A Note on the Nonlinear Vibrations of the Elastic String

By

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The nonlinear vibrations of an elastic string which are described by the equation

$$\rho h\partial^2 w/\partial t^2=(P_0+Eh/2L\cdot\int_0^L(\partial w/\partial x)^2dx)\partial^2 w/\partial x^2$$

were investigated. A proof for the existence and uniqueness of a solution with finite harmonics in the large in time and a conditionally periodic behavior is given for this equation.

1. Introduction

Free lateral 'finite' vibrations of uniform beams with the ends restrained so they remain a fixed distance apart may be described by the equation,

$$\rho h\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial x^4}=\left\{P_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2dx\right\}\frac{\partial^2 w}{\partial x^2}, \quad (-\infty<t<+\infty,\ 0\leq x\leq L) \tag{1}$$

with the initial and boundary conditions,

$$\left.\begin{aligned}&w(0,x)=w_0(x),\ \partial w(0,x)/\partial t=w_1(x),\\&w(t,0)=\partial^2 w(t,0)/\partial x^2=w(t,L)=\partial^2 w(t,L)/\partial x^2=0,\end{aligned}\right\} \tag{2}$$

where w is the lateral deflection, x is the space coordinate, t is the time, E is the Young's modulus, EI is the flexural rigidity, ρ is the mass density, h is the thickness of a beam of unit width, L is the length, and P_0 is the initial axial tension.

An equation governing free lateral 'finite' vibrations of a uniform string may be considered as a limit case, such as the no-resistance ($EI=0$) to bending of equation (1), *i.e.*,

$$\rho h\frac{\partial^2 w}{\partial t^2}=\left\{P_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2dx\right\}\frac{\partial^2 w}{\partial x^2}. \tag{3}$$

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Equations (1) and (3) have only the nonlinearity contained in the term for the axial force, which is given by the stretching of the medium line.[11] Concerning equations (1) and (3) a review has been given, for example, by Eisley.[8] An equation of 'finite' deflections of plates was derived by Berger,[3] which neglects the second invariant of strain in von Karman's equation. Free vibrations of this plate are described by the equation

$$\frac{\rho}{D}\frac{\partial^2 w}{\partial t^2}+\nabla^4 w=\left\{P+\frac{6}{Ah^2}\iint|\nabla w|^2 dA\right\}\nabla^2 w, \tag{4}$$

where D is the flexural rigidity of the plate, and A is the area of the plate, which was considered by Aggarwala.[1]

The existence and uniqueness theorem of solutions in the large in time for the problem in (1), (2) was obtained by Dickey.[7] Concerning vibrations of the string (3) a local (in time) existence theorem for general initial data was given by Dickey.[6] As regards the existence and uniqueness theorem of solutions in the large (in time) both equations (1) and (4) can be treated in the frame-work of the theory of nonlinear perturbations to linear evolution equations.[5,9,13]

Here we consider the solutions in the large (in time) for equation (3) under the assumption that the initial data of (2), do not contain infinitely higher harmonics, that is, there exists a natural number N such that the initial data may be represented as follows

$$w_0(x)=\sum_{k=1}^{N} a_k \sin\frac{\pi}{L}kx, \quad w_1(x)=\sum_{k=1}^{N} b_k \sin\frac{\pi}{L}kx, \tag{5}$$

where a_k, b_k are constants.

Then the solution of the problem (3), (2) with (5) is proved easily to exist uniquely in the large in time. This follows from the fact that the nonlinear term is an integration of good form; thus, if certain harmonics are not contained in the initial data, then these harmonics will not appear in the solution in the course of time.

Under the same assumption, the solution was examined for its behavior near the equilibrium state ($w=0$), using the Kolmogorov-Arnol'd-Moser theorem on the conservation of conditionally periodic motions of dynamic systems in the case of oscillations (a limiting degenerate case) under small perturbations of Hamiltonian functions; that is, a large part of the free vibrations starting from initial data sufficiently near the equilibrium state can be shown to be conditionally periodic oscillation.[2,4,10,12] The definition of the conditionally periodic oscillation will be discussed later. We note that the associated linear system of equation (3) is

$$\rho h \frac{\partial^2 w}{\partial t^2} = P_0 \frac{\partial^2 w}{\partial x^2},$$

and so the linear frequencies of equation (3), that is,

$$\omega = \left(\sqrt{\frac{P_0}{\rho h}}\frac{\pi}{L},\ 2\sqrt{\frac{P_0}{\rho h}}\frac{\pi}{L}, \dots\dots,\ N\sqrt{\frac{P_0}{\rho h}}\frac{\pi}{L},\ \dots\dots\right),$$

are linearly dependent on integers. Therefore, the linearized motion under the assumption (5) is only periodic oscillation (not the conditionally periodic oscillation).

