A Method for Construction of Suboptimal Nonlinear Regulators

By

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Abstract

A method is proposed for an approximate construction of the optimal state regulator for an autonomous nonlinear system with quadratic performance index. This method is based upon the instantaneous linearization technique developed by Pearson. The nonlinear system is approximated by a state-dependent linear system. First, a theorem is established to give a necessary condition for Pearson's control law to be optimal. Secondly, by making use of this condition, a systematic procedure is presented to determine a suboptimal feedback control for a second-order system. A minimax algorithm is used to design the linearized model closely approximating the original system over all state variables considered. The validity of the present method is shown by examining typical examples.

Introduction

This paper deals with an approximate design of the optimal feedback control of autonomous nonlinear systems. As a reasonable performance measure, the standard infinite time quadratic performance index is considered. The control function is required to have the following properties: The control is a function of the present state of the system only; i.e. it is in a time-invariant form and the control is close to the optimal for a prescribed set of initial states near the equilibrium state.

Some representative methods for the nonlinear regulator problem are power-series expansion, parameter optimization and instantaneous linearization. Al'brekht[1] presented the optimal feedback control of a nonlinear analytic system as a formal power series in states by considering Liapunov functions. Lukes[2] extended the work of Al'brekht and relaxed the analyticity condition to twice continuous differentiability. Recently, with the introduction of an

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extended Liapunov equation, a recursive formula for a power-series solution was given by the authors[3]. The method of power-series expansion provides an effective tool for systems with small nonlinearity. However, when the nonlinearity is large, the method may have convergence difficulties and the effect of truncation of higher-order terms is sometimes serious.

The parameter optimization technique[4,5] is based upon computer-aided adjustment of the unknown parameters included in the solution presumed properly. The optimal values of the parameters are determined so as to satisfy the necessary condition for optimality at a finite number of reference points along the nominal trajectory. Therefore the solution obtained depends strongly on the choice of the reference point or the nominal trajectory.

The method of instantaneous linearization proposed by Pearson[6] approximates the nonlinear system by a state-dependent linear model. The resultant problem may then be solved using linear techniques. However, the solution is not optimal in general because of the arbitrariness of the linearized model. Several investigations on this method were reported in Refs. 7–10). Of these, Garrard et al.[7] and Burghart[9] obtained the feedback gain matrix of power-series form for a linear model given in advance. On the other hand, Kriechbaum and Noges[10] proposed a method for determining the model by the least squares approximation. The purpose of this paper is to improve the instantaneous linearization technique from another point of view.

In this paper, we first establish a theorem which states a necessary condition for Pearson's control law to be optimal. By using this condition, a systematic procedure is presented to construct a suboptimal nonlinear regulator for a second-order system. A minimax algorithm is utilized to optimally design the state-dependent linear model for all states considered. Illustrative examples attached show several features of the method.

Problem Statement

Consider dynamical systems governed by the differential equation

$$\dot{x}=f(x)+Bu, \qquad x(0)=x^0 \in \Omega \tag{1}$$

with the associated performance index

$$J=\frac{1}{2}\int_0^\infty (x'Qx+u'Ru)dt \tag{2}$$

where x is the n dimensional state vector, u the m dimensional control vector; B is an $n \times m$ constant matrix. The n vector function f is a continuously differentiable function of x, satisfying $f(0)=0$. Ω is a given set containing the origin in the state space. Q and R are constant symmetric positive definite matrices. A prime denotes transposition of a vector or a matrix.

The problem is to find a time-invariant, feedback control law $u(x)$ which makes the performance index (2) as small as possible for all initial states in Ω. An approximate approach to the problem is developed by Pearson[6]. The basic idea is the following adoption of a linearized model of the system (1) :

$$\dot{x}=A(x)x+Bu. \tag{3}$$

Note that, when $n>1$, the $n\times n$ state-dependent matrix A can not uniquely be determined given a nonlinear function $f(x)$. Using a technique similar to that for the linear systems gives

$$u=-R^{-1}B'P(x)x \tag{4}$$

where P is the positive definite solution of

$$PA+A'P-PBR^{-1}B'P+Q=0. \tag{5}$$

The control function given by (4) is conveniently a feedback controller. The resulting feedback system, i.e.

$$\dot{x}=S(x)x, \quad S(x)\triangleq A(x)-BR^{-1}B'P(x) \tag{6}$$

is, at least locally, asymptotically stable. However, the best possible solution can not be obtained, except for a one-dimensional problem, because of the arbitrariness in the system description. Therefore, the method of instantaneous linearization should be improved on this point.

