A New Circle Criterion for the Stability of Nonlinear Control System

By

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1. Introduction

The Popov's theorem [1], as is well known, gives sufficient conditions for the absolute stability of nonlinear control systems. This theorem can be applied to a single-loop feedback system having one isolated, time-invariant nonlinear element in series with a linear transfer function, and in terms of the frequency response $G(j\omega)$ of the linear part, it determines the stability sector within which the non-linearity is to lie in order to assure the absolute stability of the system. In order to apply the Popov's theorem it is necessary that the transfer function $G(s)$ of the linear part has all poles in the open left half s-plane, therefore the Popov's theorem cannot be applied directly to a conditionally stable system having the Hurwitz sector (k_h, k_H) with $0<k_h<k_H$. The stability sectors (k_{1p}, k_{2p}), however, can be obtained by the application of the Popov's theorem to the transformed systems. When the Popov sector (0, k_p) is not equal to the Hurwitz sector, the stability sector (k_{1p}, k_{2p}) with $k_{1p}>0$ and $k_{2p}>k_{1p}$ can also be obtained by the application of the Popov's theorem to the transformed systems. It is, however, laborious to transform the system with every change of the value of k_{1p}. In these cases, it would be convenient if the stability sectors (k_1, k_2) could be determined directly, without transformations. The circle criterion [2] and the parabola test [3] serve just this purpose. In this paper, a new circle criterion directly applicable to ensure the absolute stability for the conditionally stable system is presented and some examples are shown comparing them with results obtained by other criteria.

2. System descriptions

The class of nonlinear feedback control systems considered will have the config-

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uration of Fig. 1, in which $G(s)=q(s)/p(s)$. It is assumed that $p(s)$ and $q(s)$ are polynomials of s and that the degree of $p(s)$ exceeds that of $q(s)$. It is also assumed that the non-linearity $\varphi(\sigma)$ is an arbitrary, single-valued, piecewise continuous function satisfying the condition $\varphi(0)=0$ and an additional inequality. In addition, it is assumed that the system satisfies the conditions necessary to ensure the existence and uniqueness of a solution of the differential equation governing the system.

In this paper, we say that the system is absolutely stable in the sector (k_1, k_2) if the zero solution of the governing differential equation is globally asymptotically stable for any $\varphi(\sigma)$ satisfying the inequality

$$k_1<\frac{\varphi(\sigma)}{\sigma}<k_2, \qquad (\sigma\neq 0,\ 0\leqq k_1<k_2) \tag{1}$$

3. Popov's theorem

Theorem 1 [Popov]

For the system shown in Fig. 1 to be absolutely stable in the sector $(0, k_p)$, it is sufficient that

(i) all the poles of $G(s)$ lie in the open left half s-plane, and

(ii) there exists a finite real number β such that for all $\omega\geqq 0$ the following inequality is satisfied

$$\mathrm{Re}G(j\omega)-\beta\omega\ \mathrm{Im}\ G(j\omega)+\frac{1}{k_p}\geqq 0 \tag{2}$$

In order to apply this theorem to a given system we usually use the Popov locus in which ω Im $G(j\omega)$ is plotted as a function of Re $G(j\omega)$ instead of plotting Im $G(j\omega)$ vs. Re $G(j\omega)$, as is done in the ordinary vector locus. It is clear that (2) holds if and only if the Popov locus lies to the right of a straight line passing through the $-1/k_p$ point having slope $1/\beta$ (Fig. 2).

If the nonlinear system shown in Fig. 1 is conditionally stable, it is impossible to find the stability sectors by the direct application of the Popov's theorem. In principle, there is no difficulty in finding the stability sectors. It is possible to make the trans-

