On the Probability Distribution of the Number of Crossings of a Certain Response Level in Random Vibration

By

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Synopsis

Presented are the results of analysis of a kind of threshold-value crossing problem in random vibration. Probability of crossings of a certain response level by a given number of times is formulated in terms of the "n-th passage time density", from which approximate solutions are derived under the renewal process approximation. Discussions are made in comparison with the simple Poisson process approximation.

1. Introduction

In the analysis of structural systems subjected to random loading, it is important to obtain the probability that the dynamic response of a critical member passes out of a limited domian of safe service. If the structural failure is to take place at the time when the response first exceeds this safety limit, then the problem is discussed in terms of the probability distribution of the maximum response[1),2)] or the first-passage time density[3)~5)]. However, depending on the properties of the structural materials and composition of members, the failure of a structure due to dynamic loads is not uniquely determined by the maximum response, but it would also be affected by the accumulation of damage in the repetition of stress reversals.

The present study deals with this problem in terms of the probability that a certain assigned deformation or stress level is exceeded by the response by a given number of times, from the results of which some basic properties of such phenomena are surveyed. For this purpose, the concept of the first-passage time density is generalized to the density of the n-th passage time, with the aid of which discussed are the probability distribution of the number of crossings of an assigned response level.

In solving the problem, the correlation effect between responses at different times are considered by means of a renewal process approximation to be discussed in comparison with the simple Poisson process approximation.

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2. Basic Analysis

(1) Formulation of the Problem

Let $y(t)$ be an arbitrary continuous random process, or the structural response in discussion. The analysis in this study shall be made on the number of crossings of the response level $|y(t)|=Y$ in its duration τ.

First we consider the state of response illustrated schematically in Fig. 1 in which $|y(t)|$ exceeds the level Y by $(n-1)$ times in the interval $(0,t)$ and the n-th upward crossing of the same level occurs at $t=t$, and we shall represent the state of Fig. 1 by the notation $R_p(n,Y,t)$. For such a problem, we define the probability densities $p_n(Y,t)$ and $\tilde{p}_n(Y,t)$ of the time of the n-th upward crossing of the response level $|y(t)|=Y$ conditional on the initial states, respectively, that $|y(0)|\leqq Y$ and that $|y(0)|>Y$:

$$\left.\begin{aligned} p_n(Y,t)\,dt &= \mathrm{P}[R_p(n,Y,t)\,\big|\,|y(0)|\leqq Y] \\ \tilde{p}_n(Y,t)\,dt &= \mathrm{P}[R_p(n,Y,t)\,\big|\,|y(0)|>Y] \end{aligned}\right\} \quad \cdots\cdots\cdots(1)$$

The density functions $p_n(Y,t)$ and $\tilde{p}_n(Y,t)$ are considered to be a generalization of the first-passage time density. Analogous to this terminology, $p_n(Y,t)$ and $\tilde{p}_n(Y,t)$ shall be referred to as the n-th passage time densities in this paper. It is obvious that the first passage time density coincides with the specific case where $n=1$.

On integration of $p_n(Y,t)$ and $\tilde{p}_n(Y,t)$ with respect to t in the interval $(0,\tau)$, we obtain the probabilities that there occur n- or more upward crossings of the response level $|y(t)|=Y$ in the duration τ conditional on the above-mentioned initial conditions. Hence the probability $\Phi_n(Y,\tau)$ that just n upward crossings occur in the duration τ is given by

$$\Phi_n(Y,\tau)=\begin{cases} a_0(Y)\left\{1-\int_0^\tau p_1(Y,t)\,dt\right\} \;;\; n=0 \\ a_0(Y)\left\{\int_0^\tau p_1(Y,t)\,dt-\int_0^\tau p_2(Y,t)\,dt\right\} \\ \quad +\tilde{a}_0(Y)\left\{1-\int_0^\tau \tilde{p}_1(Y,t)\,dt\right\} \;;\; n=1 \\ a_0(Y)\left\{\int_0^\tau p_n(Y,t)\,dt-\int_0^\tau p_{n+1}(Y,t)\,dt\right\} \\ \quad +\tilde{a}_0(Y)\left\{\int_0^\tau \tilde{p}_{n-1}(Y,t)\,dt-\int_0^\tau \tilde{p}_n(Y,t)\,dt\right\} \;;\; n\geqq 2 \end{cases} \quad \cdots\cdots\cdots(2)$$

where

$$a_0(Y) = \mathrm{P}\,[|y(0)| \leqq Y],\quad \tilde{a}_0(Y) = 1 - a_0(Y) = \mathrm{P}\,[|y(0)| > Y]$$

In deriving Eqs. (2), the event $|y(0)| > Y$ has been treated as the first upward crossing.

From Eqs. (2), the probablity $\Phi_P(n;\, Y, \tau)$ that the number of the crossings will not exceed n, or the probability distribution of the number of crossings of the response level Y, is written as

$$\Phi_P(n;\, Y, \tau) = \sum_{\nu=0}^{n} \Phi_\nu(Y, \tau)$$
$$= \begin{cases} a_0(Y)\left\{1 - \int_0^\tau p_1(Y, t)\, dt\right\} \;;\; n = 0 \\ 1 - \left\{a_0(Y)\int_0^\tau p_{n+1}(Y, t)\, dt + \tilde{a}_0(Y)\int_0^\tau \tilde{p}_n(Y, t)\, dt\right\} \;;\; n \geqq 1 \end{cases} \qquad \cdots\cdots\cdots(3)$$

(2) Representation of the n-th Passage Time Densities $p_n(Y,t)$ and $\tilde{p}_n(Y,t)$

The explicit representation of the n-th passage time densities can be obtained in a manner analogous to that used by M. S. Bartlett[6] for the first-recurrence time density. Divide the time interval $(0,t)$ into m small intervals $\Delta_1, \Delta_2, \ldots, \Delta_m$ of equal length $\Delta t = t/m$; i.e., $\Delta_i \equiv (t_{i-1}, t_i)$, $t_i = i\Delta t$; $i = 1, 2, \ldots, m$. And let e_i and $\bar{e}_i$ represent the following events:

$e_i \equiv$ there is an upward crossing of the response level $|y(t)| = Y$ in Δ_i

$\bar{e}_i \equiv$ there is no upward crossing of the response level $|y(t)| = Y$ in Δ_i .

