The Spread of Yield Zone from a Penny-Shaped Crack

By

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On the basis of the theory of continuous distribution of dislocations, the extent of spread of plasticity from the rim of a penny-shaped crack and the relative displacement of the two faces of the crack are determined and the analysis is reduced to a mixed boundary value problem in the potential theory.

Solutions are obtained for a penny-shaped crack in an infinite elastic-plastic medium under the conditions of uniform tension perpendicular to the plane of the crack and uniform shear stress parallel to it. The change of energy of the system accompanying the infinitesimal crack extension is calculated and discussed. It is shown that if the cohesive force is taken into account we are able to get an unstable crack condition on the energy balance consideration.

1. Introduction

In recent years, in order to make clear the mechanism of semi-brittle fracture, the problem determining distribution of stresses and displacements around a crack has attracted considerable attention. Especially the theory of continuous distributions of dislocations has been regarded to be useful in analysing a crack in an elastic-plastic solid. Bilby, Cottrell and Swinden[1] (referred to B. C. S. as B. C. S. theory) obtained, regarding the two dimensional crack and thin plastic region that spreads from its tips as a continuous distribution of straight dislocations, the solution of this problem by solving the equilibrium equation of the distribution of the dislocations. This method of replacing a crack by a continuous distribution of dislocations is applied to a variety of other similar problems. For example, Bilby, Corrtell, Smith and Swinden[2] discussed a periodic sequence of collinear cracks. Smith treated two collinear relaxed cracks[3] and two coplanar wedge shaped cracks.[4] The interaction of slip dislocations, slip bands, and cracks was studied by Yokobori and Yoshida.[5] However their investigations are confined to two-dimensional problems.

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In this paper, we develop the B. C. S. theory for a three-dimensional case and study a penny-shaped crack in an elastic-plastic solid, using a model of continuous distribution of dislocation loops. In section 2 an intergral equation with respect to the distribution function of dislocations is derived. In section 3, referring to the analogy to the equation determining the electric charge distribution on the circular disk, the integral equation is transformed to dual integral equations and solved. The extent of spread of plastic region is investigated and the displacement in the plastic region and on the crack surface is obtained under the condition of uniform tension parallel to the axis of the crack (section 3) and of uniform shear stress parallel to the plane of the crack (section 4). Finally the total energy change of the system accompanying the infinitesimal crack extension is calculated and discussed. It is shown that by taking a cohesive force into account we are able to derive a fracture condition based on an energy balance consideration.

2. Formulation of Problem

An infinite isotropic elastic medium subjected to uniformly applied tensile stress at infinity is supposed to contain a crack ($r \leq c$) accompanying thin plastic region ($c<r<a$) emitted from its rim, where r is the radius of the polar coordinate system r, θ, z and a and c are respectively outer and inner radius of the plastic region. Displacement field may be assumed to be axially symmetric and also be symmetric with a crack plane ($z=0$). So only normal displacement and normal stress component may exist on the crack plane.

In order to solve the equilibrium problem of the crack strictly, the boundary condition must be given on a surface which is also determined by the solution of the probelm. But when the crack is assumed to be very thin the boundary condition may be given on the plane $z=0$. That is, we can regard a penny-shaped crack as a circular cut on whose surface the normal displacement undergoes a jump $u_z = \pm \frac{1}{2} u(r)$, where $u(r)$ means the magnitude of the jump and is regarded to be a function of r. On the other hand, a dislocation loop is defined as an edge of a cut on whose surface displacement vector undergoes a jump $\boldsymbol{b}$ (Burgers vector). Taking into account the analogy between two circular cuts above mentioned, the crack can be replaced by a coaxial distribution of circular dislocation loops whose Burgers vector $\boldsymbol{b}$ is perpendicular to the cut surface, i.e. prismatic dislocation loops. While the plastic displacement has only z-direction component, the plastic region can also be replaced by a distribution of prismatic dislocation loops.

The displacement jump $u(r)$ normal to the plane $z=0$ may be assumed to be the sum of Burgers vector $\boldsymbol{b}$ of dislocations outside the circle of radius r. Now we define the dislocation density $D(r)$ so that the number of dislocations in the interval r and $r+dr$ is $D(r)dr$. Then we have

$$u(r) = b \int_r^a D(r)\, dr\,, \tag{1}$$

where b is the absolute value of Burgers vector $\boldsymbol{b}$. In the region $r<c$, $u(r)$ denotes the shape of the crack, and in the region $c<r<a$, $u(r)$ denotes the plastic displacement. Thus the crack problem in an elastic-plastic material may be reduced to the problem of equilibrium distribution of prismatic dislocation density $D(r)$.

