Generalized Hypo-Elasticity with Thermal Influence I. Constitutive Equations

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The constitutive equations of the generalized hypo-elastic materials with thermal influence are derived and defined from those of the isotropic thermo-elastic material with invertible stress relation. In order to remedy the infinite magnitude of propagation of thermal disturbance the deformation gradient, the temperature and the summed history of temperature gradient are adopted as independent state variables. Then the constitutive equations are defined from the heuristic basis. The restriction by the Clausius-Duhem inequality is discussed and the coefficients of constitutive equations are represented by the theorem of isotropic functions.

1. Introduction

In order to abandon the idea of natural state of material, Truesdell[1] proposed the materials which have the constitutive equation:

$$\mathring{\boldsymbol{T}}=\mathsf{H}(\boldsymbol{T})\{\boldsymbol{D}\}, \tag{1.1}$$

where

$$\mathring{\boldsymbol{T}}\equiv\dot{\boldsymbol{T}}-\boldsymbol{W}\boldsymbol{T}+\boldsymbol{T}\boldsymbol{W} \tag{1.2}$$

is the co-rotational time rate of Cauchy stress $\boldsymbol{T}$,

$$\boldsymbol{D}\equiv\frac{1}{2}(\boldsymbol{L}+\boldsymbol{L}^{T}), \qquad \boldsymbol{W}\equiv\frac{1}{2}(\boldsymbol{L}-\boldsymbol{L}^{T}) \tag{1.3}$$

are, respectively, the stretching and the spin tensor, $\boldsymbol{L}\equiv\operatorname{grad}\dot{\boldsymbol{x}}$ is the velocity gradient, and a parentheses { } indicates linear dependence upon the argument.

All variables entering (1.1) are mechanical variables. As an attempt to include the thermal influence Tokuoka[2] proposed a *Thermo-hypo-elasticity* and he also discussed the *mechanico-thermal failure conditions*. Here a generalization of hypo-elasticity with thermal influence is proposed and its constitutive equations are represented explicitly by the theorem proved by Wang.[3]

2. Constitutive Equations of a Generalized Hypo-Elasticity

Usually a thermo-elastic material is defined by that its Helmholtz free energy ψ, Cauchy stress tensor $\boldsymbol{T}$, specific entropy η and heat flux vector $\boldsymbol{q}$ are functions

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of the present values of the deformation gradient $\boldsymbol{F}$, the temperature θ and the temperature gradient $\boldsymbol{g}\equiv\mathrm{grad}\,\theta$. Refer to Coleman and Noll[4] and Coleman.[5] With respect to the propagation of thermal disturbance the constitutive equations defined above lead to a parabolic differential equation, which denotes that a thermal disturbance at any point in the body is felt instantly at every point of the body or the speed of propagation of disturbance is infinite. In order to remedy this unpleasant feature Gurtin and Pipkin[6] assumed a rigid material, where ψ, η and $\boldsymbol{q}$ are functions of θ and functionals of summed histories of θ and $\boldsymbol{g}$.

Here an isotropic thermo-elastic material is defined by the following constitutive equations:

$$\psi=\hat{\psi}\,(\boldsymbol{B},\theta,\boldsymbol{h}), \tag{2.1a}$$

$$\boldsymbol{T}=\hat{\boldsymbol{T}}\,(\boldsymbol{B},\theta,\boldsymbol{h}), \tag{2.1b}$$

$$\eta=\hat{\eta}\,(\boldsymbol{B},\theta,\boldsymbol{h}), \tag{2.1c}$$

$$\boldsymbol{q}=\hat{\boldsymbol{q}}\,(\boldsymbol{B},\theta,\boldsymbol{h}), \tag{2.1d}$$

where $\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ is the left Cauchy-Green tensor,

$$\boldsymbol{h}\equiv\int_{-\infty}^{t}\sigma(t-s)\,\boldsymbol{g}(s)\,ds \tag{2.2}$$

is a *summed history* of $\boldsymbol{g}$ up to the present t and where $\sigma(t)$ denotes the *influence function*. From here the Cartesian coordinates are adopted and a superimposed ^ denotes a function. Equations (2.1) and (2.2) show the *memory effect* with respect to the temperature gradient. In general a theory, which requires an entire past history of a specimen in order to determine its present state, has little value, since we can not know its state of long past. Then here we assume that the effect of remote past state is small in comparison with that of state in the recent past and at the present. Therefore the influence function $\sigma(s)$ is assumed to be a monotonically decreasing function with respect to s and the integral of (2.2) is taken as a fixed material particle.