If we write $e_i \cup \bar{e}_i \equiv 1$, then the usual algebraic laws are applicable in calculating the probability of compound events of the e_i's and $\bar{e}_i$'s. For example,

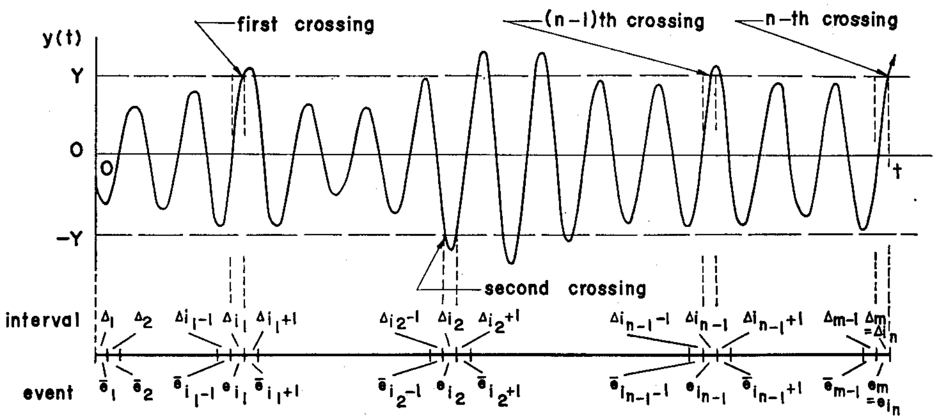

Fig. 1. Illustration of the n-th Upward Crossing of a Response Level $|y(t)| = Y$ at t.

$$\mathrm{P}\,[e_i \cap \bar{e}_j] = \mathrm{P}\,[e_i \cap (1 - e_j)] = \mathrm{P}\,[e_i] - \mathrm{P}\,[e_i \cap e_j]$$
$$\mathrm{P}\,[\bar{e}_i \cap \bar{e}_j] = \mathrm{P}\,[(1 - e_i) \cap (1 - e_j)]$$
$$= 1 - \mathrm{P}\,[e_i] - \mathrm{P}\,[e_j] + \mathrm{P}\,[e_i \cap e_j]$$

and so forth.

Since we take Δt small enough, only one crossing could take place, if any, in any of the intervals Δ_i. Hence if $G_n(Y,m)$ represents the event that the n-th upward crossing of $|y(t)|=Y$ takes place in the interval Δ_m, $m \gg n$, then by reference to Fig. 1, the probability of this event is represented by the following formula in which all symbols represented by $\cup$ to denote the union of events are replaced by the summation symbols:

$$\mathrm{P}\,[G_n(Y,m)] = \mathrm{P}\,[\bigcup_{i_1=1}^{m-n+1} \bigcup_{i_2=i_1+1}^{m-n+2} \cdots\cdots \bigcup_{i_{n-1}=i_{n-2}+1}^{m-1} (\bar{e}_1 \cap \bar{e}_2 \cdots\cdots$$
$$\cap \bar{e}_{i_1-1} \cap e_{i_1} \cap \bar{e}_{i_1+1} \cap \cdots\cdots \cap \bar{e}_{i_2-1} \cap e_{i_2} \cap \bar{e}_{i_2+1} \cap \cdots\cdots$$
$$\cap \bar{e}_{i_{n-1}-1} \cap e_{i_{n-1}} \cap \bar{e}_{i_{n-1}+1} \cap \cdots\cdots \cap \bar{e}_{m-1} \cap e_m)]$$
$$= \sum_{i_1=1}^{m-n+1} \sum_{i_2=i_1+1}^{m-n+2} \cdots\cdots \sum_{i_{n-1}=i_{n-2}+1}^{m-1} \mathrm{P}\,[\,(1 - e_1) \cap (1 - e_2) \cap \cdots\cdots$$
$$\cap (1 - e_{i_1-1}) \cap (1 - e_{i_1+1}) \cap \cdots\cdots \cap (1 - e_{i_2-1}) \cap (1 - e_{i_2+1}) \cap \cdots\cdots$$
$$\cap (1 - e_{i_{n-1}-1}) \cap (1 - e_{i_{n-1}+1}) \cap \cdots\cdots \cap (1 - e_{m-1}) \cap e_{i_1} \cap e_{i_2} \cap$$
$$\cdots\cdots \cap e_{i_{n-1}} \cap e_m]$$
$$= \sum_{i_1=1}^{m-n+1} \sum_{i_2=i_1+1}^{m-n+2} \cdots\cdots \sum_{i_{n-1}=i_{n-2}+1}^{m-1} \{\mathrm{P}\,[\,(\bigcap_{k=1}^{n-1} e_{i_k}) \cap e_m\,]$$
$$- \sum_{i_n=1}^{m-1} \mathrm{P}\,[\,(\bigcap_{k=1}^{n} e_{i_k}) \cap e_m] + \sum_{i_n=1}^{m-2} \sum_{i_{n+1}=i_n+1}^{m-1} \mathrm{P}\,[\,(\bigcap_{k=1}^{n+1} e_{i_k}) \cap e_m]$$
$$- \cdots\cdots\cdots \}$$

By letting $\Delta t \to 0$ in the above result, the n-th passage time density $p_n(Y,t)$ is obtained as

$$p_n(Y,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\, \mathrm{P}\,[G_n(Y,m) \,|\, |y(0)| \leqq Y]$$
$$= \int_0^t dt_1 \int_t^t dt_2 \cdots\cdots \int_{t_{n-2}}^t \{f_s(Y;\, t_1, t_2, \cdots\cdots, t_{n-1}, t)$$
$$- \sum_{i=0}^{\infty} (-1)^i \int_0^t dt_n \int_{t_n}^t dt_{n+1} \cdots\cdots \int_{t_{n+i-1}}^t f_s(Y;\, t_1, t_2, \cdots\cdots, t_{n+i}, t)\, dt_{n+i}\}\, dt_{n-1} \qquad \cdots\cdots\cdots(4)$$

where

$$f_s\,(Y;\,t_1,\,t_2,\,\ldots\ldots,\,t_l)\;dt_1\,dt_2\,\ldots\ldots\,dt_l$$
$$=\mathrm{P}\,[\bigcap_{i=1}^{l}\{|y(t_i)|\leqq Y\cap|y(t_i+dt_i)|>Y\}\,\|y(0)|\leqq Y] \qquad \ldots\ldots\ldots(5)$$