The same result can be obtained for both equations (1) and (4) under the assumptions of the initial data (the smallness and the analogue to (5)).

Last, we consider the following linear wave equation with time dependent wave-velocity

$$\frac{\partial^2 y}{\partial t^2} = (1 + a(t))\frac{\partial^2 y}{\partial x^2}, \quad (0 \le x \le \pi,\ 0 \le t) \tag{6}$$

$$\left.\begin{aligned} &y(t, 0) = y(t, \pi) = 0, \quad t \ge 0, \\ &y(0, x) = y_0(x),\ \partial y(0, x)/\partial t = y_1(x),\ 0 \le x \le \pi, \end{aligned}\right\}$$

where the function $a(t)$ is non-negative and bounded.

It is well known that if $a(t)$ is a Lipschitz continuous function in t, then a smooth solution for equation (6) exists in the large in time. In section 4 it is shown that if $a(t)$ has a bounded total variation in $0 \le t \le {}^{\forall}T$, then the solution of equation (6) can exist in $t \in [0, T]$, but if $a(t)$ is only a bounded measurable function, then that is, in general, not valid in the following sense. We see that the L^2-energy estimate does not hold; that is, when ${}^{\forall}C > 0$, ${}^{\forall}T > 0$, and ${}^{\forall}E > 0$ are given, there exists a bounded measurable function $a(t)$ and an initial value $y_0(x)$, $y_1(x)$ such that $\|y_0\|_2^2 + \|y_1\|_1^2 \le E$ and the corresponding solution satisfies the following inequality:

$$\|y(t, \cdot)\|_1^2 + \|\partial y(t, \cdot)/\partial t\|^2 > C \quad \text{for } t = t_0 \le T,$$

where $\|\cdot\|_m (m = 0, 1, 2)$ are norms in W_2^m (cf. section 2).

2. Existence and Uniqueness

We consider the following initial-boundary value problem which is equivalent to that mentioned in the introduction.

$$\partial^2 w/\partial t^2 = \left(1 + \alpha \frac{2}{\pi}\int_0^\pi (\partial w/\partial x)^2 dx\right)\partial^2 w/\partial x^2, \quad 0 \le x \le \pi,\ 0 \le t, \tag{7}$$

$$\left.\begin{aligned} &w(0, x) = w_0(x),\ \partial w(0, x)/\partial t = w_1(x), \quad 0 \le x \le \pi, \\ &w(t, 0) = w(t, \pi) = 0, \quad 0 \le t \end{aligned}\right\} \tag{8}$$

where α is a non-negative constant.

Let $W_2{}^l(l=1, 2,\cdots)$ be the Sobolev space of square summable functions $f(x)$ in $x\in[0, \pi]$ with square summable l-th derivatives. $W_2^0=L_2(0, \pi)$. The norm is denoted by $\|\cdot\|_l$. If a function $f(x)$ belongs to $W_2{}^l(l=1, 2,\cdots)$, then $f(x)$ has continuous derivatives up to the $(l-1)$th order. We define

$$\tilde{W}_2{}^l=\left\{f(x)\in W_2{}^l,\ f(0)=f(\pi)=0\right\} \quad (l=1, 2,\cdots),$$

and then it is also a Hilbert space with the same norm as in $W_2{}^l$; further, Sobolev's lemma or a direct calculation gives the following equivalent norm.

$$\begin{aligned}\|f\|_l{}^2&=\|d^lf/dx^l\|^2+|f(0)|^2+|f(\pi)|^2\\ &=\|d^lf/dx^l\|^2 \quad \text{for } f\in\tilde{W}_2{}^l(l=1, 2).\end{aligned}$$

Let Q_T be a class of functions $w(t, \cdot)$ such that they are bounded measurable (in t) vector-valued functions from $t\in[0, T]$ in $\tilde{W}_2{}^2$ with bounded measurable first derivatives from t in $\tilde{W}_2{}^1$ and bounded measurable second derivatives from t in L_2.

Now we suppose an essential but restrictive assumption: that the initial data, equation (8), does not contain infinitely higher harmonics; that is, that there exists a natural number N such that the initial data are represented as

$$w_0(x)=\sum_{k=1}^N a_k \sin kx,\ w_1(x)=\sum_{k=1}^N b_k \sin kx,\ (a_k, b_k: \text{constants}). \tag{9}$$

Under the assumption (9) we may easily obtain the following existence and uniqueness theorem of the solution for problem (7) and (8).

If we assume the condition (9) on the initial data (8), then there exists a solution of problem (7) (8) in the large in time, which has the form

$$w(t, x)=\sum_{k=1}^N a_k(t)\sin kx, \tag{10}$$

and it is unique in the class of functions Q_T.

