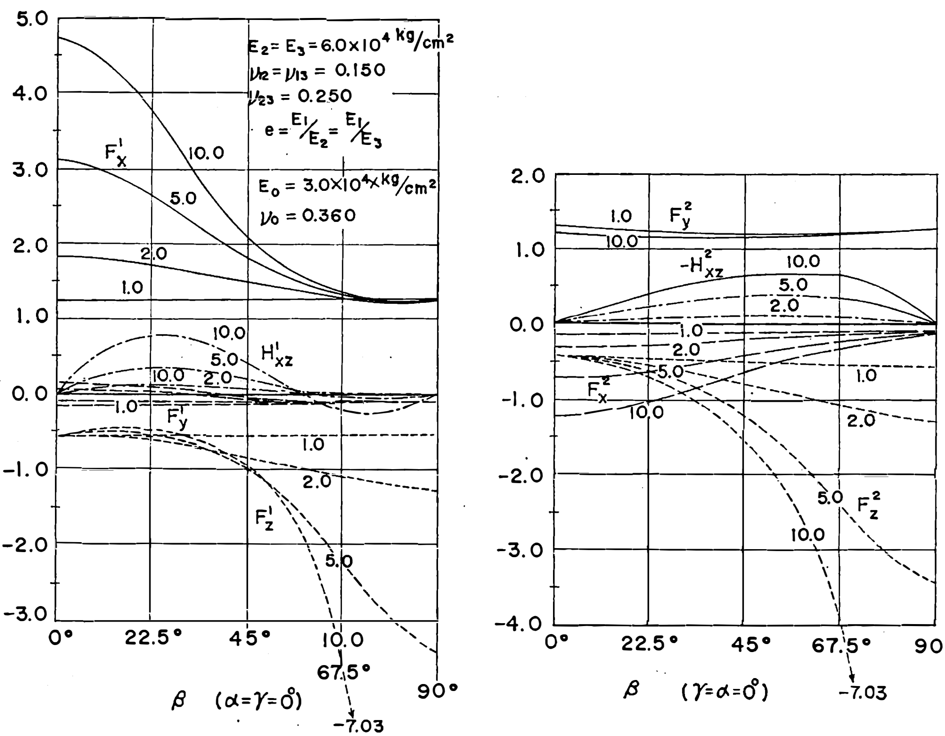
(c)

Fig. 3. Influence coefficients of stress F_j^k and H_{ij}^k when the Young's moduli E_1 and E_2 of the matrix rotates in-plane with respect to the z-axis.

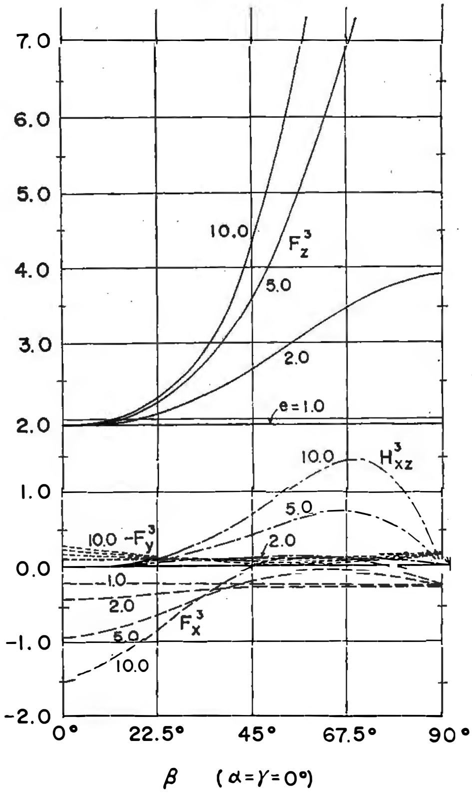
In the second case, let us consider the cross-anisotropic elastic matrix such that the elastic moduli and Poisson's ratios are given respectively by $E_2=E_3=6.0\times10^4$ kg/cm^2 and $\nu_{12}=\nu_{13}=0.150$, $\nu_{23}=0.250$ and the moduli of rigidity are defined by the following formula:

$$\frac{1}{G_{ij}}=\frac{1}{E_i}+\frac{1}{E_j}+\frac{2\nu_{ij}}{E_i},\ (i, j=1, 2, 3). \tag{3.2}$$

In this case, we assume that the principal elastic axes of the body coincide with the coordinate axes of the rectangular Cartesian system (x''', y''', z''') before performing the rotations of the coordinates.[10] The influence coefficients of stress, in the case where the principal elastic axes of the matrix are inclined to in-plane or out-of-plane for the plane perpendicular to the z-axis of the inclusion, are shown in Figs. 3~4. In the following numerical examples, it was assumed that the elastic constants of the inclusion were equal to $E_0=3.0\times10^4$ kg/cm^2 and $\nu_0=0.360$. Fig. 3 (a), (b) and (c) show the influence coefficients in which the axis of the principal elastic modulus E_1 (or E_2) rotates in-plane with respect to the z-axis (that is, corresponding to plane orthotropy), when the ratios of the elastic moduli, $e=E_1/E_2=E_1/E_3$, take several values.