As long as $y(t)$ has the same kind of probability distribution, Gaussian for example, throughout the time axis, the independent time variables in $f_s(Y;\ t_1,\ t_2,\ \ldots.\ t_l)$ can be interchanged arbitrarily whether $y(t)$ is stationary or nonstationary. Hence by reference to Eq. (A. 4) in Appendix, Eq. (4) is reduced to

$$P_n\,(Y,t)=\sum_{m=1}^{\infty}(-1)^{m-1}\binom{n+m-2}{n-1}\int_0^t dt_1\int_{t_1}^t dt_2\ldots\ldots$$
$$\ldots\ldots\int_{t_{n+m-3}}^t f_s\,(Y;\,t_1,\,t_2,\,\ldots\ldots,\,t_{n+m-2},\,t)\;dt_{n+m-2} \qquad \ldots\ldots\ldots(6)$$

In the same manner, $\tilde{p}_n(Y,\ t)$ complementary to $p_n(Y,t)$ is obtained as

$$\tilde{p}_n\,(Y,t)=\sum_{m=1}^{\infty}(-1)^{m-1}\binom{n+m-2}{n-1}\int_0^t dt_1\int_{t_1}^t dt_2\ldots\ldots$$
$$\ldots\ldots\int_{t_{n+m-3}}^t \tilde{f}_s\,(Y;\,t_1,\,t_2,\,\ldots\ldots,\,t_{n+m-2},\,t)\;dt_{n+m-2} \qquad \ldots\ldots\ldots(7)$$

where

$$\tilde{f}_s\,(Y;\,t_1,\,t_2,\,\ldots\ldots,\,t_l)\;dt_1\,dt_2\,\ldots\ldots\,dt_l$$
$$=\mathrm{P}\,[\bigcap_{i=1}^{l}\{|y\,(t_i)|\leqq Y\cap|y\,(t_i+dt_i)|>Y\}\,\|y(0)|>Y] \qquad \ldots\ldots\ldots(8)$$

For application of these results to Eqs. (2) and (3), the formulae (6) and (7) are rewritten with the aid of Eq. (A. 2) as

$$\left.\begin{aligned}&\int_0^t p_n\,(Y,t)\,dt=\sum_{m=1}^{\infty}(-1)^{m-1}\binom{n+m-2}{n-1}\int_0^t dt_1\int_{t_1}^t dt_2\ldots\ldots\\&\quad\ldots\ldots\int_{t_{n+m-3}}^t dt_{n+m-2}\int_{t_{n+m-2}}^t f_s\,(Y;\,t_1,\,t_2,\,\ldots\ldots,\,t_{n+m-2},\,t)\;dt\\&\int_0^t \tilde{p}_n\,(Y,t)\,dt=\sum_{m=1}^{\infty}(-1)^{m-1}\binom{n+m-2}{n-1}\int_0^t dt_1\int_{t_1}^t dt_2\ldots\ldots\\&\quad\ldots\ldots\int_{t_{n+m-3}}^t dt_{n+m-2}\int_{t_{n+m-2}}^t \tilde{f}_s\,(Y;\,t_1,\,t_2,\,\ldots\ldots,\,t_{n+m-2},\,t)\;dt\end{aligned}\right\} \qquad \ldots\ldots\ldots(9)$$

3. Approximate Solution for Stationary Response

The formulation made in the previous chapter is valid for a general case. However, the n-th passage time densities in Eqs. (6) and (7) involve infinite number of integrations which do not enable numerical evaluation for processes such as the structural response with a high correlation between the amplitudes at different times. Hence in this section, two types of approximation valid for a stationary response are adopted, viz., the renewal process approximation and the Poisson process approximation, the former being expected to be closer to the exact solution.

(1) Renewal Process Approximation

In this section, we shall make the following two assumptions. (1) The response $y(t)$ is a stationary random process. (2) Although the time of the n-th upward crossing $|y(t)|=Y$ in Fig. 1 is in general affected by the times of n–1 crossings prior to it, we shall treat the n-th crossing time as affected only by the time of the (n–1)-th crossing; i.e., the process of such crossings constitute a renewal process[4].

Under these assumptions the probability parameters $f_s(Y; t_1, t_2, \ldots\ldots, t_l)$ and $\tilde{f}_s(Y; t_1, t_2, \ldots\ldots, t_l)$ in Eqs. (5) and (8) assume the form

$$\left.\begin{aligned} f_s(Y; t_1, t_2, \ldots\ldots, t_l) &= f_s(Y; t_1) \prod_{i=1}^{l-1} f_c(Y; t_{i+1} - t_i) \\ \tilde{f}_s(Y; t_1, t_2, \ldots\ldots, t_l) &= \tilde{f}_s(Y; t_1) \prod_{i=1}^{l-1} f_c(Y; t_{i+1} - t_i) \end{aligned}\right\} \quad \cdots\cdots\cdots(10)$$

where

$$f_c(Y; t)\, dt = P\left[\, |y(t)| \leqq Y \cap |y(t+dt)| > Y \,\big|\, |y(0)| = Y \cap |y(dt)| > Y \right] \quad \cdots\cdots\cdots(11)$$

This $f_c(Y; t)$ represents the conditional probability of an upward crossing of $|y(t)|=Y$ in $(t, t+dt)$ on the hypothesis of the crossing at $t=0$.