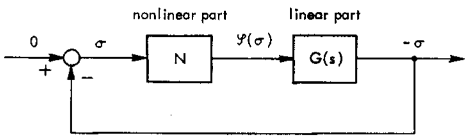

Fig. 1. Nonlinear feedback control system

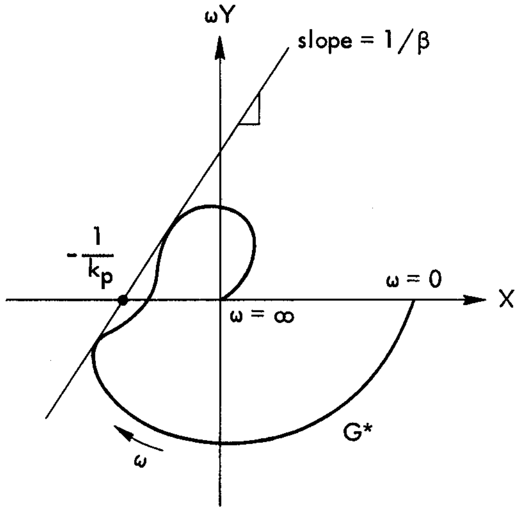

Fig. 2. Geometric representation of Popov's theorem

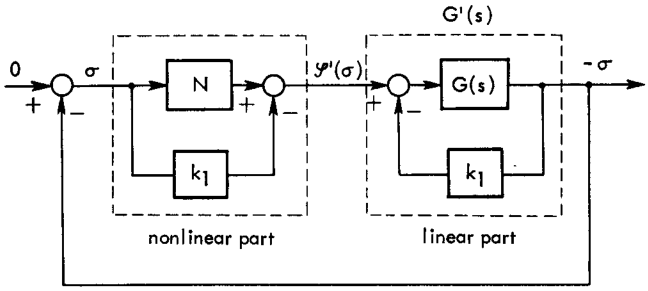

Fig. 3. Equivalent transformation of the system shown in Fig. 1

formation shown in Fig. 3. From Fig. 3,

$$
\begin{aligned}
G'(s) &= \frac{G(s)}{1+k_1 G(s)} \\
\varphi'(\sigma) &= \varphi(\sigma) - k_1\sigma
\end{aligned}
\tag{3}
$$

The Popov's theorem can be applied to the transformed system having a nonlinear characteristic $\varphi'(\sigma)$ in series with a linear transfer function $G'(s)$, provided that all the poles of $G'(s)$ lie in the open left half s-plane. If the transformed system is absolutely stable in the sector $(0, k_2-k_1)$, then the original system shown in Fig. 1 is absolutely stable in the sector (k_1, k_2). In this case, however, we must re-draw the Popov locus $G'^*(\omega)$ corresponding to $G'(s)$ with every change of the value of k_1. For example, in order to find the value k_1 to maximize the sector (k_1, k_2), a time-consuming process of trial and error is necessary.

It would thus be helpful if the stability sectors (k_1, k_2) with $0<k_1<k_2$ could be determined directly, without transformations. The new circle criterion serves just this purpose.

4. New circle criterion

Theorem 2

For the system shown in Fig. 1 to be absolutely stable in the sector (k_1, k_2), it is sufficient that

(I) all the poles of $G'(s)$ lie in the open left half s-plane and there exists a finite real number α such that,

(II) for $0\leqq\omega\leqq 1$, the Popov locus $G^*(\omega)$ lies outside of the circle which is centered on the point $C\left\{-\frac{1}{2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right), \alpha\right\}$ and has the intersections $P\left(-\frac{1}{k_1}, 0\right)$ and $Q\left(-\frac{1}{k_2}, 0\right)$ with the real axis, and

(III) for $\omega>1$, the locus $G^*(\omega)$ lies to the right of the tangent line of the circle C passing through the point Q.

Proof. Let X and Y be the real and imaginary parts of the frequency response $G(j\omega)$, that is,

$$\begin{aligned} X &= \mathrm{Re}G(j\omega) \\ Y &= \mathrm{Im}G(j\omega) \end{aligned} \tag{4}$$

From the Popov's theorem, it follows that the transformed system shown in Fig. 3 is absolutely stable in the sector $(0, k_2-k_1)$ if the condition (I) holds and there exists a finite real number β such that for all $\omega\geqq 0$

$$\mathrm{Re}G'(j\omega)-\beta\omega\,\mathrm{Im}\,G'(j\omega)+\frac{1}{k_2-k_1}\geqq 0 \tag{5}$$

Then the original system shown in Fig. 1 is absolutely stable in the sector (k_1, k_2). Therefore, in order to complete the proof of Theorem 2, it suffices to prove that (5) holds if the conditions (II) and (III) of Theorem 2 are satisfied.

Substituting (3) and (4) into (5),

$$X^2+\left(\frac{1}{k_1}+\frac{1}{k_2}\right)X+Y^2-\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\beta\omega Y+\frac{1}{k_1k_2}\geqq 0 \tag{6}$$

We consider (6) on the following two intervals of ω.