(a) (b)



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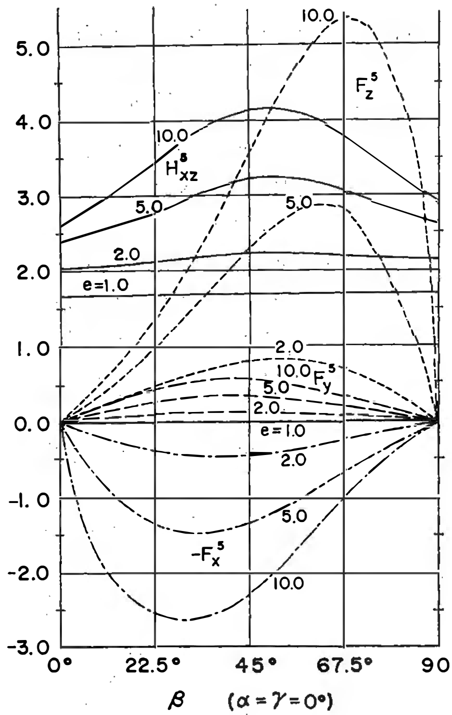
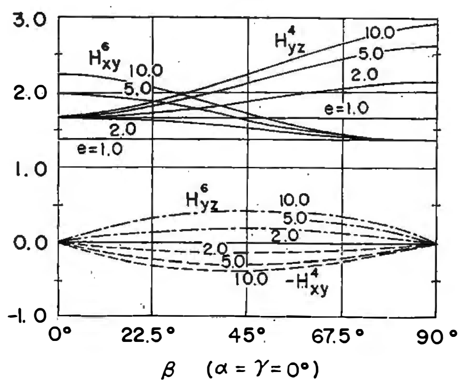


(d)



(e)

Fig. 4. Influence coefficients of stress F^k_i and H^k_{ij} when the plane containing the principal directions of E_1 and E_2 of the matrix rotates out-of-plane with respect to the y-axis.

Although the coefficients F^k_y and H^k_{yz} are not given in this figure, these coefficients are determined as in the following expressions.

$$(F^k_y)_\gamma=(F^k_x)_{\gamma=90^\circ-\gamma},\quad (H^k_{yz})_\gamma=(H^k_{xz})_{\gamma=90^\circ-\gamma}. \tag{3.3}$$

Here, γ is an angle of rotation around the z'-axis.[10] Fig. 4 (a)~(e) shows the influence coefficients in which the plane containing the principal axes of the elastic moduli, E_1 and E_2, rotates out-of-plane with respect to the y''-axis.[10]

When the elastic constants and their directions with respect to the matrix and the inclusion are known in advance, stress components applied at infinity in the matrix can be easily determined from the measured stress components in the instrumented inclusion in the following ways.

As an numerical example, consider the case that the elastic constants of the matrix and the inclusion take the values shown in Figs. 2~4 and the measured stress components in the inclusion are given as

$$\left.\begin{array}{l}\sigma'_x=2.0\,\mathrm{kg/cm^2},\ \sigma'_y=3.5\,\mathrm{kg/cm^2},\ \sigma'_z=4.3\,\mathrm{kg/cm^2},\\ \tau'_{yz}=1.5\,\mathrm{kg/cm^2},\ \tau'_{xz}=1.0\,\mathrm{kg/cm^2},\ \tau'_{xy}=-1.2\,\mathrm{kg/cm^2}.\end{array}\right\} \tag{3.4}$$

When the elastic constants and their directions with respect to the matrix are given that

(a) $E=3.0\times10^5\,\mathrm{kg/cm^2},\ \nu=0.150,$

(b) $E_1=3.0\times10^5\,\mathrm{kg/cm^2},\quad E_2=E_3=6.0\times10^4\,\mathrm{kg/cm^2},\quad \nu_{12}=\nu_{13}=0.150,\quad \nu_{23}=0.250,$ $\gamma=45°,\ \alpha=\beta=0°,$

(c) $E_1=3.0\times10^5\,\mathrm{kg/cm^2},\quad E_2=E_3=6.0\times10^4\,\mathrm{kg/cm^2},\quad \nu_{12}=\nu_{13}=0.150,\quad \nu_{23}=0.250,$ $\beta=45°,\ \alpha=\gamma=0°,$

the stress components applied at infinity in the matrix are determined by the use of Figs. 2~4 as follows.

(a) $\sigma_x^0=1.39,\quad \sigma_y^0=7.57,\quad \sigma_z^0=23.48,\ \tau_{yz}^0=9.64,\ \tau_{xz}^0=6.41,\ \tau_{xy}^0=-4.94,$

(b) $\sigma_x^0=2.61,\quad \sigma_y^0=5.61,\quad \sigma_z^0=5.09,\ \tau_{yz}^0=2.71,\ \tau_{xz}^0=1.51,\ \tau_{xy}^0=-3.19,$

(c) $\sigma_x^0=-3.35,\ \sigma_y^0=-1.11,\ \sigma_z^0=11.75,\ \tau_{yz}^0=2.75,\ \tau_{xz}^0=4.44,\ \tau_{xy}^0=-1.58.$

In other cases when the elastic constants and their directions with respect to the matrix and the cylindrical inclusion take arbitrarily different values, the same analysis as described above may be easily applied.

4. Concluding Remarks

In the present paper we have analyzed the theoretical stresses in isotropic or anisotropic elastic matrices with a cylindrical inclusion in which the length is sufficiently larger than the diameter. The results of calculation have been explicitly shown by several numerical examples. This method is very applicable for practical usage such as for stress measurements in rock masses.

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