The Basic Result

The fundamental result for this problem is as follows.

Theorem

Assume that the algebraic Riccati equation (5) has a positive definite solution. Then, in order for the control law (4) to be optimal it is necessary that

$$G(x)S(x)x=0 \quad \text{for all } x \tag{7}$$

where G is the $n\times n$ skew-symmetric matrix whose i, j element is

$$G_{ij}=\sum_{k=1}^{n}\left(\frac{\partial P_{ik}}{\partial x_j}-\frac{\partial P_{jk}}{\partial x_i}\right)x_k \quad (i, j=1, 2, \cdots\cdots, n). \tag{8}$$

Proof

We derive a necessary condition for optimality by using the minimum principle. The Hamiltonian of the problem is given by

$$H=\frac{1}{2}(x'Qx+u'Ru)+p'(Ax+Bu) \tag{9}$$

where p is the n dimensional costate vector satisfying the equation $\dot{p}=-\partial H/\partial x$ or in component forms

$$\dot{p}_i=-(Qx+A'p)_i-x'\frac{\partial A'}{\partial x_i}p \quad (i=1,2,\cdots\cdots,n) \tag{10}$$

with the boundary condition

$$\lim_{t_f\to\infty} p(t_f)=0. \tag{11}$$

The optimal control is given by

$$u=-R^{-1}B'p. \tag{12}$$

Here we assume the vector p in the form:

$$p=P(x,t)x. \tag{13}$$

Substitution of (13) into (10) and use of (3) and (12) yields

$$\sum_{j=1}^{n}\Big(\frac{\partial P}{\partial t}+PA+A'P-PBR^{-1}B'P+Q\Big)_{ij}x_j$$
$$+\sum_{j,k,l=1}^{n}\Big[\frac{\partial P_{ij}}{\partial x_k}(A-BR^{-1}B'P)_{kl}+\frac{\partial A'_{jk}}{\partial x_i}P_{kl}\Big]x_jx_l=0 \quad (i=1,2,\cdots\cdots,n). \tag{14}$$

Equating the first term in (14) to zero gives

$$\frac{\partial P}{\partial t}+PA+A'P-PBR^{-1}B'P+Q=0. \tag{15}$$

It is easily observed that P is symmetric. If the final time, t_f, is sufficiently large, the matrix P may be assumed to reduce to a form free of explicit dependence on t as $t\to-\infty$[3]. We here consider only a problem for obtaining the time-invariant matrix P. Then (15) reduces to the algebraic equation (5). The positive definite solution of (5), if it exists, guarantees asymptotic stability of the system (6). Consequently, the boundary condition (11) holds.

Differentiation of (5) with respect to x_i gives

$$P\frac{\partial A}{\partial x_i}+\frac{\partial A'}{\partial x_i}P=-\Big(\frac{\partial P}{\partial x_i}S+S'\frac{\partial P}{\partial x_i}\Big) \tag{16}$$

where the matrix S is defined by (6). From (16) we obtain

$$\sum_{j,k,l=1}^{n}\frac{\partial A'_{jk}}{\partial x_i}P_{kl}x_jx_l=-\sum_{j,k,l=1}^{n}\frac{\partial P_{jk}}{\partial x_i}S_{kl}x_jx_l. \tag{17}$$

Substitution of (5) and (17) into (14) and use of (6) results in

$$\sum_{k=1}^{n}\Big[\sum_{j=1}^{n}\Big(\frac{\partial P_{ij}}{\partial x_k}-\frac{\partial P_{kj}}{\partial x_i}\Big)x_j\Big]\dot{x}_k=0 \quad (i=1,2,\cdots\cdots,n). \tag{18}$$

Equation (18) is rewritten into the vector-matrix form:

$$G\dot{x}=GSx=0 \tag{19}$$

where G is the $n \times n$ matrix defined by (8). Then, in order for (4) to satisfy the optimality condition, (7) must hold for all x. Q.E.D.