In fact, substituting the expression (10) into equation (7) we get the following system of ordinary differential equations.

$$\ddot{a}_k(t)=-k^2a_k(t)\left\{1+\alpha\sum_{n=1}^N n^2a_n{}^2(t)\right\}, \quad (k=1, 2,\cdots, N). \tag{11}$$

The initial data for system (11) are given by the condition (9) as follows.

$$a_k(0)=a_k, \quad \dot{a}_k(0)=b_k, \quad (k=1, 2,\cdots, N). \tag{12}$$

The total energy of problem (11) (12) is the following.

$$E(t)\equiv\frac{1}{2}\sum_{k=1}^N\left\{\dot{a}_k{}^2(t)+k^2a_k{}^2(t)\right\}+\frac{\alpha}{4}\left\{\sum_{k=1}^N k^2a_k{}^2(t)\right\}^2=$$

$$=E(0)\equiv\frac{1}{2}\sum_{k=1}^{N}(b_k^2+k^2a_k^2)+\frac{\alpha}{4}\Big(\sum_{k=1}^{N}k^2a_k^2\Big)^2. \tag{13}$$

This *a priori* estimate, equation (13), allows us to conclude that the solution of problem (11) (12) and thus problem (7) (8) with (9) exists in the large in time.

The uniqueness of the solution in the class Q_T is obtained by the energy estimate for the difference of the two solutions in the usual way.

3. Behavior of Solutions near the Equilibrium State

When we assume the condition (9), the system (11), (12) is equivalent to problem (7), (8) from the argument of section 2. Therefore, we may apply the Kolmogorov-Arnol'd-Moser theorem on the conservation of conditionally periodic motions of dynamic systems in the case of oscillation (a limiting degenerate case) under small perturbations of Hamiltonian functions to system (11). The Hamiltonian function corresponding to (11) is given by (13), that is,

$$H=\frac{1}{2}\Big\{\sum_{k=1}^{N}(\dot{a}_k^2+k^2a_k^2)+\frac{\alpha}{2}\Big(\sum_{k=1}^{N}k^2a_k^2\Big)^2\Big\}. \tag{13}$$

Here we note that the linear frequencies of system (11) ($\omega=(1, 2,\cdots, N)$) are linearly dependent on integers, that is, system (11) is a resonant one with a specialized nonlinear coupling.

By the use of the variables $p_k=\dot{a}_k/\sqrt{k}$ and $q_k=\sqrt{k}\,a_k$ ($k=1,\cdots, N$) the Hamiltonian function (13) becomes

$$H=\frac{1}{2}\Big\{\sum_{k=1}^{N}k(p_k^2+q_k^2)+\frac{\alpha}{2}\Big(\sum_{k=1}^{N}kq_k^2\Big)^2\Big\}. \tag{14}$$

First, in order to transform (14) into a normal form by Birkhoff's transformations we introduce the following variables

$$\xi_k=\frac{1-i}{2}(p_k-iq_k),\ \eta_k=\frac{1-i}{2}(p_k+iq_k),\ (k=1,\cdots, N,\ i=\sqrt{-1}),$$

where the canonical transformation from (q, p) to (η, ξ) is generated by the function

$$\phi(q, \xi)=\sum_{k=1}^{N}\Big\{\frac{1}{2}\xi_k^2+(1+i)q_k\xi_k+\frac{i}{2}q_k^2\Big\}.$$

Then (14) becomes

$$H=\sum_{k=1}^{N}ik\xi_k\eta_k-\frac{\alpha}{16}\Big\{\sum_{k=1}^{N}k(\xi_k-\eta_k)^2\Big\}^2\equiv H_2+H_4.$$

Birkhoff's transformation, which normalizes the fourth order term H_4, is a canonical transformation from (η, ξ) to $(\bar{q}, \bar{p})$, which is defined by

$$\xi_k=\bar{p}_k+\partial K/\partial \eta_k,\ \ \bar{q}_k=\eta_k+\partial K/\partial \bar{p}_k,\ \ (k=1,\cdots\cdots,N),$$

where

$$K=\sum_{k_1+\cdots+k_N+l_1+\cdots+l_N=4} a_{k_1\cdots k_N l_1\cdots l_N}\cdot \bar{p}_1^{k_1}\cdots\cdots\bar{p}_N^{k_N}\cdot\eta_1^{l_1}\cdots\cdots\eta_N^{l_N},$$

$$k_j,\ l_j=0,1,\cdots\cdots,4,\quad a_{k_1\cdots k_N l_1\cdots l_N}=\text{constant}.$$