There are two kinds of prismatic dislocations, one is formed by the aggregation of interstitial atoms in a circular sheet, and another is formed by the condensation of vacancies and the surface collapse of flat cavity thus produced. Directions of Burgers vectors are opposite to each other, so the stress and deformation fields have opposite signs. We refer to the above dislocations as a positive and a negative prismatic dislocation respectively. For the purpose of expressing the crack under tension perpendicular to the crack plane, the positive dislocations are suitable for our present problem. The situation is shown in Fig. la.

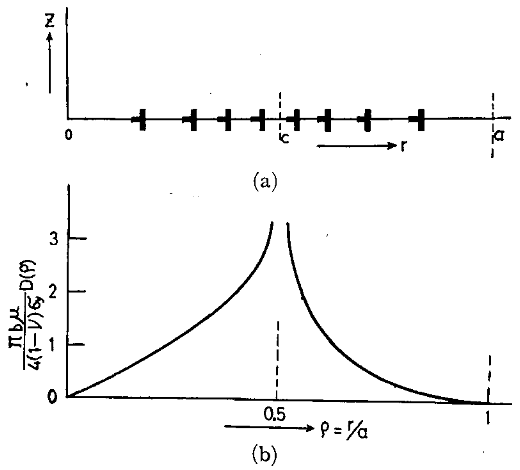

Fig. 1. Distribution of prismatic dislocations in a penny-shaped crack and its associated plastic zone.

In an infinite isotropic elastic medium, the stress field produced by a single negative prismatic dislocation loop was given by Kroupa[6]. The stress field by a positive dislocation loop may be obtained simply by changing the sign of the relevant expressions given by Kroupa. A positive dislocation with its radius R placed on a plane $z=0$ produces only a normal stress component σ_z on $z=0$ and we have

$$\sigma_z(\rho,0) = -\frac{b\mu}{\pi R(1-\nu)} \frac{1}{1-\rho^2} E(\rho) \quad 0<\rho<1\,,$$
$$\sigma_z(\rho,0) = -\frac{b\mu}{\pi R(1-\nu)} \frac{1}{\rho} \left[K\left(\frac{1}{\rho}\right) - \frac{\rho^2}{\rho^2-1} E\left(\frac{1}{\rho}\right)\right] \rho>1\,, \tag{2}$$
$$\rho = r/R\,,$$

where μ and ν are respectively the shear modulus and Poisson's ratio and E and K are complete elliptic integrals in the usual notations.
Making use of the formulae

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\,K(k)\,,$$
$$E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}\,[2E(k) - (1-k^2)\,K(k)] \tag{3}$$

and

$$\frac{\partial K(\rho)}{\partial\rho} = \frac{E(\rho)}{\rho(1-\rho^2)} - \frac{K(\rho)}{\rho} \tag{4}$$

and writing simply as

$$J(r,R) = \frac{4}{r+R} K\left(\frac{2\sqrt{rR}}{r+R}\right), \tag{5}$$

we have

$$\sigma_z(r) = -\frac{b\mu}{4\pi(1-\nu)} \frac{\partial}{\partial r}(rJ(r,R)) \quad 0<r<\infty\,. \tag{6}$$

The normal stress σ_z^{cr} (superscript cr denotes 'crack') acting on a circular region of $r\leq a$ is then given as

$$\sigma_z^{cr}(r) = -\frac{b\mu}{4\pi(1-\nu)} \int_o^a \frac{\partial}{\partial r}(rJ(r,R))\,D(R)\,dR\,. \tag{7}$$

The dislocation with radius r is exerted by the force $b\sigma_z^{cr}(r)$ per unit length which is directed to the radial direction (Peach and Koehler[7)]). In a crack ($0\leq r\leq c$) this force must be balanced with the force bp which is caused by the applied stress p, i. e.,

$$\sigma_z^{cr}(r) + p = 0 \quad 0\leq r\leq c\,, \tag{8}$$

while in the plastic region, a resistance force σ_y is exerted on dislocations. And we may assume that this force is equal to the yield stress of the material considered