Remarks 1

When $n=1$, the condition (7) is always valid. Hence the method of instantaneous linearization gives the optimal solution for a one-dimensional problem. This was previously proved in Ref. 11) by a direct calculation of the solution.

Remarks 2

If the solution of (5) is a constant matrix, the condition (7) holds automatically. Then the solution is optimal. Such an example is as follows: Consider a norm-invariant system

$$\dot{x}=f(x)+u \qquad \text{with} \quad x'f(x)=0 \quad \text{for all } x \tag{20}$$

and the performance index (2) where $Q=I$ and $R=rI$ (I; $n \times n$ identity matrix). If the state-dependent matrix A is chosen as skew-symmetric, we obtain from (5) the optimal solution $P=r^{1/2}I$. A special case of the system (20) is discussed in Ref. 3).

Remarks 3

The existence of the positive definite solution of (5) depends on the form of $A(x)$. If the associated linear system

$$\dot{x}=A(0)x+Bu \tag{21}$$

is completely controllable, there exists the unique solution of (5) which is positive definite, at least, for small x.

Remarks 4

It follows from the proof mentioned above that, in the instantaneous linearization method, the term $G\dot{x}$ is disregarded in the expression derived from the optimality condition. A similar question also arises in the method of power-series expansion, in which the higher-order terms are truncated[3]. It is noted that, if the system (6) is asymptotically stable, the term $G\dot{x}$ approaches zero as $t \to \infty$. Therefore, the effect of the error caused by the disregarded term is dominant only in the initial transient interval. Consequently, the norm of the matrix G at $t=0$ or, equivalently, at $x=x^0$ may provide a good measure of suboptimality of the control.

An Approximate Solution for a Second-Order System

It is generally difficult to obtain the exact solution satisfying both (5) and (7). The solution of (5) depends on the state-dependent linear model. Then we are now skillfully to choose the form of A in such a way that the solution of (5) satisfies (7) approximately. In this section our consideration is confined to the problem of optimizing the second-order system

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2) + u \end{aligned}\right\} \tag{22}$$

with respect to the performance index

$$J = \frac{1}{2}\int_0^\infty (q_1 x_1^2 + q_2 x_2^2 + u^2)dt \tag{23}$$

where q_1, $q_2 > 0$ and $f(0, 0) = 0$.

The state-dependent matrix A for the system (22) is here assumed in the form:

$$A(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ a(x_1, x_2) & b(x_1, x_2) \end{pmatrix} \tag{24}$$

where a and b are the, as yet, unspecified functions satisfying

$$a(x_1, x_2)x_1 + b(x_1, x_2)x_2 = f(x_1, x_2). \tag{25}$$

Thus the solution of (5) is given by

$$\left.\begin{aligned} &P = \begin{pmatrix} P_1 & P_3 \\ P_3 & P_2 \end{pmatrix} \\ &\text{where} \\ &P_3 = a + \sqrt{a^2 + q_1}, \\ &P_2 = b + \sqrt{b^2 + 2a + q_2 + 2\sqrt{a^2 + q_1}}, \\ &P_1 = P_2 P_3 - aP_2 - bP_3, \end{aligned}\right\} \tag{26}$$

which is the unique positive definite solution for arbitrary functions a and b. From (4) the suboptimal control is given by

$$u = -f - x_1\sqrt{a^2 + q_1} - x_2\sqrt{b^2 + 2a + q_2 + 2\sqrt{a^2 + q_1}}. \tag{27}$$

The condition (7) for this problem is equivalent to

$$g \triangleq \left(\frac{\partial P_1}{\partial x_2} - \frac{\partial P_3}{\partial x_1}\right)x_1 + \left(\frac{\partial P_3}{\partial x_2} - \frac{\partial P_2}{\partial x_1}\right)x_2 = 0. \tag{28}$$

The functions a and b are determined as follows: First, taking into account the system nonlinearity, we assume the proper forms of a and b as the functions of x including the unknown parameters $\alpha_j (j = 1, 2, \cdots\cdots, N)$:

$$a = a(x_1, x_2 ; \alpha_j), \quad b = b(x_1, x_2 ; \alpha_j). \tag{29}$$

Then P_i given by (26) also includes the parameters α_j and consequently g defined by (28) is a function of x_i and α_j. Secondly, the unknown parameters