Noll[7] proved that an isotropic elastic material with invertible stress relation is hypo-elastic. Following his process we will now derive the rate-type constitutive equations of (2.1).

Taking the material time derivative of (2.1) and referring $\dot{\boldsymbol{B}}=\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B}+\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W}$, we have

$$\dot{\psi}=\mathrm{tr}\,(\hat{\psi}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B}))+\mathrm{tr}\,(\hat{\psi}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W}))+\hat{\psi}_{\theta}\dot{\theta}+\hat{\psi}_{\boldsymbol{h}}\cdot\dot{\boldsymbol{h}}, \tag{2.3a}$$

$$\dot{\boldsymbol{T}}=\mathrm{tr}\,(\hat{\boldsymbol{T}}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B}))+\mathrm{tr}\,(\hat{\boldsymbol{T}}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W}))+\hat{\boldsymbol{T}}_{\theta}\dot{\theta}+\hat{\boldsymbol{T}}_{\boldsymbol{h}}\cdot\dot{\boldsymbol{h}}, \tag{2.3b}$$

$$\dot{\eta}=\mathrm{tr}\,(\hat{\eta}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B}))+\mathrm{tr}\,(\hat{\eta}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W}))+\hat{\eta}_{\theta}\dot{\theta}+\hat{\eta}_{\boldsymbol{h}}\cdot\dot{\boldsymbol{h}}, \tag{2.3c}$$

$$\dot{\boldsymbol{q}}=\mathrm{tr}\,(\hat{\boldsymbol{q}}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B}))+\mathrm{tr}\,(\hat{\boldsymbol{q}}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W}))+\hat{\boldsymbol{q}}_{\theta}\dot{\theta}+\hat{\boldsymbol{q}}_{\boldsymbol{h}}\cdot\dot{\boldsymbol{h}}. \tag{2.3d}$$

On the other hand the functions $\hat{\psi}$, $\hat{\boldsymbol{T}}$, $\hat{\eta}$ and $\hat{\boldsymbol{q}}$ must obey the principle of frame-indifference, which demands that they obey the following identities:

$$\hat{\psi}(\boldsymbol{B}^*, \theta, \boldsymbol{h}^*) = \hat{\psi}(\boldsymbol{B}, \theta, \boldsymbol{h}), \tag{2.4a}$$
$$\hat{\boldsymbol{T}}(\boldsymbol{B}^*, \theta, \boldsymbol{h}^*) = \boldsymbol{Q}\hat{\boldsymbol{T}}(\boldsymbol{B}, \theta, \boldsymbol{h})\boldsymbol{Q}^{\mathrm{T}}, \tag{2.4b}$$
$$\hat{\eta}(\boldsymbol{B}^*, \theta, \boldsymbol{h}^*) = \hat{\eta}(\boldsymbol{B}, \theta, \boldsymbol{h}), \tag{2.4c}$$
$$\hat{\boldsymbol{q}}(\boldsymbol{B}^*, \theta, \boldsymbol{h}^*) = \boldsymbol{Q}\hat{\boldsymbol{q}}(\boldsymbol{B}, \theta, \boldsymbol{h}) \tag{2.4d}$$

for all symmetric tensors $\boldsymbol{B}$, all positive scalars θ, all vectors $\boldsymbol{h}$ and all orthogonal tensors $\boldsymbol{Q}$ and where

$$\boldsymbol{B}^* \equiv \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{\mathrm{T}}, \tag{2.5a}$$

$$\boldsymbol{h}^* \equiv \int_{-\infty}^{t} \boldsymbol{Q}(s)\sigma(t-s)\boldsymbol{g}(s)\,ds. \tag{2.5b}$$