(B. C. S.[1]). Hence,

$$\sigma_z^{cr}(r) + p = \sigma_y \quad c < r \leq a\,. \tag{9}$$

Thus we have

$$\frac{b\mu}{4\pi(1-\nu)} \int_o^a \frac{\partial}{\partial r}\left(rJ(r, R)\right) D(R)\, dR = \begin{cases} p & 0 \leq r \leq c \\ p-\sigma_y & c < r \leq a\,. \end{cases} \tag{10}$$

Integrating both sides with respect to r we have

$$\frac{b\mu}{4\pi(1-\nu)} \int_o^a J(r, R)\, D(R)\, dR = \begin{cases} p + \dfrac{c_o}{r} & 0 \leq r \leq c \\ (p - \sigma_y) + \dfrac{c\,\sigma_y}{r} + \dfrac{c_o}{r} & c < r < a\,, \end{cases} \tag{11}$$

where c_o is an integral constant and must vanish by no permission of singularity at $r=0$.

Hence we have

$$\int_o^a J(r, R)\, D(R)\, dR = f_1(r)\,, \tag{12}$$

where

$$f_1(r) = \begin{cases} A_o\, p & 0 \leq r \leq c \\ A_o(p - \sigma_y) + A_o\, c\, \sigma_y/r & c \leq r \leq a \end{cases} \tag{13}$$

and

$$A_o = 4\pi\,(1 - \nu)/b\mu\,.$$

Referring to the expression

$$J(r, R) = \frac{4}{r+R}\, K\left(\frac{2\sqrt{rR}}{r+R}\right) = \int_o^{2\pi} \frac{d\varphi}{(r^2+R^2-2rR\cos\varphi)^{1/2}}\,, \tag{14}$$

we would like to remark that equation (12) is analogous to the equation of electro-static problem. That is, if we denote $d(R)$ the axially symmetric surface charge density distributed on a circular disk with radius a, its electro-static potential is given by

$$\int_o^a d(R)\, R\, J(r, R)\, dR = f_1(r)\,. \tag{15}$$

Hence, the integral equation (12) may be regarded to be equivalent to the equation of the charge density whose potential on the disk is given by the prescribed value $f_1(r)$. The comparison (15) with equation (12) yields relation

$$D(R) = R\,d(R)\,. \tag{16}$$

3. Reduction to Dual Integral Equations

By means of the method adopted by Sneddon[8], equation (12) can be reduced to dual integral equations. The electric potential $V(r, z)$ due to a charged disk with radius a and lying on a plane $z=0$ is subjected to the boundary conditions in half-space

$$V(r, 0) = f_1(r) \qquad 0 \leq r \leq a \tag{17}$$

$$\frac{\partial V(r, 0)}{\partial z} = 0 \qquad r > a \tag{18}$$

on $z=0$ and

$$V = 0 \tag{19}$$

when r^2+z^2 tends to infinity in the absence of an external field.

The axially symmetric electric potential $V(r, z)$ must satisfy Laplace's equation and then has a general solution

$$V(r, z) = \text{const} \times J_o(\xi r)\, e^{-\xi z}\,, \tag{20}$$

which satisfies condition (19) when z tends to positive infinity, where ξ is a positive parameter. As a required solution, we may take a superposition of the general solution

$$V(r, z) = \int_0^\infty G(\xi)\, J_o(r)\, e^{-\xi z}\, d\xi\,, \tag{21}$$

where $G(\xi)$ is a function of ξ and must satisfy the boundary conditions (17) and (18),

$$\begin{aligned} &\int_0^\infty G(\xi)\, J_o(\xi r)\, d\xi = f_1(r) \qquad 0 \leq r \leq a\,, \\ &\int_0^\infty \xi G(\xi)\, J_o(\xi r)\, d\xi = 0 \qquad r > a\,. \end{aligned} \tag{22}$$

By means of $G(\xi)$ the charge density $d(r)$ is given by

$$d(r) = \frac{1}{2\pi} \int_0^\infty \xi\, G(\xi)\, J_o(\xi r)\, d\xi\,. \tag{23}$$