(i) Case 1: $\omega\in[0, 1]$

From (6), we have

$$X^2+\left(\frac{1}{k_1}+\frac{1}{k_2}\right)X+\omega^2Y^2-\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\beta\omega Y+\frac{1}{k_1k_2}\geqq(\omega^2-1)Y^2 \quad (7)$$

Since $(\omega^2-1)Y^2\leqq 0$, it follows that if the inequality

$$X^2+\left(\frac{1}{k_1}+\frac{1}{k_2}\right)X+\omega^2Y^2-\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\beta\omega Y+\frac{1}{k_1k_2}\geqq 0 \quad (8)$$

holds, then (7) holds. Furthermore, let

$$\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\beta=2a \quad (9)$$

then, from (8) we obtain

$$\left\{X+\frac{1}{2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)\right\}^2+(\omega Y-a)^2\geqq\frac{1}{4}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)^2+a^2 \quad (10)$$

This inequality represents the outside of the circle which is centered on the point $C\left\{-\frac{1}{2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right), a\right\}$ and has the intersections $P\left(-\frac{1}{k_1}, 0\right)$ and $Q\left(-\frac{1}{k_2}, 0\right)$ with the real axis. Thus, if the condition (II) of Theorem 2 is satisfied, then (5) holds for all $\omega \in [0, 1]$. (Fig. 4)

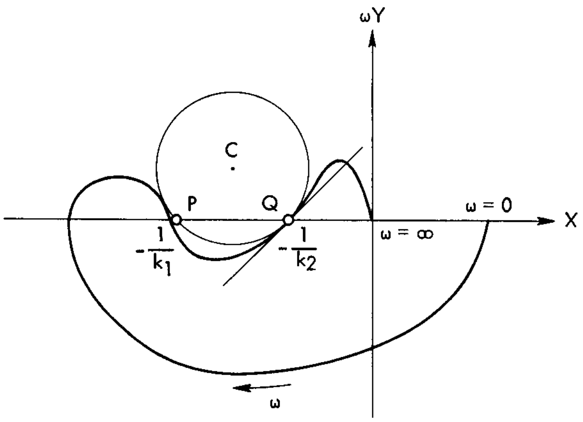

Fig. 4. Geometric representation of a new circle criterion

(ii) Case 2: $\omega \in [1, \infty]$

Since $Y^2\geqq 0$, it follows that if the inequality

$$X^2+\left(\frac{1}{k_1}+\frac{1}{k_2}\right)X-2a\omega Y+\frac{1}{k_1k_2}\geqq 0 \quad (11)$$

holds, then (6) holds.

From (11), we have

$$\left(X+\frac{1}{k_2}\right)^2+\left(\frac{1}{k_1}-\frac{1}{k_2}\right)X-2a\omega Y+\frac{1}{k_2}\left(\frac{1}{k_1}-\frac{1}{k_2}\right) \geqq 0 \tag{12}$$

Since $\left(X+\frac{1}{k_2}\right)^2 \geqq 0$, it follows that if the inequality

$$\left(\frac{1}{k_1}-\frac{1}{k_2}\right)X-2a\omega Y+\frac{1}{k_2}\left(\frac{1}{k_1}-\frac{1}{k_2}\right) \geqq 0 \tag{13}$$

holds, then (12) holds. The above inequality represents the right region of the tangent line of the circle C passing through the point Q. Thus, if the condition (III) of Theorem 2 is satisfied, then (5) holds for all $\omega > 1$.

Therefore, (5) holds for all $\omega \geqq 0$ if the conditions (II) and (III) of Theorem 2 are satisfied. This completes the proof of Theorem 2.

In order to apply this criterion to a given system we use the Popov locus $G^*(\omega)$ of the original system, and for the change of the value of k_1 or k_2 it is possible to obtain the stability sectors only by re-drawing the circle, without re-drawing the Popov locus. Therefore the application of the new circle criterion is less laborious than the application of the Popov's theorem to the transformed system.

Remarks:

(1) When all the poles of $G(s)$ lie in the open left half s-plane, in the limit as $k_1 \to 0$, the circle C and its tangent line passing through the point Q reduce to the Popov line.

(2) Although the tangent line appeared in the condition (II) of Theorem 2 can be replaced by the parabola given by (II), it is troublesome to draw the parabola.