Now we may fix $\boldsymbol{B}$, θ, σ, $\boldsymbol{g}$ and choose $\boldsymbol{Q}=\boldsymbol{Q}(\alpha)$ as a function of a parameter α such that $\boldsymbol{Q}(\alpha)=\boldsymbol{1}$ for $\alpha \leqq t$ and $\dot{\boldsymbol{Q}}(t)=\boldsymbol{W}$. If (2.4) are differentiated with respect to α and put $\alpha=t$, we have

$$\mathrm{tr}(\hat{\psi}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W})) + \hat{\psi}_{\boldsymbol{h}}\cdot \boldsymbol{W}\boldsymbol{h} = 0, \tag{2.6a}$$
$$\mathrm{tr}(\hat{\boldsymbol{T}}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W})) + \hat{\boldsymbol{T}}_{\boldsymbol{h}}\cdot \boldsymbol{W}\boldsymbol{h} = \boldsymbol{W}\boldsymbol{T}-\boldsymbol{T}\boldsymbol{W}, \tag{2.6b}$$
$$\mathrm{tr}(\hat{\eta}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W})) + \hat{\eta}_{\boldsymbol{h}}\cdot \boldsymbol{W}\boldsymbol{h} = 0, \tag{2.6c}$$
$$\mathrm{tr}(\hat{\boldsymbol{q}}_{\boldsymbol{B}}(\boldsymbol{W}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{W})) + \hat{\boldsymbol{q}}_{\boldsymbol{h}}\cdot \boldsymbol{W}\boldsymbol{h} = \boldsymbol{W}\boldsymbol{q}. \tag{2.6d}$$

Then we have by substitution (2.6) into (2.3) that

$$\dot{\psi} = \mathrm{tr}(\hat{\psi}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B})) + \hat{\psi}_{\theta}\dot{\theta} + \hat{\psi}_{\boldsymbol{h}}\cdot\mathring{\boldsymbol{h}}, \tag{2.7a}$$
$$\mathring{\boldsymbol{T}} = \mathrm{tr}(\hat{\boldsymbol{T}}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B})) + \hat{\boldsymbol{T}}_{\theta}\dot{\theta} + \hat{\boldsymbol{T}}_{\boldsymbol{h}}\cdot\mathring{\boldsymbol{h}}, \tag{2.7b}$$
$$\dot{\eta} = \mathrm{tr}(\hat{\eta}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B})) + \hat{\eta}_{\theta}\dot{\theta} + \hat{\eta}_{\boldsymbol{h}}\cdot\mathring{\boldsymbol{h}}, \tag{2.7c}$$
$$\mathring{\boldsymbol{q}} = \mathrm{tr}(\hat{\boldsymbol{q}}_{\boldsymbol{B}}(\boldsymbol{B}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{B})) + \hat{\boldsymbol{q}}_{\theta}\dot{\theta} + \hat{\boldsymbol{q}}_{\boldsymbol{h}}\cdot\mathring{\boldsymbol{h}}, \tag{2.7d}$$

where

$$\mathring{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{W}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{W}, \tag{2.8a}$$
$$\mathring{\boldsymbol{h}} = \dot{\boldsymbol{h}} - \boldsymbol{W}\boldsymbol{h}, \tag{2.8b}$$
$$\mathring{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \boldsymbol{W}\boldsymbol{q} \tag{2.8c}$$

are, respectively, the co-rotational time derivatives of $\boldsymbol{T}$, $\boldsymbol{h}$ and $\boldsymbol{q}$, and

$$\dot{\boldsymbol{h}}(t) = \sigma(0)\boldsymbol{g}(t) + \int_{-\infty}^{t} \frac{d}{dt}\sigma(t-s)\boldsymbol{g}(s)\,ds. \tag{2.9}$$