If we solve explicitly for ξ, η in terms of $\bar{p}$, $\bar{q}$, we obtain

$$\xi_k=\bar{p}_k+\partial K^*/\partial\bar{q}_k+\cdots,\ \ \eta_k=\bar{q}_k-\partial K^*/\partial\bar{p}_k+\cdots,\ \ (k=1,\cdots,N),$$

where K^* denotes the function obtained by replacing η with $\bar{q}$ in K and the power series converges in the neighborhood of the origin. The modified value of H, obtained by substitution, is

$$H(\bar{p},\bar{q})=H_2(\bar{p}_1+\partial K^*/\partial\bar{q}_1+\cdots,\ \cdots,\ \bar{q}_N-\partial K^*/\partial\bar{p}_N+\cdots)+H_4,$$

where the arguments of H_4 are the same as those of H_2. To terms of the fourth degree inclusive we find

$$H(\bar{p},\bar{q})=\sum_{k=1}^{N} ik\bar{p}_k\bar{q}_k+\sum_{k=1}^{N} ik\Big(\bar{q}_k\frac{\partial K^*}{\partial\bar{q}_k}-\bar{p}_k\frac{\partial K^*}{\partial\bar{p}_k}\Big)+H_4(\bar{p}_1,\cdots,\bar{q}_N)+$$

$$+\text{higher degree terms}\equiv H_2(\bar{p},\bar{q})+H_4(\bar{p},\bar{q})+\tilde{H}(\bar{p},\bar{q}).$$

Thus the forms of H_2 are unmodified while H_4 takes the form

$$\sum_{n=1}^{N} in\Big(\bar{q}_n\cdot\frac{\partial K^*}{\partial\bar{q}_n}-\bar{p}_n\cdot\frac{\partial K^*}{\partial\bar{p}_n}\Big)+H_4(\bar{p}_1,\cdots,\bar{q}_N)=$$

$$=\sum_{k_1+\cdots+l_N=4}\Big[a_{k_1\cdots l_N}\Big\{\sum_{n=1}^{N} in(l_n-k_n)\Big\}+h_{k_1\cdots k_N l_1\cdots l_N}\Big]\bar{p}_1^{k_1}\cdots\bar{p}_N^{k_N}\bar{q}_1^{l_1}\cdots\bar{q}_N^{l_N}, \tag{15}$$

where $h_{k_1\cdots k_N l_1\cdots l_N}$ is the coefficient in the original H_4 analogous to $a_{k_1\cdots l_N}$. Easy calculation gives the follwing:

$$H_4(\bar{p},\bar{q})=-\frac{\alpha}{16}\Big\{\sum_{k=1}^{N} k(\bar{p}_k-\bar{q}_k)^2\Big\}^2=$$

$$=-\frac{\alpha}{16}\Big\{\sum_{m=1}^{N} 6m^2\bar{p}_m^2\bar{q}_m^2+\sum_{m\neq n} 4mn\bar{p}_m\bar{q}_m\bar{p}_n\bar{q}_n+$$

$$+\sum_{m=1}^{N} m^2(\bar{p}_m^4-4\bar{p}_m^3\bar{q}_m-4\bar{p}_m\bar{q}_m^3+\bar{q}_m^4)+$$

$$+\sum_{m\neq n} mn(\bar{p}_m^2\bar{p}_n^2+\bar{p}_m^2\bar{q}_n^2+\bar{q}_m^2\bar{p}_n^2+\bar{q}_m^2\bar{q}_n^2-$$

$$-2\bar{p}_m^2\bar{p}_n\bar{q}_n-2\bar{p}_m\bar{q}_m\bar{p}_n^2-2\bar{p}_m\bar{q}_m\bar{q}_n^2-2\bar{q}_m^2\bar{p}_n\bar{q}_n)\Big\}. \tag{16}$$

Although the linear frequencies of system (11) $(\omega=(1,2,\cdots,N))$ are linearly dependent on integers, we can take convenient values for the coefficients

$a_{k_1\cdots k_N l_1\cdots l_N}$ in (15) such that the terms in the latter two summations in (16) vanish in $\bar{H}_4$, when the terms in the former two summations in (16) remain unchanged. In fact, for example, the coefficient of the term p_m^4 in $\bar{H}_4$ turns out to be

$$a_{0\cdots 4\cdots 00\cdots 0}\left\{im(4-0)\right\}-\frac{\alpha}{16}m^2,$$

therefore we may take $a_{0\cdots 4\cdots 0\cdots 0}=-i\alpha m/64$, and then the term vanishes in $\bar{H}_4$. For another example, since the coefficient of the term of $p_m^2\bar{q}_n^2$ $(m\neq n)$ turns out to be