α_j should be determined so that the solution (26) may satisfy (28) as precisely as possible for all $x \epsilon \Omega$, where Ω is a prescribed set in the state space. For this end, we consider the following minimax problem:

$$g^2(x_1^*, x_2^*; \alpha_j^*) = \min_{\alpha_j} \max_{(x_1, x_2) \epsilon \Omega} g^2(x_1, x_2; \alpha_j). \tag{30}$$

That is, the parameters α_j are adjusted so as to minimize the effect of the worst-case error caused by the approximation.

There have been a number of studies on the algorithm for minimax problems[12~15]. Of these, a computationally simple procedure has been developed by Heller and Cruz[15]. The algorithm is valid for both saddle point and nonsaddle point minimax problems. The various calculations in the algorithm can be implemented using standard linear and quadratic programming. Therefore it is conveniently applicable to the present problem, i.e., to obtain the solution of (30).

Remarks 5

When the system nonlinearity $f(x)$ is characterized by a polynomial function in the state, the simple representation of a and b is proposed in the following way: If f is a polynomial of the degree q in x, a and b should be assumed to be polynomials of the degree $q-1$ in x. This is conjectured from the result obtained by the method of power-series expansion[3].

Remarks 6

Due to the discussion in Remarks 4, (30) may be approximately replaced by

$$g^2(x_1^*, x_2^*; \alpha_j^*) = \min_{\alpha_j} \max_{(x_1, x_2) \epsilon \partial\Omega} g^2(x_1, x_2; \alpha_j) \tag{31}$$

where $\partial\Omega$ is the boundary of the region Ω. This may reduce much work for performing the maximization numerically.

Illustrative Examples

Two examples are presented to illustrate the application of the present method. The suboptimal controls are compared with those obtained by other methods.

Example 1

As an example of (22), consider the Duffing type of equation given by

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^3 + u \end{aligned}\right\} \tag{32}$$

with the performance index

$$J=\frac{1}{2}\int_0^\infty (x_1^2+x_2^2+u^2)dt. \tag{33}$$

The set of the initial states to be controlled is given by

$$\Omega=\{(x_1, x_2) : x_1^2+x_2^2 \leq r^2\}. \tag{34}$$

The functions a and b are here assumed in the form:

$$a=-x_1^2-\alpha_1 x_1 x_2-\alpha_2 x_2^2, \quad b=\alpha_1 x_1^2+\alpha_2 x_1 x_2, \tag{35}$$

α_1 and α_2 being adjustable parameters. For simplicity, only the quadratic terms are included in a and b. The nonlinear function in (32) is independent of x_2. However it is noted that $\alpha_1=\alpha_2=0$ is not the optimal values of the parameters. In fact, putting $\alpha_1=\alpha_2=0$ in (35) yields $a=-x_1^2$ and $b=0$. Thus P_i given by (26) do not contain x_2 and consequently

$$g=-(x_2+x_1 P_2)\frac{\partial P_2}{\partial x_1} \neq 0, \tag{36}$$

which does not satisfy the optimality condition of the theorem. On the other hand, when $\alpha_1=\alpha_2=0$ the control law (27) with (35) stabilizes the system (32) in the large. Therefore, from the viewpoint of the stability property, the values of α_1 and α_2 are expected to be small.

The optimal values α_1^* and α_2^* are determined by solving the minimax problem (31). For convenience (31) is rewritten, in terms of polar coordinates, as

$$\Gamma(\theta^*, \alpha_1^*, \alpha_2^*) \triangleq \min_{\alpha_j} \max_{\theta \in \Theta} g^2(r\cos\theta, r\sin\theta\,; \alpha_1, \alpha_2). \tag{37}$$

In (37) the maximization is performed, for simplicity, with respect to θ in a set Θ of a finite number of points on $\partial\Omega$, i.e.,

$$\Theta \triangleq \{\theta_n(n=1, 2, \cdots\cdots, L) : \theta_n=(n-1)\pi/L\}. \tag{38}$$

Note that only the upper-half state plane is considered because of the symmetry of the problem.