(3) Change of time-scale: The behavior of the system shown in Fig. 1 is governed by the differential equation

$$G\left(\frac{d}{dt}\right)\varphi(\sigma)+\sigma=0 \tag{14}$$

Letting $\tau=\nu t$ with $\nu>0$,

$$G\left(\nu\frac{d}{d\tau}\right)\varphi(\sigma)+\sigma=0 \tag{15}$$

The transfer function of the linear part of (15) obtained by the Laplace transformation with respect to τ is $G(\nu s)$. Therefore, the system having the transfer function $G(s)$ is absolutely stable if the system having the transfer function $G(\nu s)$ in place of $G(s)$ is absolutely stable. This implies that it is possible to shift suitably the part of the Popov locus $G^*(\omega)$ corresponding to $\omega>1$.

5. Example

Example 1. Consider the system shown in Fig. 1 with

$$G(s)=\frac{s^2-0.1}{(s^2+1)(s+1)} \tag{16}$$

The Popov locus $G^*(\omega)$ is plotted for $\omega\geqq 0$ in Fig. 5. When $k_2=2$, the coordinate of

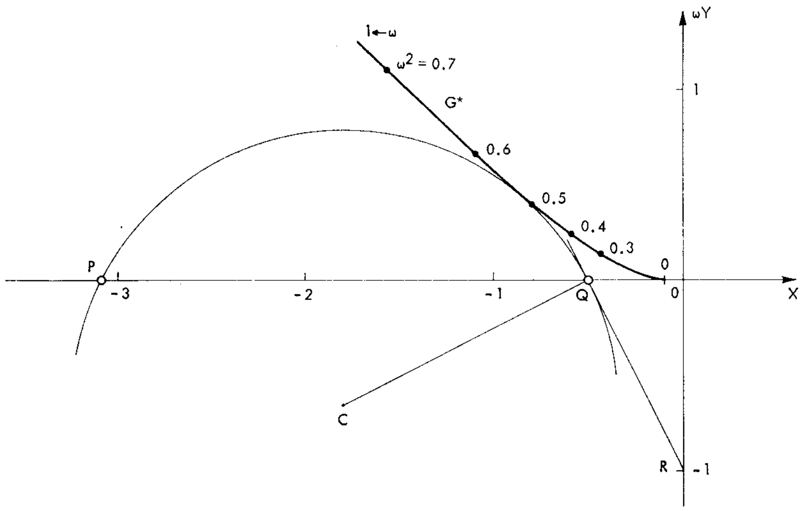

Fig. 5. Popov locus for Eq. (20)

the point Q is $(-1/2, 0)$. Thus, in this case, the circle C is centered on the straight line vertical to $\overline{\mathrm{QR}}$ with R(0, -1) and passing through the point Q. Using the trial and error method, we obtain the point P(-3.09, 0), that is, the value of k_1 is $1/3.09=0.324$. Fig. 6 shows the relation of k_2-k_1 vs. k_2 which has been obtained by repeating the same process with every change of the value of k_2. The result obtained by using the Popov's theorem on the transformed systems and the result obtained by the parabola test are also shown in Fig. 6. In addition, the result obtained by the circle criterion is plotted by the dashed line.

Example 2. Consider the system shown in Fig. 1 with

$$G(s)=\frac{(1+11s)^2}{100s^3(1+s)^2} \tag{17}$$

The vector locus is shown in Fig. 7. This system has the Hurwitz sector (0.059, 1.058),