On substitution from Eqs. (10), the right-hand side members of Eqs. (9) become multiple convolution integrals. Hence the Laplace transforms of Eqs. (9) are obtained as

$$\left.\begin{aligned} \mathfrak{L}\left[\int_0^t p_n(Y, t)\, dt\right] &= \frac{F_s(Y; s)}{s} \sum_{m=1}^{\infty} (-1)^{m-1} \binom{n+m-2}{n-1} \{F_c(Y; s)\}^{m+n-2} \\ \mathfrak{L}\left[\int_0^t \tilde{p}_n(Y, t)\, dt\right] &= \frac{\tilde{F}_s(Y; s)}{s} \sum_{m=1}^{\infty} (-1)^{m-1} \binom{n+m-2}{n-1} \{F_c(Y; s)\}^{m+n-2} \end{aligned}\right\} \quad \cdots\cdots\cdots(12)$$

in which

$$\mathfrak{L}[f(t)] = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

is the Laplace transform of $f(t)$, and $F_s(Y;s)$, $\tilde{F}_s(Y;s)$ and $F_c(Y;s)$ are the Laplace transforms of $f_s(Y;t)$, $\tilde{f}_s(Y;t)$ and $f_c(Y;t)$, respectively. By virtue of Eq. (A.7), the infinite series in Eqs. (12) are simplified as

$$\left.\begin{aligned} \mathfrak{L}\left[\int_0^t p_n(Y,t)\,dt\right] &= \frac{F_s(Y;s)\{F_c(Y;s)\}^{n-1}}{s\{1+F_c(Y;s)\}^n} \\ \mathfrak{L}\left[\int_0^t \tilde{p}_n(Y,t)\,dt\right] &= \frac{\tilde{F}_s(Y;s)\{F_c(Y;s)\}^{n-1}}{s\{1+F_c(Y;s)\}^n} \end{aligned}\right\} \qquad \cdots\cdots\cdots(13)$$

Thus the inverse transforms of Eqs. (13) are the following multiple integral equations in terms of $p_n(Y,t)$ and $\tilde{p}_n(Y,t)$:

$$\left.\begin{aligned} p_n(Y,t) &+ \sum_{i=1}^{n}\binom{n}{i}\int_0^t dt_1\int_{t_1}^t dt_2\cdots\cdots\int_{t_{i-1}}^t p_n(Y;t_1)f_c(Y;t-t_i) \\ &\qquad\cdot\prod_{m=1}^{i-1} f_c(Y;t_{m+1}-t_m)\,dt_i \\ &= \int_0^t dt_1\int_{t_1}^t dt_2\cdots\cdots\int_{t_{n-2}}^t f_s(Y;t_1)f_c(Y;t-t_{n-1}) \\ &\qquad\cdot\prod_{m=1}^{n-2} f_c(Y,t_{m+1}-t_m)\,dt_{n-1} \\ \tilde{p}_n(Y,t) &+ \sum_{i=1}^{n}\binom{n}{i}\int_0^t dt_1\int_{t_1}^t dt_2\cdots\cdots\int_{t_{i-1}}^t \tilde{p}_n(Y;t_1)f_c(Y;t-t_i) \\ &\qquad\cdot\prod_{m=1}^{i-1} f_c(Y;t_{m+1}-t_m)\,dt_i \\ &= \int_0^t dt_1\int_{t_1}^t dt_2\cdots\cdots\int_{t_{n-2}}^t \tilde{f}_s(Y;t_1)f_c(Y;t-t_{n-1}) \\ &\qquad\cdot\prod_{m=1}^{n-2} f_c(Y;t_{m+1}-t_m)\,dt_{n-1} \end{aligned}\right\}$$

$$\cdots\cdots\cdots(14)$$

The multiple integrals appearing in Eqs. (14) are at most n-fold, for which numerical evaluation is much easier than for Eqs. (9) though not simple enough.

The terms $f_s(Y;t)$ and $\tilde{f}_s(Y;t)$ are given by

$$\begin{aligned} f_s(Y;t) &= \frac{1}{dt}\,\mathrm{P}\left[\,|y(t)|\leqq Y\cap|y(t+dt)|>Y\,\big|\,|y(0)|\leqq Y\right] \\ &= \frac{1}{dt\,a_o(Y)}\,\mathrm{P}\left[\,|y(t)|\leqq Y\cap|y(t+dt)|>Y\cap|y(0)|\leqq Y\right] \\ &= Q(Y)/a_o(Y) \qquad\qquad \cdots\cdots\cdots(15) \end{aligned}$$

$$\tilde{f}_s(Y;t) = \frac{1}{dt\,\tilde{a}_o(Y)}\,\mathrm{P}\left[\,|y(t)|\leqq Y\cap|y(t+dt)|>Y\cap|y(0)|>Y\right]$$

$$
\begin{aligned}
&= \frac{1}{dt\,\tilde{a}_o(Y)} \{\mathrm{P}\,[\,|y(t)| \leqq Y \cap |y(t+dt)| > Y] \\
&\quad - \mathrm{P}\,[\,|y(t)| \leqq Y \cap |y(t+dt)| > Y \cap |y(0)| < Y]\,\} \\
&= \{N_c(Y) - Q\,(Y)\}/a_o(Y) \qquad \cdots\cdots\cdots(16))
\end{aligned}
$$

where $Q\,(Y)$ and $N_c(Y)$ are the probabilistic parameters related to the threshold-value crossings[1),2),4)] which are represented for a stationary Gaussian process by

$$
N_c(Y) = \frac{1}{\pi}\frac{\sigma_{\dot{y}}}{\sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{Y}{\sigma_y}\right)^2\right\} \qquad \cdots\cdots\cdots(17)
$$

$$
\begin{aligned}
Q\,(Y) &= \frac{1}{\mu_{s1}\sqrt{2\pi^3 C_s}}\frac{\sigma_{\dot{y}}}{\sigma_y} \exp\left\{-\frac{1}{2}\left(\mu_{s4} - \frac{\mu_{s5}^2}{\mu_{s6}}\right)\left(\frac{Y}{\sigma_y}\right)^2\right\} \\
&\times \int_0^{Y/\sigma_y} \left[\exp\left(-\frac{u_{s1}}{2}\right)\left\{\exp\left(-\frac{v_{s1}^2}{2}\right) + \sqrt{\frac{\pi}{2}}\,v_{s1}\left(1 + \mathrm{erf}\left(\frac{v_{s1}}{\sqrt{2}}\right)\right)\right\}\right. \\
&\left. + \exp\left(-\frac{u_{s2}}{2}\right)\left\{\exp\left(-\frac{v_{s2}^2}{2}\right) + \sqrt{\frac{\pi}{2}}\,v_{s2}\left(1 + \mathrm{erf}\left(\frac{v_{s2}}{\sqrt{2}}\right)\right)\right\}\right] d\xi_1
\end{aligned}
$$