As an example, let $r=2$ and $L=20$. Figure 1 illustrates the convergence rate of Γ obtained by the minimax gradient technique. Figure 2 shows the stepwise paths toward the minimax point of Γ on the $\alpha_1\alpha_2$ plane. In this figure are also shown the iso-max g^2 curves in the neghborhood of the solution. The solution of the problem (37) is thus obtained as

$$\left.\begin{array}{l}\alpha_1^*=0.0582, \quad \alpha_2^*=0.0250, \\ \theta^*=0,\ 7\pi/20 \text{ and } 8\pi/20, \quad \Gamma=1.0215,\end{array}\right\} \tag{39}$$

which is a nonsaddle point solution[15].

Now the quality of the control law obtained here is examined. Figure 3

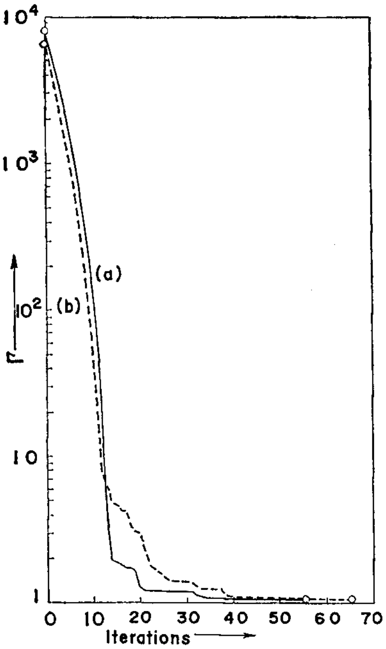

Fig. 1. Convergence of $\Gamma(\theta, \alpha_1, \alpha_2)$.
(a) The case where the initial estimate of the parameters is $\alpha_1^0=\alpha_2^0=1$.
(b) The case where the initial estimate of the parameters is $\alpha_1^0=\alpha_2^0=-1$.

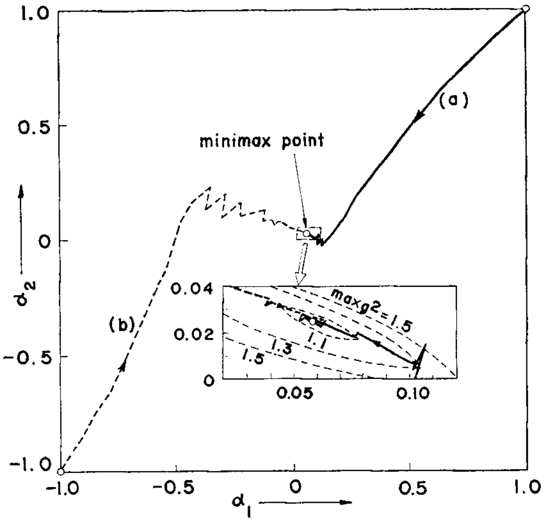

Fig. 2. Convergence of the parameters α_1 and α_2.
(a) The case where the initial estimate of the parameters is $\alpha_1^0=\alpha_2^0=1$.
(b) The case where the initial estimate of the parameters is $\alpha_1^0=\alpha_2^0=-1$.
Equivalue lines of max g^2 are shown by dotted lines.

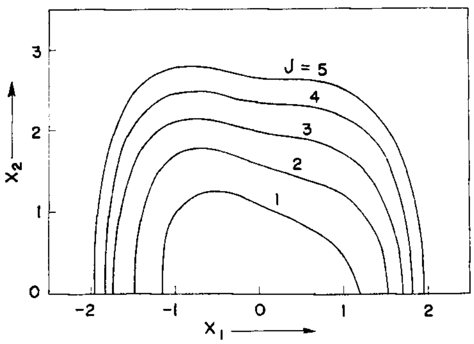

Fig. 3. Iso-J curves on the x_1x_2 plane.