$$a_{0\cdots 2\cdots 00\cdots 2\cdots 0}\left\{im(-2)+in(2)\right\}-\frac{\alpha}{16}mn,$$

we may take $a_{0\cdots 2\cdots 00\cdots 2\cdots 0}=i\alpha mn/32(m-n)$, $(m\neq n)$, and then the term vanishes in $\bar{H}_4$. The other terms (except $p_m^2\bar{q}_m^2$, $p_m\bar{q}_m p_n\bar{q}_n$) can be treated in the same way. The terms of $p_m^2\bar{q}_m^2$, $p_m\bar{q}_m p_n\bar{q}_n$ remain unchanged by any $a_{k_1\cdots l_N}$, because the multiplier of $a_{k_1\cdots l_N}$ is zero. Thus we obtain the following expression of the Hamiltonian function:

$$H(p, \bar{q})=\sum_{n=1}^{N} inp_n\bar{q}_n-\frac{\alpha}{8}\sum_{m,n=1}^{N}(2+\delta_n^m)mnp_m\bar{q}_m p_n\bar{q}_n+\tilde{H}(p, \bar{q}),$$

where $\tilde{H}(p, \bar{q})$ begins with the sixth degree terms with respect to p and $\bar{q}$, because the original Hamiltonian function does not contain the third degree terms, and so $\tilde{H}(p, \bar{q})$ does not contain the fifth degree terms. By the transformation $p_k=\frac{1-i}{2}(\bar{p}_k-i\bar{\bar{q}}_k)$, $\bar{q}_k=\frac{1-i}{2}(\bar{p}_k+i\bar{\bar{q}}_k)$ we get

$$H(\bar{p}, \bar{\bar{q}})=\frac{1}{2}\sum_{n=1}^{N}n(\bar{p}_n^2+\bar{\bar{q}}_n^2)+\frac{\alpha}{32}\sum_{m,n=1}^{N}(2+\delta_n^m)\cdot$$
$$\cdot mn(\bar{p}_m^2+\bar{\bar{q}}_m^2)(\bar{p}_n^2+\bar{\bar{q}}_n^2)+\tilde{H}(\bar{p}, \bar{\bar{q}}),$$

and by the transformation using action-angle variables

$$\bar{p}=\sqrt{2\tau_k}\cos Q_k,\ \bar{\bar{q}}_k=\sqrt{2\tau_k}\sin Q_k\quad (k=1,\cdots\cdots, N),$$

we get the following

$$H(\tau, Q)=\sum_{n=1}^{N}n\tau_n+\frac{\alpha}{8}\sum_{m,n=1}^{N}(2+\delta_n^m)mn\tau_m\tau_n+$$
$$+\tilde{H}(\tau, Q)\equiv H_0(\tau)+\tilde{H}(\tau, Q), \tag{17}$$

where by virtue of the above reduction by Birkhoff's transformation to this normal form, there exists $\varepsilon_0>0$ such that $\tilde{H}(\tau, Q)$ is analytic in the region $G=\left\{|\tau_n-\varepsilon_0|<\varepsilon_0,\ |I_m Q|<1\right\}$ and $\tilde{H}$ begins with the sixth degree terms with respect to $\bar{p}$, $\bar{\bar{q}}$, that is,

$$|\tilde{H}(\tau, Q)|\leq C|\tau|^3 \text{ in } G. \tag{18}$$

In order to introduce the definition of conditionally periodic motions of the dynamic system, we consider the Hamiltonian system with the Hamiltonian $H_0(\tau)$ defined in (17) in the action-angle variables $(\tau_1,\cdots,\tau_N, Q_1,\cdots,Q_N)$.

$$H_0(\tau)=\sum_{n=1}^{N} n\tau_n+\frac{\alpha}{8}\sum_{m,n=1}^{N}(2+\delta_n^m)mn\tau_m\tau_n.$$

Since the canonical equation corresponding to it is

$$\frac{dQ_m}{dt}=\frac{\partial H_0}{\partial\tau_m},\quad \frac{d\tau_m}{dt}=-\frac{\partial H_0}{\partial Q_m}\qquad (m=1,\cdots\cdots,N),$$

the integration is at once the following

$$\tau_m(t)=\tau_m(0),\ Q_m(t)=\omega_m t+Q_m(0)\qquad (m=1,\cdots\cdots,N),$$

$$\omega_m=m+\frac{\alpha}{4}\sum_{n=1}^{N}(2+\delta_n^m)nm\tau_n(0).$$

We put $\tau=(\tau_1,\cdots,\tau_N)$, $\omega=(\omega_1,\cdots,\omega_N)$.