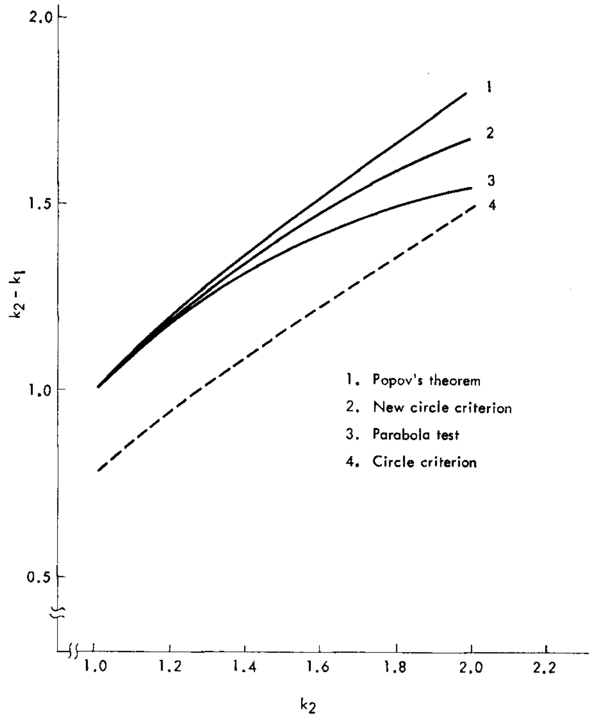

Fig. 6. Stability sectors of Example 1

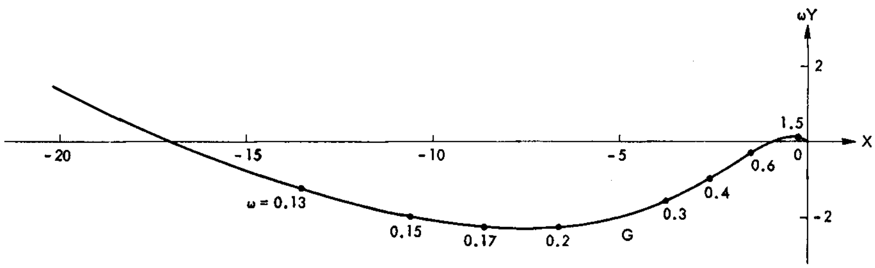

Fig. 7. Vector locus for Eq. (21)

that is, all the poles of $G'(s)$ lie in the open left half s-plane for all $k_1 \in (0.059, 1.058)$. Now, we make the change of time-scale for $\nu=0.2$. The form of vector locus then does not change at all. The angular frequency at any point on the vector locus for $\nu=0.2$ is 5 times as large as that on the original vector locus. The Popov locus for $\nu=0.2$ is shown in Fig. 8. The stability sector for $k_2=0.5$ is found to be (0.064, 0.5), because the circle intersects with the X-axis at $Q\left(-\frac{1}{0.5}, 0\right)$ and $P\left(-\frac{1}{0.064}, 0\right)$. In this case, while the parabola test can be applied, it is very difficult to obtain the best

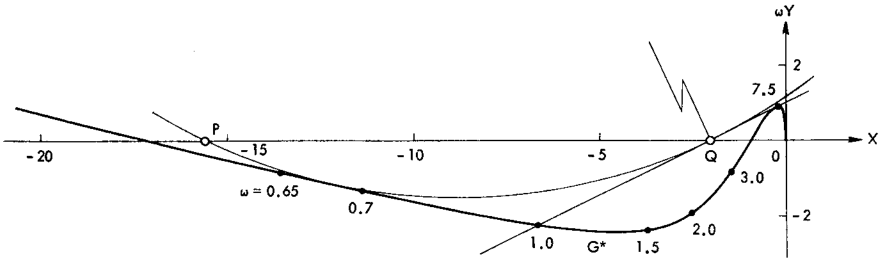

Fig. 8. Popov locus for Eq. (21) ($\nu=0.2$)

parabola graphically by means of trial and error method. In Example 1, although the best parabola have been obtained half analytically from the fact that the best parabola passes through the point R(0, -1), it is in general also very difficult to obtain the best parabola analytically. For $k_2=0.5$, the stability sector obtained by the circle criterion is (0.25, 0.5).

Example 3. Consider the system shown in Fig. 1 with

$$G(s)=\frac{(2s+1)(s+1)}{2s^3} \tag{18}$$

The Popov locus $G^*(\omega)$ is shown in Fig. 9. In this case, the Hurwitz sector is $\left(\frac{1}{3},\ \infty\right)$, and the stability sector obtained by the new circle criterion is also $\left(\frac{1}{3},\ \infty\right)$. In addition, the stability sector obtained by the parabola test is also $\left(\frac{1}{3},\ \infty\right)$. Using the circle criterion, we obtain, for say $k_1=0.4$, the stability sector (0.4, 1.07).