$$\cdots\cdots\cdots(18)$$

where

$$
\begin{aligned}
&\left.\begin{aligned} u_{s1} &= u_{s1}(\xi_1) \\ u_{s2} &= u_{s2}(\xi_1)\end{aligned}\right\} = \left(\mu_{s1} - \frac{\mu_{s3}^2}{\mu_{s6}}\right)\xi_1^2 \pm \left(\mu_{s2} - \frac{\mu_{s5}\,\mu_{s3}}{\mu_{s6}}\right)\frac{Y}{\sigma_y}\,\xi_1 \\
&\left.\begin{aligned} v_{s1} &= v_{s1}(\xi_1) \\ v_{s2} &= v_{s2}(\xi_1)\end{aligned}\right\} = \mp\frac{1}{\sqrt{\mu_{s6}}}\left(\mu_{s3}\,\xi_1 \pm \mu_{s5}\frac{Y}{\sigma_y}\right) \\
&\mu_{s1} = 1/C_s\,,\ \mu_{s2} = -\,\rho_{yy}/C_s\,,\ \mu_{s3} = -\,\rho_{y\dot{y}}/C_s\,, \\
&\mu_{s4} = (1 - \rho_{y\dot{y}}^2)/C_s\,,\ \mu_{s5} = \rho_{yy}\,\rho_{y\dot{y}}/C_s\,,\ \mu_{s6} = (1 - \rho_{yy}^2)/C_s \\
&C_s = 1 - \rho_{yy}^2 - \rho_{y\dot{y}}^2 \\
&\sigma_y^2 = \mathrm{E}\,[y^2(t)]\,,\ \sigma_{\dot{y}}^2 = \mathrm{E}\,[\dot{y}^2(t)] \\
&\rho_{yy} = \mathrm{E}\,[y(0)\,y(t)]/\sigma_y^2,\ \rho_{y\dot{y}} = \mathrm{E}\,[y(0)\,\dot{y}(t)]/(\sigma_y\,\sigma_{\dot{y}})
\end{aligned}
$$

Likewise, $f_c(Y,t)$ is represented by

$$
\begin{aligned}
f_c(Y,t) &= \frac{1}{N_c(Y)}\left\{\int_0^\infty d\dot{y}(0)\int_0^\infty \dot{y}(0)\,\dot{y}(t)\,\phi_c(Y,\dot{y}(0),\,Y,\dot{y}(t))\,d\dot{y}(t)\right. \\
&+ \int_0^\infty d\dot{y}(0)\int_\infty^0 \dot{y}(0)\,|\dot{y}(t)|\,\phi_c(Y,\dot{y}(0),\,-\,Y,\dot{y}(t))\,d\dot{y}(t) \\
&+ \int_{-\infty}^0 d\dot{y}(0)\int_0^\infty |\dot{y}(0)\,|\dot{y}(t)\,\phi_c(-\,Y,\dot{y}(0),\,Y,\dot{y}(t))\,d\dot{y}(t) \\
&\left.+ \int_{-\infty}^0 d\dot{y}(0)\int_{-\infty}^0 |\dot{y}(0)\,\dot{y}(t)\,|\,\phi_c(-\,Y,\dot{y}(0),\,-\,Y,\dot{y}(t))\,d\dot{y}(t)\right\}
\end{aligned}
$$

$$\cdots\cdots\cdots(19)$$

where $\phi_c(y(0), \dot{y}(0), y(t), \dot{y}(t))$ is the joint probability density of the four independent variables. For the Gaussian distribution, Eq.(19) yields

$$f_c(Y,t) = \frac{1}{2\pi\,\mu_{c5}\sqrt{C_c}}\frac{\sigma_{\dot{y}}}{\sigma_y}\Bigg[\exp\left\{-\frac{\nu_1}{2}\left(\frac{Y}{\sigma_y}\right)^2\right\}$$

$$\cdot\int_0^\infty \xi_1 \exp\left(-\frac{u_{c1}}{2}\right)\left\{\exp\left(-\frac{v_{c1}^2}{2}\right)+\sqrt{\frac{\pi}{2}}\,v_{c1}\left(1+\mathrm{erf}\left(\frac{v_{c1}}{\sqrt{2}}\right)\right)\right\}d\xi_1$$

$$+\exp\left\{-\frac{\nu_2}{2}\left(\frac{Y}{\sigma_y}\right)^2\right\}\int_0^\infty \xi_1\exp\left(-\frac{u_{c2}}{2}\right)\left\{\exp\left(-\frac{v_{c2}^2}{2}\right)\right.$$

$$\left.+\sqrt{\frac{\pi}{2}}\,v_{c2}\left(1+\mathrm{erf}\left(\frac{v_{c2}}{\sqrt{2}}\right)\right)\right\}d\xi_1 \qquad \text{.........(20)}$$

where

$$\left.\begin{aligned}u_{c1}&=u_{c1}(\xi_1)\\ u_{c2}&=u_{c2}(\xi_1)\end{aligned}\right\} = \left(\mu_{c5}-\frac{\mu_{c6}^2}{\mu_{c5}}\right)\xi_1^2 - 2\left(1\pm\frac{\mu_{c6}}{\mu_{c5}}\right)(\mu_{c2}\mp\mu_{c4})\frac{Y}{\sigma_y}\xi_1$$

$$\left.\begin{aligned}v_{c1}&=v_{c1}(\xi_1)\\ v_{c2}&=v_{c2}(\xi_1)\end{aligned}\right\} = \mp\frac{1}{\sqrt{\mu_{c5}}}\left\{\mu_{c6}\,\xi_1 \pm (\mu_{c2}\mp\mu_{c4})\frac{Y}{\sigma_y}\right\}$$

$$\left.\begin{aligned}\nu_1\\ \nu_2\end{aligned}\right\} = 2\,(\mu_{c1}\pm\mu_{c3}) - \frac{(\mu_{c2}\mp\mu_{c4})^2}{\mu_{c5}} - 1$$

$$\mu_{c1} = (1-\rho_{\dot{y}\dot{y}}^2-\rho_{y\dot{y}}^2)/C_c,\quad \mu_{c2} = -\rho_{y\dot{y}}\,(\rho_{yy}-\rho_{\dot{y}\dot{y}})/C_c$$