Each torus $\tau=\tau(0)$ (=constant vector) is invariant. If the frequency $\omega=\omega(\tau(0))$ is linearly independent of integers (that is, from

$$\omega_1 k_1+\cdots\cdots+\omega_N k_N=0 \text{ with integers } k_i(i=1,\cdots,N)$$

it follows that $k_i=0$ $(i=1,\cdots,N)$), then the motion is called *conditionally periodic* with n-frequencies $\omega_1,\cdots,\omega_N$. Then the trajectory $\tau(t)$, $Q(t)$ fills the torus $\tau=\tau(0)$ everywhere densely.

Now we remember the theorem of Kolmogorov-Arnol'd-Moser, of which we need especially the theorem that treats the motion near the equilibrium state in the theory of oscillations. It may be described as follows: The motion is supposed to be described by the canonical equation

$$\dot{p}_k=-\partial H/\partial q_k,\ \dot{q}_k=\partial H/\partial p_k\quad (k=1,2,\cdots,N),\tag{19}$$

where the Hamiltonian function is assumed to be of the form

$$H=H_0(p,q)+H_1(p,q),$$

where

$$H_0=\sum_{k=1}^{N}\lambda_k\tau_k+\sum_{k,l=1}^{N}\lambda_{kl}\tau_k\tau_l,\qquad 2\tau_k=p_k^2+q_k^2,$$

λ_k, $\lambda_{kl}=\lambda_{lk}$ are constants,

H_1 is analytic with respect to p and q in the domain $G=\left\{|\tau_k-\varepsilon_0|<\varepsilon_0,\ k=1,\cdots,N\right\}$ and it satisfies

$|H_1|\le C|\tau|^{5/2}$ in G.

If the condition

$$\det(2\lambda_{kl})=\det(\partial^2 H_0/\partial\tau_k\partial\tau_l)\neq 0 \text{ in } G\tag{20}$$

is valid, then for any $\kappa>0$ it is possible to find $\varepsilon>0(\varepsilon_0>\varepsilon)$ such that: I. The domain $G_\varepsilon=\{|\tau_k-\varepsilon|<\varepsilon\}$ consists of two sets f_ε and F_ε, one of which, F_ε, is invariant with respect to the motions of equation (19) and the other, f_ε, is small: mes $f_\varepsilon<\kappa$ mes F_ε, where mes denotes the ordinary Lebesgue measure. II. F_ε consists of invariant n-dimensional analytic tori T_ω given by the parametrically represented equations

$$p_k=\sqrt{2(\tau_k^\omega+f_k^\omega(Q))}\cos(Q_k+g_k^\omega(Q))$$

$$q_k=\sqrt{2(\tau_k^\omega+f_k^\omega(Q))}\sin(Q_k+g_k^\omega(Q)),$$

where

$$f_k^\omega(Q_j+2\pi)=f_k^\omega(Q_j),\quad g_k^\omega(Q_j+2\pi)=g_k^\omega(Q_j),$$

$Q=(Q_1,\cdots,Q_N)$ is the angular parameter and

$\tau^\omega=(\tau_1^\omega,\cdots,\tau_N^\omega)$ is constant depending on

the number of the torus ω. III. The invariant tori T_ω differ little from the tori $\tau=\tau^\omega=$constant, *i. e.*, $|f_k^\omega|$, $|g_k^\omega|<\kappa\varepsilon$. IV. The motion determined by eguation (19) on the torus T_ω is conditionally periodic with n-frequencies ω: $\dot{Q}=\omega=\partial H_0/\partial\tau^\omega$, $\dot{\tau}=0$.

In order to apply the above theorem to our case it remains only to verify that the condition (20) holds. It can be easily calculated so that

$$\det\left\{(2+\delta_n^m)mn\right\}=(2N+1)\prod_{m=1}^{N}m^2\neq 0.$$

Therefore we can conclude that

A large part of the vibrations of equation (7) starting from initial data sufficiently near the equilibrium state and satisfying the condition (9) is a conditionally periodic motion.

At last we remark that the same result holds for the free vibrations of a nonlinear beam or plate (which is described by equations (1) or (4) in the introductory section) under the two assumptions of the initial data (the smallness and the analogue to (9)).

4. Wave Equation with a Time-Dependent Wave Velocity

We consider the wave equation with a time-dependent coefficient:

$$\partial^2 y/\partial t^2=b(t)\partial^2 y/\partial x^2,\quad 0\le t,\ 0\le x\le\pi, \tag{21}$$

where the function $b(t)$ is bounded and measurable and

$b(t)\ge\delta=$constant>0.