If we assume that for any fixed values of θ and $\boldsymbol{h}$ the stress—deformation relation (2.1b) is invertible,

$$\boldsymbol{B} = \hat{\boldsymbol{B}}(\boldsymbol{T}, \theta, \boldsymbol{h}), \tag{2.10}$$

then, substituting (2.10) into (2.7) we have

$$\dot{\psi} = \boldsymbol{E}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\boldsymbol{D}\} + E(\boldsymbol{T}, \theta, \boldsymbol{h})\dot{\theta} + \boldsymbol{e}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\mathring{\boldsymbol{h}}\}, \tag{2.11a}$$
$$\mathring{\boldsymbol{T}} = \mathsf{H}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\boldsymbol{D}\} + \boldsymbol{H}(\boldsymbol{T}, \theta, \boldsymbol{h})\dot{\theta} + \mathsf{K}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\mathring{\boldsymbol{h}}\}, \tag{2.11b}$$
$$\dot{\eta} = \boldsymbol{M}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\boldsymbol{D}\} + M(\boldsymbol{T}, \theta, \boldsymbol{h})\dot{\theta} + \boldsymbol{m}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\mathring{\boldsymbol{h}}\}, \tag{2.11c}$$
$$\mathring{\boldsymbol{q}} = \mathsf{P}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\boldsymbol{D}\} + \boldsymbol{p}(\boldsymbol{T}, \theta, \boldsymbol{h})\dot{\theta} + \boldsymbol{P}(\boldsymbol{T}, \theta, \boldsymbol{h})\{\mathring{\boldsymbol{h}}\}, \tag{2.11d}$$

where coefficients $\boldsymbol{E}$, E, $\boldsymbol{e}$, H. $\boldsymbol{H}$, K, $\boldsymbol{M}$, M and $\boldsymbol{m}$ have special forms in terms of the derivatives of $\hat{\psi}$, $\hat{\boldsymbol{T}}$, $\hat{\eta}$ and $\hat{\boldsymbol{q}}$ and hence satisfy various integrability conditions.

Coleman and Noll[4] interpreted the Clausius-Duhem inequality:

$$\rho(\dot{\psi}+\eta\dot{\theta})-\mathrm{tr}(\boldsymbol{T}\boldsymbol{D})+\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g}\leqq 0, \tag{2.12}$$

where ρ is the density, as a requirement on the constitutive functions $\hat{\psi}$, $\hat{\boldsymbol{T}}$, $\hat{\eta}$ and $\hat{\boldsymbol{q}}$ for all admissible processes. Here a process is said to be admissible if it is compatible with (2.11), the law of balance of linear momentum:

$$\mathrm{div}\ \boldsymbol{T}+\rho\boldsymbol{b}=\rho\ddot{\boldsymbol{x}}, \tag{2.13}$$

the law of conservation of mass:

$$\dot{\rho}+\rho\,\mathrm{tr}\,\boldsymbol{D}=0, \tag{2.14}$$

and the law of balance of energy:

$$\rho(\psi+\eta\theta)^{\cdot}=\mathrm{tr}(\boldsymbol{T}\boldsymbol{D})-\mathrm{div}\,\boldsymbol{q}+\rho s, \tag{2.15}$$

where $\boldsymbol{b}$ and s are, respectively, the body force per unit mass and the heat supply per unit mass.

Relations (2.11) are derived from (2.1) of an isotropic thermo-elastic material with invertible stress relation. Now we use them as a heuristic basis for defining a generalized hypo-elasticity with thermal influence. Here we regard the coefficients in (2.11) as arbitrary isotropic functions of arguments. Then we have the following definition:

Definition 1. *The relations (2.11) are said to be the constitutive equations of a generalized hypo-elastic material with thermal influence if they satisfy the Clausius-Duhem inequality (2.12) for any admissible process, where the coefficients in (2.11) are isotropic scalar, vector or tensor functions of* $\boldsymbol{T}$, θ, $\boldsymbol{h}$ *and* $\boldsymbol{D}$ *or* $\mathring{\boldsymbol{h}}$ *and linear in* $\boldsymbol{D}$ *or* $\mathring{\boldsymbol{h}}$.