Introducing new variables

$$\rho = r/a\,, \qquad k = a\xi \tag{24}$$

and putting

$$G(\xi)/a = A(k) \qquad f(\rho)/A_o = \begin{cases} p & 0 \le \rho \le \chi \\ (p-\sigma_y) + \chi\sigma_y/\rho & \chi \le \rho \le 1\,, \end{cases} \tag{25}$$

$$\chi = c/a\,,$$

the integral equation (12) is finally reduced to the dual integral equations

$$\int_0^\infty A(k)\, J_o(\rho k)\, dk = f(\rho) \qquad 0 \le \rho \le 1\,,$$

$$\int_0^\infty kA(k)\, J_o(\rho k)\, dk = 0 \qquad \rho > 1\,. \tag{26}$$

Using the formal solution given by Titchmarsh[9)]

$$A(k) = \frac{2}{\pi}\left\{\cos k \int_0^1 \frac{uf(u)}{\sqrt{1-u^2}}\,du + k\int_0^1 \sin kt\, dt \int_0^t \frac{uf(u)}{\sqrt{t^2-u^2}}\,du\right\}, \tag{27}$$

we immediately have

$$\frac{\pi}{2A_o} A(k) = (p - \sigma_y \sqrt{1-\chi^2}\,)\,\frac{\sin k}{k} + \frac{\sigma_y\,\chi^2}{k}\int_\chi^1 \frac{\sin kt}{t^2\sqrt{t^2-\chi^2}}\,dt\,. \tag{28}$$

Substituting equation (28) into equation (25) and referring equation (16) and (23), we have

$$\frac{D(\rho)}{4A_o} = \rho\,(p - \sigma_y\sqrt{1-\chi^2}\,)\int_0^\infty \sin k\, J_o(\rho k)\, dk$$

$$+ \rho\,\sigma_y\,\chi^2 \int_\chi^1 dt \int_0^\infty \frac{\sin kt\, J_o(\rho k)}{t^2\sqrt{t^2-\chi^2}}\,dk\,. \tag{29}$$

The relation between the applied stress p and the radius of the plastic zone χ can be determined by the condition that the dislocation density $D(\rho)$ must be zero at the rim of the plastic zone, i.e., at $\rho=1$. Referring to the formula

$$\int_0^\infty \sin kt\, J_o(\rho k)\, dk = \begin{cases} \dfrac{1}{\sqrt{t^2-\rho^2}} & 0<\rho<t \\ 0 & t<\rho\,, \end{cases} \tag{30}$$

we see that the first term of right hand side of equation (29) approaches infinite when $\rho \to 1$. On the other hand the second term becomes

$$\rho \sigma_y \varkappa^2 \int_\rho^1 \frac{dt}{t^2 \sqrt{t^2-\varkappa^2}\sqrt{t^2-\rho^2}}, \tag{31}$$

which tends to zero when ρ approaches 1 as we may readily prove by taking the limit. Thus the relation

$$p = \sigma_y \sqrt{1-\varkappa^2} \tag{32}$$

must hold and this relation specifies the applied stress p by a function of plastic zone radius $\varkappa$. Fig. 2 shows the relation (32).

This relation was first obtained by Olesiak and Wnuk.[10] They adopt the hypothesis of Dugdale[11] and the problem is reduced to the mixed boundary value problem in the classical theory of elasticity. But their analysis does not consider explicitly the equilibrium of dislocations.

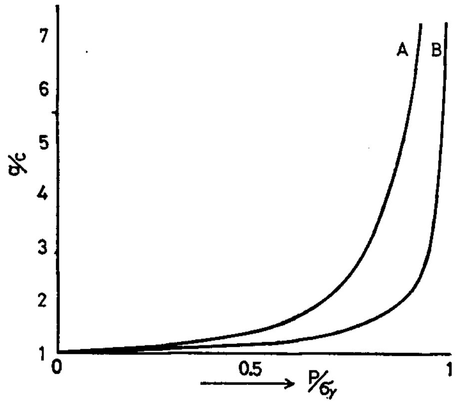

Fig. 2. Comparison of lengths of plastic zones as deduced from the B. C. S. theory (A) and from Eq. (32) (B).