$$\mu_{c3} = (\rho_{yy}\,\rho_{\dot{y}\dot{y}}^2+\rho_{y\dot{y}}^2\,\rho_{\dot{y}\dot{y}}-\rho_{yy})/C_c$$

$$\mu_{c4} = -\rho_{y\dot{y}}\,(1-\rho_{yy}\,\rho_{\dot{y}\dot{y}}-\rho_{y\dot{y}}^2)/C_c,\quad \mu_{c5} = (1-\rho_{y\dot{y}}^2-\rho_{yy}^2)/C_c$$

$$\mu_{c6} = -\rho_{\dot{y}\dot{y}}-\rho_{yy}^2\,\rho_{\dot{y}\dot{y}}-\rho_{yy}\,\rho_{y\dot{y}}^2)/C_c$$

$$C_c = (1-\rho_{yy}^2)\,(1-\rho_{\dot{y}\dot{y}}^2)-\rho_{y\dot{y}}^2\,(2-2\rho_{yy}\,\rho_{\dot{y}\dot{y}}-\rho_{y\dot{y}}^2)$$

$$\rho_{\dot{y}\dot{y}} = \mathrm{E}\,[\dot{y}(0)\,\dot{y}(t)]/\sigma_{\dot{y}}^2$$

(2) Poisson Process Approximation

It can readily be verified that in the limit as $t\to\infty$, the parameters $f_s(Y;t)$, $\tilde{f}_s(Y;t)$ and $f_c(Y;t)$ are all asymptotic to $N_c(Y)$ which is independent of the conditions at $t=0$. If we use this limiting value, the process of upward crossings of $|y(t)|=Y$ becomes a Possion process, for which the n-th passage time densities $p_n(Y,t)$ and $p_n(Y,t)$ are obtained as[7)]

$$p_n(Y,t) = \tilde{p}_n(Y,t) = \frac{N_c(Y)\,\{N_c(Y)t\}^{n-1}}{(n-1)\,!}\exp\{-N_c(Y)\,t\} \qquad \text{.........(21)}$$

and

$$\int_0^t p_n(Y, t)\, dt = \int_0^t \tilde{p}_n(Y, t)\, dt = 1 - \exp\{-N_c(Y)t\} \sum_{i=0}^{n-1} \frac{\{N_c(Y)t\}^i}{i\,!} \qquad \cdots\cdots\cdots(22)$$

(3) Application to a Single-Degree-of-Freedom System

The two approximate methods discussed in the foregoing sections have been applied to the stationary random response of a linear structure with a single degree of freedom. In this case, the correlation coefficients ρ_{yy} and $\rho_{y\dot{y}}$ and the standard deviation ratio $\sigma_{\dot{y}}/\sigma_y$ involved in Eqs. (15)–(20) are approximately given by[8)]

$$\rho_{yy} = \rho_{yy}(t) \cong e^{-h_n\omega_n t}(\cos \bar{\omega}_n t + \bar{h}_n \sin \bar{\omega}_n t)$$

$$\rho_{y\dot{y}} = \rho_{y\dot{y}}(t) \cong \frac{e^{-h_n\omega_n t}}{\sqrt{1-h_n^2}} \sin \bar{\omega}_n t, \quad \sigma_{\dot{y}}/\sigma_y \cong \omega_n$$

in which ω_n is the natural circular frequency, h_n is the damping factor, $\bar{\omega}_n = \sqrt{1-h_n^2}\,\omega_n$ is the frequency of damped free vibration, and $\bar{h}_n = h_n/\sqrt{1-h_n^2}$. Likewise, $\rho_{\dot{y}\dot{y}}$ is obtained as

$$\rho_{\dot{y}\dot{y}} = \rho_{\dot{y}\dot{y}}(t) \cong e^{-h_n\omega_n t}(\cos \bar{\omega}_n t - \bar{h}_n \sin \bar{\omega}_n t)$$

Using these results, some numerical values have been obtained for $f_s(Y;t)$, $\tilde{f}_s(Y;t)$, $f_c(Y;t)$, the n-th passage time densities $p_n(Y,t)$ and $\tilde{p}_n(Y,t)$, and thereby the probability distribution $\Phi_p(n;Y,\tau)$ of the number of upward crossings of $|y(t)|=Y$ in the duration τ. The damping factor h_n was taken as 0.1.

Figs. 2 and 3 show the parameters $f_s(Y;t)$ and $f_c(Y;t)$ plotted against t normalized with respect to the natural period T_n which proves that the effect of the initial condition dominates the fluctuation of them for small t. For example, high peaks of $f_c(Y;t)$ at $t=0.5T_n$ is a consequence of the highly periodic fluctuation of the response $y(t)$ since these peaks mean that the upward crossing of $y(t)=\pm Y$ at $t=0$ makes another crossing which is quite likely to take place at $t=0.5T_n$ on the

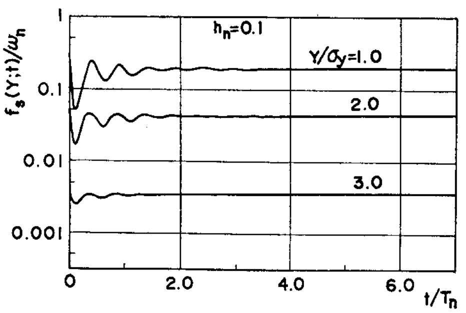

Fig. 2. Probability Parameter $f_s(Y; t)$.

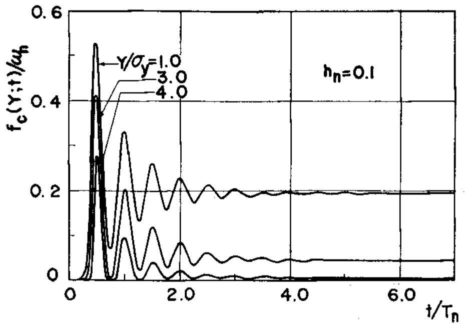

Fig. 3. Probability Parameter $f_c(Y;t)$.

other side of the neutral position of the oscillator.