We say a function, $y(t)$, belonging to the class Q_T, is the solution for the initial-boundary value problem (21) with

$$\left.\begin{aligned} &y(t,0)=y(t,\pi)=0,\ 0\le t\\ &y(0,x)=y_0(x),\ \partial y(0,x)/\partial t=y_1(x),\ 0\le x\le\pi \end{aligned}\right\} \tag{22}$$

if it satisfies equation (21) almost everywhere and also satisfies the initial conditions. ($y_0(x)\in\tilde{W}_2^2$, $y_1(x)\in\tilde{W}_2^1$)

We show that there exists a solution in the large in time in the case that $b(t)$ has a locally bounded total variation in t, and then the following estimate is valid:

$$\|y\|_{m+1}^2+\|\partial y/\partial t\|_m^2\le C(\|y_0\|_{m+1}^2,\ \|y_1\|_m^2) \tag{23}$$
$$\text{for } 0\le t\le{}^{\forall}T,\ m=0,1,$$

where C depends only on the total variation of $b(t)$ in $[0,T]$ and does not depend on each $b(t)$. But in the case that $b(t)$ is only bounded and measurable in $[0,+\infty)$, it is, in general, not valid; that is, the following energy estimate does not hold:

$$\|y\|_1^2+\|\partial y/\partial t\|^2\le C(\|y_0\|_2^2,\ \|y_1\|_1^2) \tag{24}$$
$$\text{for } 0\le t\le T,$$

where C is dependent only on $\operatorname{ess.sup}_{0\le t\le T}|b(t)|$ and independent of each $b(t)$.

First, in order to prove the latter, we consider the following initial data.

$$y_0^{(n)}(x)=\alpha\sin nx,\ y_1^{(n)}(x)=\beta\sin nx, \tag{25}$$
$$\text{where } \alpha,\ \beta \text{ are constants.}$$

If $n^2\alpha^2+\beta^2$, $n^4\alpha^2+n^2\beta^2$ are kept fixed, then $\|y_0^{(n)}\|_{m+1}^2+\|y_1^{(n)}\|_m^2$ $(m=0,1)$ are constant (independent of n). For initial (25) the solution of (21) is given by the ordinary differential equation:

$$\begin{aligned} &\ddot{a}_n(t)+n^2b(t)a_n(t)=0,\ 0\le t\\ &a_n(0)=\alpha,\ \dot{a}_n(0)=\beta, \end{aligned} \tag{26}$$

where $a_n(t)$ and $\dot{a}_n(t)$ must be continuous in t in order that $y=a_n(t)\sin nx$ and $\partial y/\partial t=\dot{a}_n(t)\sin ux$ are continuous in $\tilde{W}_2^1$ and L^2, respectively.

Now we suppose that

$$b(t)=\begin{cases} b_0, & t_{2i}<t<t_{2i+1}\\ b_1, & t_{2i+1}<t<t_{2i+2} \end{cases}\quad (i=0,1,2,\cdots\cdots)$$

where $t_0=0$, b_0, b_1 are constants, and $b_0>b_1>0$ and also t_i are determined below appropriately. In the interval $0\le t\le t_1$, the equation

$$\ddot{a}_n(t)+n^2b_0a_n(t)=0$$

gives

$$n^2\dot{a}_n^2(t)+n^4b_0a_n^2(t)=n^2\dot{a}_n^2(0)+n^4b_0a_n^2(0)\equiv e_1^2.$$

We start with $\dot{a}_n(0)=0$ and $a_n^2(0)=e_1^2/n^4b_0$.

If we integrate the above first order differential equation with respect to t, we obtain

$$t=\mp\int_{e_1/n^2\sqrt{b_0}}^{a} da_n/\sqrt{e_1^2/n^2-n^2b_0a_n^2}.$$

Here we define t_1 as follows:

$$t_1=-\int_{e_1/n^2\sqrt{b_0}}^{0} da_n/\sqrt{e_1^2/n^2-n^2b_0a_n^2}=\frac{\pi}{2}\frac{1}{\sqrt{b_0}n}$$

Then $\dot{a}_n^2(t_1)=e_1^2/n^2$ and $a_n^2(t_1)=0$. In the interval $t_1<t<t_2$ the equation is $\ddot{a}_n(t)+n^2b_1a_n(t)=0$, and so the energy equality is

$$\dot{a}_n^2(t)+n^2b_1a_n^2(t)=e_1^2/n^2.$$

Here we define t_2 as follows:

$$t_2-t_1=-\int_{0}^{-e_1/n^2\sqrt{b_1}} da_n/\sqrt{e_1^2/n^2-n^2b_1a_n^2}=\frac{\pi}{2}\frac{1}{\sqrt{b_1}n},$$

where $\dot{a}_n^2(t_2)=0$, $a_n^2(t_2)=e_1^2/n^4b_1$. The time $t_i(i=3,4,\cdots\cdots)$ is determined in the same way