The dislocation density is then expressed in the usual notation for elliptic integrals

$$D(\rho) = \frac{4(1-\nu)\sigma_y}{\pi b \mu} \begin{cases} \dfrac{\varkappa}{\rho}\{F(\mu_1, t_1) - E(\mu_1, t_1)\} + \dfrac{\rho}{\varkappa^2}\sqrt{\dfrac{1-\varkappa^2}{1-\rho^2}} & 0<\rho<\varkappa \\ \dfrac{1}{\rho}\{F(\mu_2, t_2) - E(\mu_2, t_2)\} + \dfrac{\varkappa^2}{\rho^2}\sqrt{\dfrac{1-\rho^2}{1-\varkappa^2}} & \varkappa<\rho<1, \end{cases} \tag{33}$$

where

$$\mu_1 = \text{arc sin} \sqrt{\frac{1-\varkappa^2}{1-\rho^2}} \qquad t_1 = \frac{\rho}{\varkappa}$$

$$\mu_2 = \text{arc sin} \sqrt{\frac{1-\rho^2}{1-\varkappa^2}} \qquad t_2 = \frac{\varkappa}{\rho} . \tag{34}$$

Fig. 1b shows the quantity $\frac{\pi\, b\, \mu}{4(1-\nu)\sigma_y} D(\rho)$ as a function of ρ for the case $\chi=1/2$. The form of this distribution function agrees with our qualitative expectations and is similar to the one by the B. C. S. theory.

Substituting (29) into equation (1) and interchanging the order of integration, we obtain the shape of the crack as

$$u_z = \frac{2(1-\nu)\sigma_y\, a}{\pi\, \mu} \begin{cases} -\rho^2 \sqrt{\frac{1-\varkappa^2}{1-\rho^2}} + \varkappa\, E\,(\mu_1, t_1) & 0<\rho<\varkappa \\ -\varkappa_2 \sqrt{\frac{1-\rho^2}{1-\varkappa^2}} + \frac{\varkappa^2-\rho^2}{\rho} F\,(\mu_2, t_2) + \rho E\,(\mu_2, t_2) & \varkappa<\rho<1 . \end{cases} \tag{35}$$

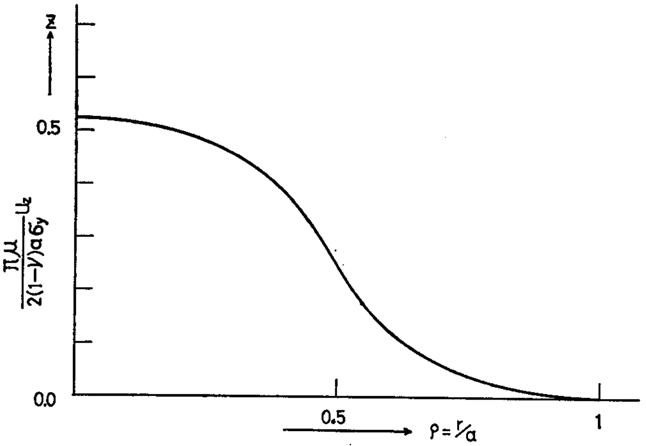

Fig. 3. Normal displacement of the crack surface and plastic zone of Fig. 1.

The crack opening criterion was discussed by B. C. S. in detail and their criterion is

$$\frac{\Phi\,(c)}{c} = \frac{4(1-\nu)\,\sigma_y}{\pi\, \mu}\, ln \left(\frac{1}{\chi}\right), \tag{36}$$

where $\Phi(c)=2u_z(c)$ denotes the crack opening displacement. On the other hand our theory gives

$$\frac{\Phi\,(c)}{c} = \frac{4(1-\nu)\,\sigma_y}{\pi\, \mu}\,(1-\varkappa) . \tag{37}$$

In the B. C. S. theory, $\Phi(c)/c$ becomes infinite as χ approaches zero (i.e. as the applied stress p approaches yield stress σ_y), but in the penny-shaped crack $\Phi(c)/c$ still remains finite value $4(1-\nu)\sigma_y/\pi\mu$.

In the limit of small applied stress, $p \ll \sigma_y$, from equation (32) we have

$$\chi \simeq 1 - \frac{1}{2}\left(\frac{p}{\sigma_y}\right)^2 . \tag{38}$$

Combining this with equation (37) we obtain the fracture stress as

$$\sigma = \sqrt{\frac{\pi}{2}\,\frac{\mu\gamma_c}{(1-\nu)c}}\,, \tag{39}$$

where $\gamma_c = \sigma_y\,\Phi(c)$ is the plastic work done per unit area of fracture.