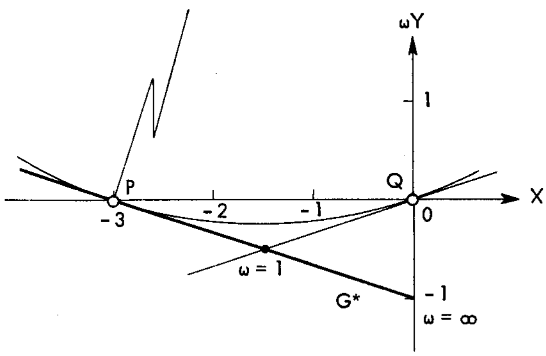

Fig. 9. Popov locus for Eq. (22)

6. Comparison of new circle criterion with Popov's theorem

In general, the stability sector (k_{1c}, k_{2c}) obtained by the new circle criterion is equal to, or smaller than the stability sector (k_{1p}, k_{2p}) obtained by using the Popov's theorem on the transformed system, provided $k_{1c}=k_{1p}$ or $k_{2c}=k_{2p}$. The difference between (k_{1c}, k_{2c}) and (k_{1p}, k_{2p}) can be estimated by means of the following method.

In the Popov's theorem, the function

$$F_p(\omega, \beta, k_1, k_2) = X^2+\left(\frac{1}{k_1}+\frac{1}{k_2}\right)X+Y^2-\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\beta\omega Y + \frac{1}{k_1 k_2} \tag{19}$$

is considered. The left side of the equation

$$\inf_{\omega\geqq 0} F_p(\omega, \beta, k_1, k_2) = 0 \tag{20}$$

is the function of β, k_1 and k_2. If k_2 has a maximum k_{2p} at $\beta=\beta_p$ for a fixed $k_1=k_{1p}$ such that all the poles of $G'(s)$ lie in the open left half s-plane, then from the Popov's theorem the stability sector for $k_1=k_{1p}$ is (k_{1p}, k_{2p}), and there exists more than one ω_p such that

$$\lim_{\omega\to\omega_p} F_p(\omega, \beta_p, k_{1p}, k_{2p}) = 0 \tag{21}$$

These are denoted by the set $\{\omega_p\}$. The relation of k_{1p} and k_{2p} is as follows:

$$\begin{aligned}\frac{1}{k_{2p}-k_{1p}} &= \min_{\beta} \sup_{\omega\geqq 0}\{-\mathrm{Re}G'(j\omega)+\beta\omega\,\mathrm{Im}\,G'(j\omega)\}\\ &\geqq \min_{\beta} \max_{i\in\{1,2\}}\{-\mathrm{Re}G'(j\omega_i)+\beta\omega_i\,\mathrm{Im}\,G'(j\omega_i)\}\end{aligned} \tag{22}$$

Let

$$\begin{aligned}&X_i = \mathrm{Re}G(j\omega_i),\quad Y_i = \mathrm{Im}\,G(j\omega_i)\\ &G'(j\omega_i) = \frac{G(j\omega_i)}{1+k_{1p}G(j\omega_i)}\\ &\qquad i=1,\ 2\end{aligned} \tag{23}$$

and choose the values of ω_1 and ω_2 to satisfy

$$\begin{aligned}\omega_1 Y_1 > 0\\ \omega_2 Y_2 < 0\end{aligned} \tag{24}$$

From (22), (23) and (24), it follows that

$$1+(k_{1p}+k_{2p})X_1+k_{1p}k_{2p}(X_1^2+Y_1^2)$$
$$-\frac{\omega_1 Y_1}{\omega_2 Y_2}\{1+(k_{1p}+k_{2p})X_2+k_{1p}k_{2p}(X_2^2+Y_2^2)\}\geqq 0 \tag{25}$$

From (25) and the inequality $0\leqq k_h\leqq k_{1p}\leqq k_{2p}\leqq k_H$, it follows that

$$\begin{aligned} k_{2p} &\leqq H_2(k_{1p}) \\ k_{1p} &\geqq H_1(k_{2p}) \end{aligned} \tag{26}$$

The equations hold in (25) and (26) if the equation holds in (22). On the other hand, in the new circle criterion, the function

$$F_c(\omega, \beta, k_1, k_2)=$$
$$\begin{cases} X^2+\left(\frac{1}{k_1}+\frac{1}{k_2}\right)X+\omega^2Y^2-\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\beta\omega Y+\frac{1}{k_1k_2}, & 1\geqq\omega\geqq 0 \\ \left(\frac{1}{k_1}-\frac{1}{k_2}\right)X-\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\beta\omega Y+\frac{1}{k_2}\left(\frac{1}{k_1}-\frac{1}{k_2}\right), & \omega>1 \end{cases} \tag{27}$$

is considered. The left side of the equation

$$\inf_{\omega\geqq 0} F_c(\omega, \beta, k_1, k_2)=0 \tag{28}$$

is the function of β, k_1 and k_2. If k_2 has a maximum k_{2c} at $\beta=\beta_c$ for a fixed $k_1=k_{1c}$ such that all the poles of $G'(s)$ lie in the open left half s-plane, then from the new circle criterion the stability sector for $k_1=k_{1c}$ is (k_{1c}, k_{2c}), and there exists more than one ω_c such that

$$\lim_{\omega\to\omega_c} F_c(\omega, \beta_c, k_{1c}, k_{2c})=0 \tag{29}$$

These are denoted by the set $\{\omega_c\}$.