Any constitutive equations of a material must be in frame-indifferent. This fact is satisfied if, and only if all coefficients in the constitutive equations are isotropic functions. Refer to Truesdell and Noll.[8]

The dependence of the constitutive equations upon $\boldsymbol{h}$ indicates that the present rates of state variables depend not only upon the present values of state variables but also upon the past history of the temperature gradients. If we define the constitutive equations as following, we have a thermo-hypo-elastic material with no effect of past history. That is,

$$\dot{\psi}=\boldsymbol{E}(\boldsymbol{T},\theta)\{\boldsymbol{D}\}+E(\boldsymbol{T},\theta)\dot{\theta}+\boldsymbol{e}(\boldsymbol{T},\theta)\{\boldsymbol{g}\}, \tag{2.16a}$$

$$\mathring{\boldsymbol{T}}=\mathsf{H}(\boldsymbol{T},\theta)\{\boldsymbol{D}\}+\boldsymbol{H}(\boldsymbol{T},\theta)\dot{\theta}+\mathsf{K}(\boldsymbol{T},\theta)\{\boldsymbol{g}\}, \tag{2.16b}$$

$$\dot{\eta}=\boldsymbol{M}(\boldsymbol{T},\theta)\{\boldsymbol{D}\}+M(\boldsymbol{T},\theta)\dot{\theta}+\boldsymbol{m}(\boldsymbol{T},\theta)\{\boldsymbol{g}\}, \tag{2.16c}$$

$$\mathring{\boldsymbol{q}}=\mathsf{P}(\boldsymbol{T},\theta)\{\boldsymbol{D}\}+\boldsymbol{p}(\boldsymbol{T},\theta)\dot{\theta}+\boldsymbol{P}(\boldsymbol{T},\theta)\{\boldsymbol{g}\}. \tag{2.16d}$$

Then we have another definition.

Definition 2. *The relations (2.16) are said to be the constitutive equations of a* *thermo-hypo-elastic material if they satisfy (2.12) for any admissible process, where the coefficients in (2.16) are isotropic scalar, vector or tensor functions of* $\boldsymbol{T}$, θ *and* $\boldsymbol{D}$ *or* $\boldsymbol{g}$ *and linear in* $\boldsymbol{D}$ *or* $\boldsymbol{g}$.

In the following sections we will discuss the material defined in Definition 2.

3. Thermodynamic Restriction

Substituting (2.16a) into (2.12) we have

$$\rho\,[\boldsymbol{E}(\boldsymbol{T},\theta)-\frac{1}{\rho}\boldsymbol{T}]\{\boldsymbol{D}\}+\rho[E\,(\boldsymbol{T},\theta)+\eta]\dot{\theta}+\rho[\boldsymbol{e}(\boldsymbol{T},\theta)+\frac{1}{\theta}\boldsymbol{q}]\cdot\boldsymbol{g}\leqq 0. \tag{3.1}$$

Definition 2 demands that inequality (3.1) must hold for every admissible process. As functions of time $\boldsymbol{T}$, θ and $\boldsymbol{g}$ can be chosen arbitrarily, then the values of $\boldsymbol{T}$, θ, $\boldsymbol{D}$, $\dot{\theta}$ and $\boldsymbol{g}$ at an instant time can also be assigned arbitrarily. Therefore we have from (3.1) that

$$\boldsymbol{E}\,(\boldsymbol{T},\theta)=\frac{1}{\rho}\boldsymbol{T}, \tag{3.2a}$$

$$E\,(\boldsymbol{T},\theta)=-\eta, \tag{3.2b}$$

$$\boldsymbol{e}\;(\boldsymbol{T},\theta)=-\frac{1}{\theta}\boldsymbol{q}. \tag{3.2c}$$

Substituting (3.2) into (2.11a) we have

$$\dot{\psi}=\frac{1}{\rho}\,\mathrm{tr}\,(\boldsymbol{T}\boldsymbol{D})-\eta\dot{\theta}-\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g} \tag{3.3}$$

and so we have

Theorem 1. *The constitutive equations of the thermo-hypo-elastic material are (2.11b, c and d) and (3.3).*

Relation (3.2) assures that the inequality (3.1) reduces to an equation, then we can say that

Theorem 2. *The thermo-hypo-elastic material defined is non-dissipative for its thermo-mechanical process.*

From (2.15) and (3.3) we have the law of balance of energy:

$$\rho\theta\dot{\eta}=\frac{\rho}{\theta}\boldsymbol{q}\cdot\boldsymbol{g}-\mathrm{div}\,\boldsymbol{q}+\rho s. \tag{3.4}$$

Here we have fifteen equations (2.16b, c and d), (2.13), (2.14) and (3.4) for fifteen unknowns: $\boldsymbol{T}$, θ, η, $\boldsymbol{q}$, $\boldsymbol{x}$ and ρ. After solving the problem by the assigned initial and boundary conditions we can obtain the rate of free energy from (3.3). In this sense we can regard (2.11b, c and d) as *independent constitutive equations*.