Equation (39) is essentially expressed by the form

$$\sigma = \sqrt{\frac{E\gamma}{c}} \tag{40}$$

given by Orowan[12] and Irwin[13], where γ represents the work required to increase the area of the crack by unit amount and E is the Young's modulus.

Our result (39) differs from that of the B. C. S. theory only by the numerical coefficient. Their value is $2/\pi$ instead of ours $\pi/2$.

4. Penny-Shaped Crack Opened by Simple Shear

In this section, we consider the problem of a penny-shaped shear crack $(0<r\leq c)$ which is opened by uniform applied shear stress parallel to the crack plane and accompanies plastic region $(c\leq r\leq a)$. As the same considerations discussed in the previous sections, we may reduce the problem of the penny-shaped shear crack in an elastic-plastic solid to the equilibrium problem of circular dislocation loops whose Burgers vector are parallel to the crack plane.

In order to simplify the problem, we assume that Poisson's ratio ν equals zero. Then the stress state produced by a single dislocation loop depends on only the distance from the center of the loop. At a distance r from the center of a dislocation loop with radius R, shear stress is given by

$$\tau\,(r, R) = \frac{b\mu}{4\pi}\int_0^{2\pi}\frac{rR\cos\varphi - R^2}{(r^2+R^2-2rR\cos\varphi)^{3/2}}\,d\varphi \tag{41}$$

on the plane $z=0$ (Leibfried[14]).

With the notation of (14), equation (41) may be rewritten as

$$\tau(r, R) = -\frac{b\mu}{4\pi}\frac{\partial}{\partial r}(rJ(r, R))\,, \tag{42}$$

which was given by Leibfried[14] who considered the equilibrium distribution of circular dislocation loops piling up against a circular obstacle.

Equation (42) coincides with equation (12) by the restriction $\nu=0$ and then again we can have the same dual integral equations (26) when we assume

$$A_0 = \frac{4\pi}{b\mu} \tag{43}$$

Thus, in the same manner as the previous section, we have the relation between the applied shear stress τ and the radius of the plastic region $\varkappa=c/a$ as

$$\tau = \tau_y\sqrt{1-\varkappa^2}\,, \tag{44}$$

where τ_y denotes the shear yield stress.

The dislocation density $D(\rho)$ is given by substituting $4\tau_y/\pi b\mu$ for the numerical coefficient of equation(33). The relative displacement of the faces of the crack is given by substituting $4\tau_y\, a/\pi\mu$ for the numerical coefficient of equation (35).

When ν is not zero, the region of the dislocation distribution becomes elliptic, but its deviation from circle may be small.

Up to the present we have regarded the dislocation models of cracks as mathematical models; the dislocation models of cracks were constructed from the analogy between the definition of crack plane and the definition of dislocation. However, with respect to the dislocation model of the shear crack we may construct the model from a physical view point.

Suppose that in a grain a Frank-Read source began to operate by certain shear stress. Dislocation loops emitted from the source expand on a slip plane and pile up against a grain boundary. As the applied shear stress is increased, the increased number of dislocations pile up. We assume that when the number of dislocations becomes a certain value, i.e., when the amount of slip in this grain become a critical value, then a shear crack is nucleated over entire slip plane in the grain.

When normal stress also acts on a crack surface, the crack becomes a cleavage crack. We may consider that the initial microcrack with grain diameter in length is initiated by this mechanism. When the crack is nucleated, dislocation loops are emitted into the neighbouring grains from the rim of the crack thus produced.

Thus the above discussion leads to the dual integral equation (26) with $\nu=0$ and A_0 is given by equation (43). In the case that the applied stress is small and so plastic zone is small compared with grain diameter, we must substitute τ_y with friction stress τ_i of the crystal. In the case that the applied stress is close to the yield stress of the material, the plastic region becomes 'far reaching' and equation (44) will remain approximately valid. In both cases crack diameter $2c$ denotes grain diameter.

The idea that a crack is opened on a slip plane was first proposed by Gilman[15] and Bullough (quoted by Gilman). Their model assumes that a crack is initiated on a part of a slip plane when many edge dislocations are blocked by an obstacle at one side of the slip plane. By assuming that a crack is opened on an entire slip plane bounded by a grain boundary in the three dimensional case, we can illustrate how the real initial microcracks are initiated.