The numerical results for the n-th passage time density $p_n(Y,t)$ are plotted in Fig. 4 both for the renewal-process and the Poisson-process approximations. It is

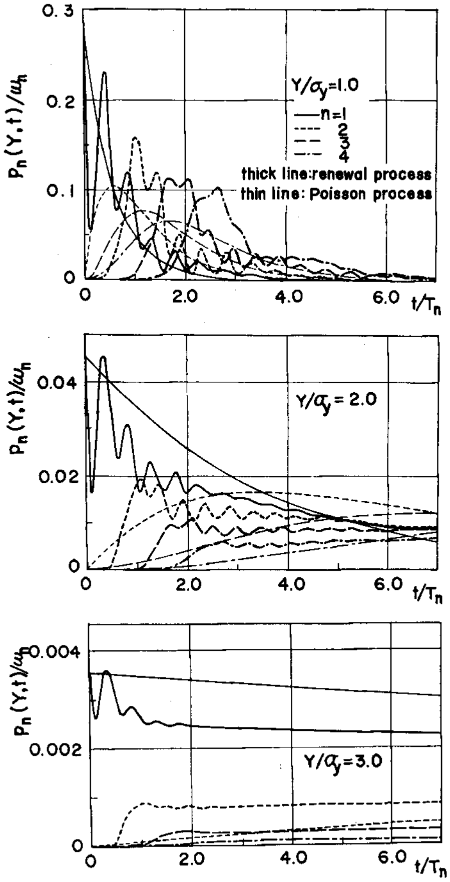

Fig. 4. n-th Passage Time Probability Density $p_n(Y,t)$, $(h_n=0.1)$.

obvious from this figure that the modes of variation of $p_n(Y,t)$ derived from the two methods are quite different. This result is natural if we consider that we have seen in Fig. 3 that two successive crossings of the response level Y are in high correlation. Hence when compared with the renewal process approximation which accounts for this correlation, the Poisson process approximation introduced by neglecting its effect is considered to be in greater error.

On the basis of the probabilistic parameters discussed above, there have been calculated the probability distribution $\Phi_p(n;Y,\tau)$ of the number of crossings of the response level Y through the duration τ which are plotted in Fig. 5 against Y/σ_y for $n=0$ and 3 and various durations. The meaning of $\Phi_p(n;Y,\tau)$ for $n=0$ is same as the probability distribution of the maximum response discussed in earlier papers[1),2)], since it represents the probability that there is no crossing the response level Y.

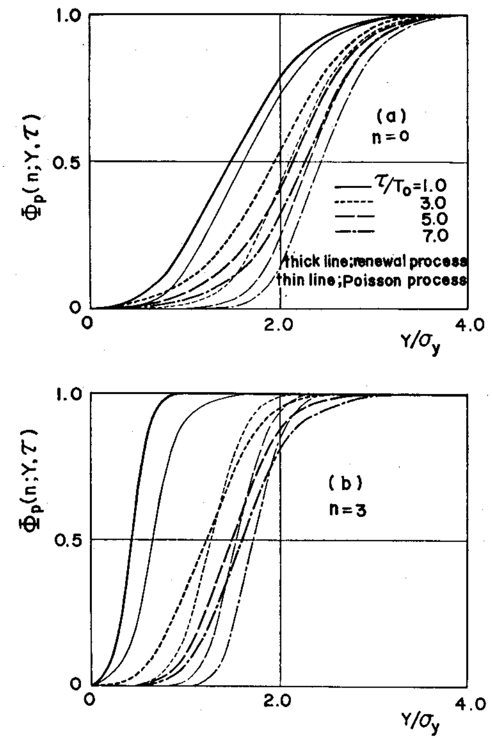

Fig. 5. Probability Distribution of the Number of Crossings of a Response Level Y, ($h_n=0.1$).

In Fig. 5(b) for $n=3$, it is noted that the curves obtained from the two methods cross each other for larger τ/T_n, and as a result, the probability $\Phi_p(n;Y,\tau)$ based on the renewal process approximation assumes smaller values than that on the Poisson process approximation for high levels of Y/σ_y or $\Phi_p(n;Y,\tau)$. In discussing the probability distribution of the maximum response, the Possion process approximation always results in the non-excess probability lower than those based on more accurate methods or the simulated values[1),2)], which implies for the design purposes that when using the simple Poisson process approximation, we are on the safe side, though the result may be inaccurate. On the contrary, the above discussions on Fig. 5(b) assert that when we are concerned with the probability distribution of the number of crossings of a high response level, the Poisson process approximation may give values on the dangerous side as the number of crossings to be considered increases.

4. Conclusion

In this study, discussions have been made on some basic aspects of the probability that crossings of a certain response level take place by a given number of times in a duration of ramdom vibration, from which following conclusions have been derived.
(1) The formulation of the problem in this chapter can be made in terms of the probability density referred to as the n-th passage time density by the author.
(2) The approximate method of solution was developed by means of the renewal process approximation, which is considered to offer a result far closer to the exact solutions than the simple Poisson process approximation.
(3) The effect of narrow frequency band fluctuation is exhibited in the n-th passage time density due to the renewal process approximation, by which this method of analysis is considered to explain the phenomena fairly well.
(4) When the probability of structural safety is discussed in terms of the probability distribution of the number of crossings of a high response level, the simple Poisson process approximation may give results on the dangerous side.

It is noted finally that the numerical computations have not been made for a sufficient range of parameters. Hence it is still required to cover a wider range of them especially of longer durations, from which broader conclusions on the present problem could be deduced.

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Appendix. Derivation of Some Mathematical Expressions

(1) Conversion of Ranges of Integration in Eq. (4)

Let $f(t_1, t_2, \ldots, t_m)$ be a function which remains unchanged by an arbitrary interchange of independent variables, and we shall deal with a multiple integral of the form

$$\int_0^t dt_1 \int_{t_1}^t dt_2 \cdots\cdots \int_{t_{l-2}}^t dt_{l-1} \int_0^t dt_l \int_{t_l}^t dt_{l+1} \cdots\cdots$$
$$\cdots\cdots \int_{t_{m-2}}^t dt_{m-1} \int_{t_{m-1}}^t f(t_1, t_2, \ldots\ldots, t_m)\, dt_m$$
$$= F^{1\;2\cdots\cdots l-1\;\; l\;\; l+1\cdots\cdots m-1\;\; m}_{0\;1\cdots\cdots l-2\;\; 0\;\;\; l\;\;\cdots\cdots m-2\;\; m-1} \qquad \cdots\cdots\cdots(\mathrm{A}\cdot 1)$$

in which F^i_j means the integration with respect to t_i in the interval (t_j, t), with $t_0=0$.