$$t_{2i+1}-t_{2i}=\frac{\pi}{2}\frac{1}{\sqrt{b_0}n},$$

$$t_{2i+2}-t_{2i+1}=\frac{\pi}{2}\frac{1}{\sqrt{b_1}n},$$

and there we obtain the following:

$$\dot{a}_n^2(t_{2i})=0,\quad a_n^2(t_{2i})=\frac{e_1^2}{n^4b_0}\left(\frac{b_0}{b_1}\right)^i,$$

$$\dot{a}_n^2(t_{2i+1})=\frac{e_1^2}{n^2}\left(\frac{b_0}{b_1}\right)^i,\quad a_n^2(t_{2i+1})=0,$$

$$(i=0,1,2,\cdots\cdots)$$

Therefore we arrive at, for $i=1,2,\cdots\cdots$

$$t_{2i}=\frac{\pi}{2}\left(\frac{1}{\sqrt{b_0}}+\frac{1}{\sqrt{b_1}}\right)\frac{i}{n}$$

$$e^2(t_{2i})\equiv\dot{a}_n^2(t_{2i})+n^2b_0a_n^2(t_{2i})=\frac{e_1^2}{n^2}\left(\frac{b_0}{b_1}\right)^i.$$

When $C>0$, $T>0$, $e_1>0$ are given arbitrarily, we may chose the least integer $i\geq i_0$ such that

$$\frac{e_1^2}{n^2}\left(\frac{b_0}{b_1}\right)^{i_0}=C,\quad \text{i.e.,}$$

$$i_0=(\log C+2\log n-2\log e_1)/\log(b_0/b_1).$$

Then for $[i_0]+1\geq i\geq[i_0]$ ($[\cdot]$ is the Gauss notation),

$$t_{2i}=\frac{\pi}{2}\left(\frac{1}{\sqrt{b_0}}+\frac{1}{\sqrt{b_1}}\right)\frac{i}{n}\leq\frac{\pi}{2}\left(\frac{1}{\sqrt{b_0}}+\frac{1}{\sqrt{b_1}}\right)\left(\frac{\log C+2\log n-2\log e_1}{\log b_0/b_1}+1\right)\frac{1}{n}.$$

Therefore, we choose n sufficiently large such that $t_{2i}<T$. Thus we have constructed the function $b(t)$ and the initial values y_0 and y_1, for which the energy inequality (24) dose not hold.

Next we consider the case that $b(t)$ is a bounded function of bounded total variation in $[0, {}^{r}T]$. It is sufficient that we consider only the n-harmonic i.e., the ordinary differential equation (26), because superposition is possible.

Let $b^k(t)$ $(k=1, 2, \cdots\cdots)$ be a step function in $[0, T]$ such that

$$b^k(t)=b\left(\frac{T}{k}i\right) \quad \text{for} \quad \frac{T}{k}i\leq t<\frac{T}{k}(i+1)$$
$$(i=0, 1, 2, \cdots\cdots, k-1).$$

Then $b^k(t)$ converges to $b(t)$ at continuous points of $b(t)$ as $k\rightarrow\infty$ and $\underset{0\leq\tau\leq t}{\text{tot. var.}}\, b^k(\tau)\leq\underset{0\leq\tau\leq t}{\text{tot. var.}}\, b(t)$. Multiplying by $\dot{a}_n(t)$ the differential equation

$$\ddot{a}_n(t)+n^2b^k(t)a_n(t)=0$$

and integrating the result in $t\in[0, t]$, we arrive at

$$\begin{aligned}e^2{}_{(k)}(t)&\equiv\dot{a}_n(t)+n^2b^k(t)a_n{}^2(t)\\&\leq e^2{}_{(k)}(0)\exp\left\{C\sum_{\frac{T}{k}i<t}\left|b^k\left(\frac{T}{k}i\right)-b^k\left(\frac{T}{k}i-0\right)\right|\right\}\\&=e^2{}_{(k)}(0)\exp\{C\underset{0\leq\tau\leq t}{\text{tot. var.}}\, b^k(\tau)\}\\&\leq e^2{}_{(k)}(0)\exp\{C\underset{0\leq\tau\leq t}{\text{tot. var.}}\, b(\tau)\},\end{aligned}$$

where C is dependent only on $\sup b(t)$ and $\inf b(t)$. Thus we get the following *a priori* estimate for the case of $b(t)$:

$$e_n{}^2(t)\equiv\dot{a}_n{}^2(t)+n^2b(t)a_n{}^2(t)\leq e_n{}^2(0)e^{C\underset{0\leq\tau\leq t}{\text{tot. var.}}\, b(\tau)}$$

at the continuous points of $b(t)$.

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