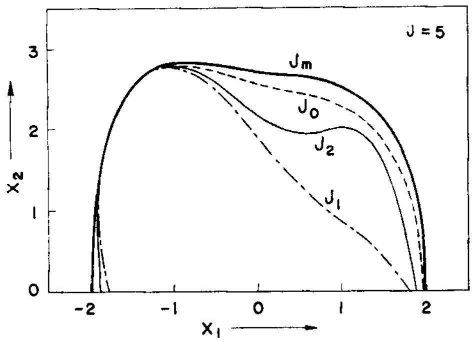

Fig. 4. Comparison of the iso-J curve obtained by the present method (J_m; thick line) with those by the power-series solution, where
J_0: by the zero-order control,
J_1: by the first-order control,
J_2: by the second-order control.

shows the iso-J curves on the x_1x_2 plane. The curves are shown only on the upper-half plane, because they are symmetrical about the origin. In Fig. 4 the result by the present procedure is compared with that obtained previously, in which the suboptimal control was calculated up to the second order by the method of power-series expansion[3]. In this example the higher-order approximations do not necessarily improve the performance of the control. The present method generates a control law which gives a better performance index than any of these approximations.

Example 2

The validity of the present method is compared with a number of other known methods using a system described by the van der Pol equation:

$$\left.\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=-x_1+(1-x_1^2)x_2+u.\end{aligned}\right\} \quad (40)$$

The performance index of (33) is used. The set of the states to be controlled is given by (34) where $r=2$.

This example was also solved by Pearson[6]. His assumption for the functions a and b is

$$a=-1, \quad b=1-x_1^2. \quad (41)$$

On the other hand, Kriechbaum and Noges[10] proposed the least squares approximation of the nonlinear function by hyperplane through the origin and obtained

$$a=-\frac{2}{7}x_1x_2-1, \quad b=-\frac{5}{42}x_1^2+1. \quad (42)$$

Then we simply assume the form of a and b as follows:

$$a=-1-\alpha x_1x_2, \quad b=1+(\alpha-1)x_1^2. \quad (43)$$

After applying the minimax algorithm the adjustable parameter is obtained:

$$\alpha^*=0.4013. \quad (44)$$

There have been a number of numerical results of various methods applied to this example. Typical solutions have been compared. Table 1, quoted from Ref. 10, shows the values of the performance index obtained by use of the various control laws. The method developed here is added to this comparison. Pearson's method, Kriechbaum et al.'s method and the present technique give a very similar performance for all initial conditions used. Especially the result obtained by the present method is slightly better for the initial states in Ω.

Table 1. Comparison of values of performance index for various approximations.

Method	Performance index: Initial conditions $x_1(0)=0.5$, $x_2(0)=0.5$	$x_1(0)=1.0$, $x_2(0)=1.0$	$x_1(0)=3.0$, $x_2(0)=3.0$	$x_1(0)=4.5$, $x_2(0)=4.5$
Linearization	0.8008	2.7026	18.2409	57.1329
Power-series expansion (up to the first order)	0.7971	2.5771	∞	∞
Approximate solution to Hamilton-Jacobi equation (Garrard et al.)	0.7982	2.6335	20.2017	270.5881
Parameter optimization (Durbeck)	0.9194	3.1259	18.3033	63.4517
Instantaneous linearization (Pearson)	0.7977	2.6188	15.1945	52.1445
Linear approximation by hyperplane through origin (Kriechbaum et al.)	0.7992	2.6427	15.4599	52.7139
Present method	0.7974	2.5732	15.6317	55.7642

Conclusion

A technique for the suboptimal design of a nonlinear state regulator for an autonomous nonlinear system with quadratic performance index has been developed. The method is based upon the approximation of the nonlinear system by the state-dependent linear model. A theorem is established to give the optimality condition for the control law derived from the instantaneous linearization approach. The straightforward application of this condition to the synthesis of the optimal control is generally difficult; however, the condition may effectively be used to specify the linearized model for a class of nonlinear systems.

A systematic procedure is proposed to construct the suboptimal control laws for second-order systems. It requires the determination of the adjustable parameters included in the linearized model. This parameter optimization is carried out by the minimax algorithm in such a way that the optimality condition holds as precisely as possible for all state variables investigated. Consequently the effectiveness of the proposed control law may be assured within a prescribed region in the state space. In addition, using the theorem derived simplifies the parameter selection procedure because the parameter optimization is performed with respect to a criterion function instead of a criterion functional. By examining typical examples, the present technique compares favourably with other suboptimal control methods. An extension of the present results to higher-order systems is being investigated.

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