5. Energy Considerations

In the consideration of the extension of a crack, the energy change associated with the deformation of the crack shape plays an important role. Here the energy deviation of a penny-shaped crack in simple tension is calculated. If we do not assume (32), the displacement of the crack surface is given by

$$u_z(r) = \frac{2(1-\nu)\,a\sigma_y}{\pi\,\mu}\begin{cases} \dfrac{p}{\sigma_y}\sqrt{1-\rho^2} - \sqrt{\dfrac{1-\varkappa^2}{1-\rho^2}} + \varkappa E(\mu_1, t_1) & 0<\rho<\varkappa \\ \sqrt{1-\rho^2}\left[\dfrac{p}{\sigma_y} - \dfrac{1}{\sqrt{1-\varkappa^2}}\right] + \dfrac{\varkappa^2-\rho^2}{\rho^2}F(\mu_2, t_2) & \\ \qquad + \rho E(\mu_2, t_2) & \varkappa<\rho<1 \, . \end{cases} \tag{45}$$

Now let δW_e and δL be respectively the deviations of the strain energy and potential energy of the external load according to the deviations δc and δa of c and a, and let δW_p be the work dissipated by the plastic deformation by the crack extension. We can calculate easily

$$\delta W_e = 2\frac{d}{dc}\left[\frac{p}{2}\int_o^a 2\pi\, r\, u_z(r)\, dr - \frac{\sigma_y}{2}\int_c^a 2\pi\, r\, u_z(r)\, dr\right]\delta c\,, \tag{46}$$

$$\delta L = -4\pi p\int_o^a\left(r\frac{du_z}{dc}\delta c\right)dr \tag{47}$$

(For equation (47), refer to Yokobori, Kamei and Ichikawa[16]). According to Goodier and Field[17], δW_p is given by

$$\delta W_p = 4\pi\sigma_y\int_c^a\left(r\frac{du_z}{dc}\delta c\right)dr\,,$$

i.e.,

$$\delta W_p = 4\,\pi\,\sigma_y \left[\frac{d}{dc} \int_c^a r\,u_z(r)\,dr + c\,u_z(c) \right] \delta c \tag{48}$$

By means of the formulae:

$$\begin{aligned} &\int_0^{\chi} \rho E(\mu_1, t_1)\,d\rho = \frac{\sqrt{1-\chi^2}}{3\chi}\,(\chi^2+2) - \frac{2}{3\chi}\,(1-\chi^3) \\ &\int_{\chi}^{1} E(\mu_2, t_2)\,d\rho = \frac{2}{9}\,(1-\chi^3) \\ &\int_{\chi}^{1} \rho^2 F(\mu_2, t_2)\,d\rho = \frac{2}{9}\,(1-\chi^3) + \frac{1}{3}\,\chi^2\,(1-\chi^2) \\ &\int_{\chi}^{1} F(\mu_2, t_2)\,d\rho = 1 - \chi\,, \end{aligned} \tag{49}$$

we can obtain the results:

$$\begin{aligned} &\int_0^c r\,u_z(r)\,dr = \frac{A_1}{3} \left[\frac{p}{\sigma_y} \{1-(1-\chi^2)^{3/2}\} - (1-\chi^2)^{3/2} + 2\chi^3 - 3\chi^2 + 1 \right] \\ &\int_c^a r\,u_z(r)\,dr = \frac{A_1}{3} \left[(1-\chi^2)^{3/2} \left(\frac{\sigma_y}{p} - \frac{1}{\sqrt{1-\chi^2}} \right) + 2\chi^2\,(1-\chi) \right], \end{aligned} \tag{50}$$

where

$$A_1 = \frac{2(1-\nu)\,\sigma_y\,a^3}{\pi\,\mu}\,.$$

If the load is assumed to maintain a constant value during the crack extension, we have

$$\frac{\partial a}{\partial c} = \frac{a}{c}\,. \tag{51}$$

Substituting equations (50) into (46) (47) and (48), and referring (51), we then have

$$\delta W_e = 2\pi\,A_1 \frac{\sigma_y}{\chi^3} \left\{ \left(\frac{p}{\sigma_y} \right)^2 - 2\,(1-\chi^2)^{3/2} \left(\frac{p}{\sigma_y} \right) + 2\chi^3 - 3\chi^2 + 1 \right\} \delta c\,, \tag{52}$$

$$\delta L = -\,4\pi\,A_1 \left(\frac{p}{\sigma_y} \right) \frac{\sigma_y}{\chi^3} \left\{ \frac{p}{\sigma_y} - (1-\chi^2)^{3/2} \right\} \delta c \tag{53}$$

and

$$\delta W_p = 4\pi\,A_1 \frac{\sigma_y}{\chi^3} \left\{ \sqrt{1-\chi^2}\,\frac{p}{\sigma_y} + 2\chi^2 - \chi^3 - 1 \right\} \delta c\,. \tag{54}$$