It can be readily verified by changing the order of integrations that

$$I_m = \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots\cdots \int_{t_{m-1}}^t f(t_1, t_2, \ldots\ldots, t_m)\, dt_m$$
$$= \int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots\cdots \int_0^{t_2} f(t_1, t_2, \ldots\ldots, t_m)\, dt_1 \qquad \cdots\cdots\cdots(\mathrm{A}\cdot 2)$$

Since the independent variables of $f(t_1, t_2, \ldots, t_m)$ are interchangeable, we obtain, by virtue of Eq. (A.2),

$$
\begin{aligned}
F^{1\,2\,3\cdots\cdots m}_{0\,0\,2\cdots\cdots m-1} &= \int_0^t dt_1 \int_0^t dt_2 \int_{t_2}^t dt_3 \cdots\cdots \int_{t_{m-1}}^t f(t_1, t_2, \cdots\cdots, t_m)\, dt_m \\
&= I_m + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_{t_2}^t dt_3 \cdots\cdots \int_{t_{m-1}}^t f(t_1, t_2, \cdots\cdots, t_m)\, dt_m \\
&= 2I_m + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_{t_2}^{t_1} dt_3 \int_{t_3}^t dt_4 \cdots\cdots \int_{t_{m-1}}^t f(t_1, t_2, \cdots\cdots, t_m)\, dt_m \\
&= \cdots\cdots\cdots\cdots \\
&= m \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots\cdots \int_{t_{m-1}}^t f(t_1, t_2, \cdots\cdots, t_m)\, dt_m \\
&= m\, F^{1\,2\,3\cdots\cdots m}_{0\,1\,2\cdots\cdots m-1} \qquad \cdots\cdots\cdots(\text{A}\cdot 3)
\end{aligned}
$$

Hence we have

$$
F^{1\,2\,3\,4\cdots\cdots m}_{0\,0\,0\,3\cdots\cdots m-1} = m\,(m-1)\, F^{1\,2\,3\cdots\cdots m}_{0\,1\,2\cdots\cdots m-1}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$
\begin{aligned}
&F^{1\,2\cdots\cdots l-1\;\; l\;\; l+1\cdots\cdots m}_{0\,0\cdots\cdots 0\;\;\; 0\;\;\; l\;\cdots\cdots m-1} \\
&\qquad = m(m-1)\cdots\cdots(m-l+2)\, F^{1\,2\,3\cdots\cdots m}_{0\,1\,2\cdots\cdots m-1}
\end{aligned}
$$

and by the same operation in the reverse order,

$$
\begin{aligned}
&F^{1\,2\cdots\cdots l-2\;\; l-1\;\; l\;\; l+1\cdots\cdots m}_{0\,0\cdots\cdots 0\;\;\; l-2\;\; 0\;\; l\;\cdots\cdots m-1} \\
&\qquad = \frac{m(m-1)\cdots\cdots(m-l+2)}{2}\, F^{1\,2\,3\cdots\cdots m}_{0\,1\,2\cdots\cdots m-1} \\
&F^{1\,2\cdots\cdots l-3\;\; l-2\;\; l-1\;\; l\;\; l+1\cdots\cdots m}_{0\,0\cdots\cdots 0\;\;\; l-3\;\; l-2\;\; 0\;\; l\;\cdots\cdots m-1} \\
&\qquad = \frac{m(m-1)\cdots\cdots(m-l+2)}{3\cdot 2}\, F^{1\,2\,3\cdots\cdots m}_{0\,1\,2\cdots\cdots m-1}
\end{aligned}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$
F^{1\,2\cdots\cdots l-1\;\; l\;\; l+1\cdots\cdots m}_{0\,1\cdots\cdots l-2\;\; 0\;\; l\;\cdots\cdots m-1} = \binom{m}{l-1} F^{1\,2\,3\cdots\cdots m}_{0\,1\,2\cdots\cdots m-1}
$$

Thus we obtain

$$
\begin{aligned}
&\int_0^t dt_1 \int_{t_1}^t dt_2 \cdots\cdots \int_{t_{l-2}}^t dt_{l-1} \int_0^t dt_l \int_{t_l}^t dt_{l+1} \cdots\cdots \int_{t_{m-1}}^t f(t_1, t_2, \cdots\cdots, t_m)\, dt_m \\
&= \binom{m}{l-1} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots\cdots \int_{t_{m-1}}^t f(t_1, t_2, \cdots\cdots, t_m)\, dt_m \qquad \cdots\cdots\cdots(\text{A}\cdot 4)
\end{aligned}
$$

(2) Reduction of a Kind of Infinite Series

An infinite series S_n of the form

$$S_n = \sum_{m=1}^{\infty} \binom{n+m-2}{n-1} (-d)^{m-1} \qquad \text{.........(A·5)}$$

is considered. By an application of the binomial theorem, Eq. (A.5) is reduced to

$$(1+d)^n S_n = 1 + \sum_{k=1}^{n} \left\{ \sum_{i=0}^{n} (-1)^{k+i} \binom{n}{i} \binom{n+k-i-1}{n-1} \right\} d^k$$
$$+ \sum_{k=n+1}^{\infty} \left\{ \sum_{i=0}^{n} (-1)^{k+i} \binom{n}{i} \binom{n+k-i-2}{n-1} \right\} d^k$$

It can be proved by means of the mathematical reduction that

$$\left.\begin{aligned} &\sum_{i=0}^{k} (-1)^i \binom{n}{i} \binom{n+k-i-1}{n-1} = 0 \; ; \; k = 1, 2, \ldots\ldots, n \\ &\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+k-i-2}{n-1} = 0 \; ; \; k = n+1, n+2, \ldots\ldots\ldots \end{aligned}\right\} \qquad \text{.........(A·6)}$$

Hence we obtain

$$S_n = \frac{1}{(1+d)_n} \qquad \text{.........(A·7)}$$