4. Representation of Coefficients of Constitutive Equations

Wang[3] proved a new representation theorem for isotropic functions. From the theorem we can represent the coefficient functions in the constitutive equations (2.16b, c and d) with their argument variables $\boldsymbol{T}$ and $\boldsymbol{D}$ or $\boldsymbol{g}$. We have

$$\begin{aligned}\mathsf{H}(\boldsymbol{T},\theta)\{\boldsymbol{D}\} = {} & \alpha_1\boldsymbol{D}+\alpha_2(\boldsymbol{TD}+\boldsymbol{DT})+\alpha_3(\boldsymbol{T}^2\boldsymbol{D}+\boldsymbol{DT}^2)\\ & +[\alpha_4\mathrm{tr}(\boldsymbol{D})+\alpha_5\mathrm{tr}(\boldsymbol{TD})+\alpha_6\mathrm{tr}(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{1}+[\alpha_7\mathrm{tr}(\boldsymbol{D})\\ & +\alpha_8\mathrm{tr}(\boldsymbol{TD})+\alpha_9\mathrm{tr}(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{T}+[\alpha_{10}\mathrm{tr}(\boldsymbol{D})+\alpha_{11}\mathrm{tr}(\boldsymbol{TD})\\ & +\alpha_{12}\mathrm{tr}(\boldsymbol{T}^2\boldsymbol{D})]\boldsymbol{T}^2,\end{aligned} \tag{4.1a}$$

$$\boldsymbol{H}(\boldsymbol{T},\theta)=\alpha_{13}\boldsymbol{1}+\alpha_{14}\boldsymbol{T}+\alpha_{15}\boldsymbol{T}^2, \tag{4.1b}$$

$$\mathsf{K}(\boldsymbol{T},\theta)\{\boldsymbol{g}\}=\boldsymbol{0}, \tag{4.1c}$$

$$\boldsymbol{M}(\boldsymbol{T},\theta)\{\boldsymbol{D}\}=\beta_1\mathrm{tr}(\boldsymbol{D})+\beta_2\mathrm{tr}(\boldsymbol{TD})+\beta_3\mathrm{tr}(\boldsymbol{T}^2\boldsymbol{D}), \tag{4.2a}$$

$$M(\boldsymbol{T},\theta,\boldsymbol{h})=\beta_4, \tag{4.2b}$$

$$\boldsymbol{m}(\boldsymbol{T},\theta,\boldsymbol{h})\{\mathring{\boldsymbol{h}}\}=0, \tag{4.2c}$$

$$\mathsf{P}(\boldsymbol{T},\theta,\boldsymbol{h})\{\boldsymbol{D}\}=\boldsymbol{0}, \tag{4.3a}$$

$$\boldsymbol{p}(\boldsymbol{T},\theta,\boldsymbol{h})=\boldsymbol{0}, \tag{4.3b}$$

$$\boldsymbol{P}(\boldsymbol{T},\theta,\boldsymbol{h})\{\mathring{\boldsymbol{h}}\}=\gamma_1\boldsymbol{g}+\gamma_2\boldsymbol{Tg}+\gamma_3\boldsymbol{T}^2\boldsymbol{g}. \tag{4.3c}$$

Here α's, β's and γ's are invariant scalar functions of θ and three principal invariants of $\boldsymbol{T}$, i.e., $\mathrm{tr}(\boldsymbol{T})$, $\mathrm{tr}(\boldsymbol{T}^2)$ and $\mathrm{tr}(\boldsymbol{T}^3)$.

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