All the conditions of the Popov's theorem are satisfied if all the conditions of the new circle criterion are satisfied. Therefore, for $k_{1c}=k_{1p}$, we have

$$k_{2c}\leqq k_{2p}\leqq H_2(k_{1p}) \tag{30}$$

and for $k_{2c}=k_{2p}$, we have

$$k_{1c}\geqq k_{1p}\geqq H_1(k_{1p}) \tag{31}$$

When $\omega_1\epsilon\{\omega_p\}$ and $\omega_2\epsilon\{\omega_p\}$, the equation holds in (22), therefore the equations hold in (26). If the terms deleted on the way to generate the new circle criterion are small, then

$$F_p(\omega, \beta, k_1, k_2)\fallingdotseq F_c(\omega, \beta, k_1, k_2) \tag{32}$$

Thus there exist elements of the set $\{\omega_p\}$ whose values are nearly equal to the values of elements of $\{\omega_c\}$. It follows that

$$\begin{aligned} k_{2p} &\fallingdotseq H_2(k_{1p}) \\ k_{1p} &\fallingdotseq H_1(k_{2p}) \end{aligned} \tag{33}$$

with $\omega_1 \epsilon \{\omega_c\}$ and $\omega_2 \epsilon \{\omega_c\}$.

Therefore, for say $k_{2c}=k_{2p}$, the difference between (k_{1c}, k_{2c}) and (k_{1p}, k_{2p}) can be estimated by comparing k_{1c} with $H_1(k_{2p})$ obtained from (25) with ω_1, $\omega_2 \epsilon \{\omega_c\}$.

In Example 1, for $k_{2c}=k_{2p}=2$, from Fig. 5 we obtain $\omega_1=\sqrt{0.5}$ and $\omega_2=\infty$. Hence from (25) we obtain

$$k_{1p} \geqq 0.179$$

Thus the sector (k_{1p}, k_{2p}) is equal to, or smaller than (0.179, 2). On the other hand, the sector (k_{1c}, k_{2c}) is (0.324, 2). Therefore the difference of both is small.

In Example 2, for $k_{2c}=k_{2p}=0.5$, from Fig. 8 we obtain $\omega_1=7.5$ and $\omega_2=0.7$. Hence from (25) we obtain

$$k_{1p} \geqq 0.063$$

Thus the sector (k_{1p}, k_{2p}) is equal to, or smaller than (0.063, 0.5). On the other hand, the sector (k_{1c}, k_{2c}) is (0.064, 0.5). Therefore the difference of the both sectors is too small.

In Example 3, from the fact that (k_{1c}, k_{2c}) is equal to the Hurwitz sector $\left(\frac{1}{3}, \infty\right)$, it is clear that (k_{1p}, k_{2p}) is also $\left(\frac{1}{3}, \infty\right)$.

7. Conclusion

A new circle criterion to determine the stability sectors within which the nonlinearity is to lie in order to assure the absolute stability of nonlinear control system has been derived and several examples have been given. In order to determine the stability sectors (k_1, k_2) of conditionally stable system, while it is possible in principle to apply the Popov's theorem to the transformed system, it is very laborious to re-draw the Popov locus with every change of the value of k_1 or k_2. However, the new circle criterion, using one Popov locus $G^*(\omega)$, determines directly the stability sectors (k_1, k_2). Moreover, this criterion not only often gives better results compared with the parabola test, but also is more easily applied.

While the stability sectors obtained by the new circle criterion are usually smaller than the stability sectors obtained by the application of the Popov's theorem to the transformed systems, the difference of both sectors can be easily estimated and is often small.

This criterion, as shown in the examples, usually gives better results than the circle criterion. However, while the circle criterion can be applied in general to time-varying nonlinear control systems, the new circle criterion, as well as the Popov's theorem and the parabola test, can be applied only to time-invariant nonlinear control systems.

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