The total energy change is given as the sum of (50) (51) and (52) such that

$$\delta E = -2\pi A_1 \frac{\sigma_y}{\chi^3} \left(\frac{p}{\sigma_y} - \sqrt{1-\chi^2} \right)^2 \delta c\,. \tag{55}$$

Thus, if relation (32) holds, no total change of energy occurs, then the crack is in equilibrium at any applied stress. If relation (32) does not hold, the energy deviation is negative, then the crack must be unstable. It is found that relation (32), which is determined uniquely, is also the condition for the equilibrium crack. And under such a condition there is no critical fracture stress where the crack becomes unstable. On the other hand, if the thickness of the layer of the plastic zone is sufficiently small in comparison with the crack length, according to the modified Griffith theory by Orowan[12] and Irwin[13], a crack becomes unstable when the applied stress exceeds a critical value, so that our result is in remarkable contrast to the one by Orowan and Irwin. The inconsistency between our result and the one by Orowan and Irwin may be caused by the neglected cohesion force in our model. If we take the force of cohesion into account which is effective over a narrow annular region of the crack rim, then the total energy change may be modified by

$$\delta E = \left\{ -2\pi A_1 \frac{\sigma_y}{\chi^3} \left(\frac{p}{\sigma_y} - \sqrt{1-\chi^2} \right)^2 + 4\pi c \gamma \right\} \delta c\,, \tag{56}$$

where γ is the surface energy of the material i.e. the work per unit area done by the cohesive force during the crack extension. Again, the crack may be in the equilibrium state, $\delta E=0$, under a modified relation (32) by the perturbation of the cohesive force. When the applied stress p is increased, the total change of energy associated with an infinitesimal crack extension excluding contributions from the force of cohesion decreases and this decrease of the energy must be compensated by the increase of the surface energy, so that $\delta E=0$ holds necessarily. But γ has a maximum value γ_c, which corresponds to the maximum force of cohesion. If p is increased beyond the point where γ reaches γ_c, δE becomes negative and unstable crack propagation occurrs. Hence the critical applied stress p_c is given by

$$\frac{p_c}{\sigma_y} = \sqrt{1-\chi^2} + \frac{\chi^{3/2}}{\sigma_y} \sqrt{\frac{\pi \mu \gamma_c}{(1-\nu)\, c}}\,. \tag{57}$$

when χ in (57) may be determined as a function of p/σ_y, the fracture stress may be determined.(*)

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* The discussion which takes the cohesive forces into account in the B. C. S. model was at first assumed by Yokobori, Kamei and Ichikawa[16]. They adopt the macroscopic theory of plasticity of Hult and McClintock[18] as the relation between χ and the applied stress p and derive the fracture stress. But in their method the stress becomes infinite at both ends of the plastic zones so the nature of afully relaxed crack in the B. C. S. model is lost.

Taking the cohesive force into account, we are thus able to avoid the inconsistency with the Orowan and Irwin's theory. (The equation (40) is in agreement with the result by Orowan and Irwin, but in view of energy balance it is entirely different, as has been discussed.) A more detailed treatment taking the cohesive force into consideration will appear in the following paper.

6. Concluding Remarks

An alternative approach to the elastic-plastic crack problems are proposed, which is an extension of the B. C. S. theory to the three dimensional case. A penny-shaped crack associated with thin plastic zone around its rim is represented by a coaxial continuous distribution of circular dislocation loops. The equilibrium equation of dislocation loops is reduced to dual integral equations and solved. Solutions are obtained for a penny-shaped crack under the conditions of uniform tension perpendicular to the plane of the crack and uniform shear stress parallel to it. The energy change of the system associated with an infinitesimal extension of the crack is calculated and discussed. It is shown that if the cohesive force is taken into account we are able to get unstable crack condition on the energy balance consideration.

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